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NON-LINEAR THEORY OF THIN ELASTIC SHELLS

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Х. М. МУШТАРИ, К. З. ГАЛИМОВ
Kh. M. MUSHTARI, K. Z. GALIMOV

НЕЛИНЕЙНАЯ ТЕОРИЯ УПРУГИХ ОБОЛОЧЕК

NELINEINAYA TEORIYA UPRUGIKH OBOLOCHEK

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A METHOD OF SOLVING BOUNDARY VALUE PROBLEMS FOR NON-LINEAR EQUATIONS IN THE THEORY OF SHALLOW SHELLS

This book deals with the general theory of elastic shells under large displacements and small deformations, and with the application of this theory to the study of the stability and large deflections of parts of shell structures.

The book is intended for scientists, engineers and research students whose work deals with the calculation of the strength and stability of shells; it may also serve as a textbook for senior university students specializing in the theory of elasticity.

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PREFACE

In modern engineering, particularly in aircraft, engines, ships, etc, that is wherever it is necessary to reduce the weight, thin walled structures, the main components of which are thin bars, plates and shells, are widely used.

The characteristic property of these parts is their flexibility, i. e., their relatively small resistance to bending and torsion; therefore, when deformed under load, the displacements of the elements of such structures are comparable to their linear dimensions. The classical theory of elasticity, in particular the theory of shells, is based on the assumption that the displacements of points in the body are infinitestimal and this enables one to neglect, within mathematically strict limits, the squares and higher orders of displacements in comparison with the first order. This so-called linear theory was dealt with by V. Z. Vlasov /0.4, 0.5/, A. L. Goldenvaiser /0.8/, A. I. Lurie /0.10/, A. Love /0.11/, V. V. Novozhilov /0.15/, and others.

The tneory of flexible bars, plates, and shells must be free of such geometrical hypotheses. In this respect it is "geometrically non-linear". Apart from this it may be "physically non-linear" if the stress-strain relation of the body is non-linear. The basis of the general theory of elasticity which takes into account both geometrical and physical non-linearity is given in V. V. Novozhilov's monograph /0.14/, which also gives a rich bibliography of Soviet and foreign works (up to 1946 inclusive).

This monograph, the only one of its kind, deals only with the three-dimensional problem of the theory of elasticity, barely touching upon the theory of flexible bodies. We therefore thought it necessary to deal with the special problem of flexible plates and shells in our monograph. Owing to the magnitude of the problem, we tackled only the geometrical non-linearity.

The reader may find the theory of physical non-linearity in the well-known monograph by A. A. Ityushin /0.9/ and in journals, all of them being based, however, on the assumption of small displacements.

One of the most important problems of the theory of flexible shells is the investigation of the stability of plates and shells. This problem is of interest to us and therefore we have given it particular attention. We did not intend to compile a monograph to replace the well-known work of S. P. Timoshenko /0.16/, "The Stability of Elastic Systems" in which the simpler cases of loss of stability of plates and shells, which are considered classical examples now, were adequately treated. In many aspects, however, Timoshenko's book no longer reflects our present state of knowledge in this field.

We hope that our monograph will largely fill this gap and provide a useful help to undergraduate and post-graduate students in universities who wish to specialize in the theory of elasticity, for post-graduates in other engineering faculties, and for engineers and scientists who have to design structures and calculate strength and stability.

We have dealt with the general non-linear theory of shells without using the tensor calculus, unlike a number of Soviet and foreign papers dealing with this theory /0.1, 0.7, 0.12, 0.19-0.22/; nevertheless it is possible that some parts of the book will be difficult for the beginner because the problems dealt with are very intricate. Such parts are marked by a star ★ at the beginning and the end of the text and can be omitted at first reading*. The greater part of the book, sections 25-65, deals with the application of the general theory. For the convenience of the reader who is mainly interested in this part, section 25 contains a short summary of the preceding material, insofar as it is indispensable for an understanding of the following.

In selecting the material for this monograph, preference was given to those problems which the first of the authors and his students and co-workers had dealt with for 20 years. In dealing with the material, great attention was also given to the contributions of many other Soviet and foreign scientists. We had to refer very often to the monographs on investigations by I. G. Bubnov /0.3/, P. F. Papkovich /0.17/, Wei-Tsang Chien /0.19/, and others. At the end of the book a bibliography of references is given; the numbers of the relevant chapters are shown by Roman numerals. Monographs and general literature are marked by the prefix "0".

Sections 14-23, 25-26, 35-62, were written by Kh. M. Mushtari who also edited the book as a whole. Sections 2-13, 24, 63-65 are by K. Z. Galimov and sections 27-34 by I. V. Svirskii.

The following members of the Kazan' Branch of the Academy of Sciences of the USSR have given valuable help in compiling the material for the monograph: M. S. Kornishin, A. V. Sachenkov, R. G. Surkin, F. S. Isanbaeva, N. I. Krivosheev, and N. S. Ganiev to whom the authors hereby wish to express their thanks.

Finally it should be noted that we do not deal in this monograph with the theories of non-isotropic and laminar shells, supported shells or the dynamic problems of the theory of shells. As these are problems of a specific kind, we feel they should be dealt with in a special monograph so as to avoid superficial treatment of the subject.

Kazan', February 1956

Institute of Physics and Engineering
of the Kazan' Branch of the
Academy of Sciences, USSR

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* In parts covering several pages every page carries the star at the beginning. In the Russian original these parts are in small type - Translator.

§ 1. Fundamental Concepts and Notations*

A body is called a shell if it is bounded by two curved surfaces, the distance between them being small in comparison with the other dimensions of the body. The geometrical locus of all points equidistant from the two boundary surfaces of the shell is called the middle surface. The distance between surfaces, measured normal to the middle surface, is called the thickness of the shell. In all cases, except where stipulated otherwise, the thickness of the shell is assumed to be constant.

Notations for the description of the geometry of deformation of the shell.

σ --Middle surface of shell before deformation;

$R_1 = 1/k_1$, $R_2 = 1/k_2$ --the principal radii of curvature of the middle surface σ, i. e., the greatest and the smallest radii of curvature of the normal sections;

α_1, α_2 --orthogonal curvilinear coordinates, giving the position of a point on the middle surface before and after deformation;

α, β --the same quantities for the case when lines of curvature are taken as coordinate lines;

$A_1\,d\alpha_1$, $A_2\,d\alpha_2$, and correspondingly $A d\alpha$, $B d\beta$--line elements of the coordinate lines;

$\bar{e}_1$, $\bar{e}_2$, $\bar{m}$--unit vectors tangent to the lines α_1 and α_2 and the outward normal to the surface σ which form a right-handed orthogonal triad (see Figure 1);

w^0--projection in the direction of the vector $\bar{m}$ of the initial displacement which transforms a surface of simple geometrical form into a surface σ^0 before the application of the load;

ε_1^0, ε_2^0, $2\varepsilon_{12}^0$--relative elongations and shear corresponding to the above displacement;

$\varkappa_1^0$, $\varkappa_2^0$, $\varkappa_{12}^0$ --changes of curvature and twist of the reference surface σ caused by the initial displacement which characterizes the surface σ^0;

u_1^I, u_2^I, w^I --the projections on the unit vectors $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$ of the vector of displacement, due to a load carrying the surface σ^0 into the surface σ^I;

u^I, v^I, w^I --the same quantities for lines of curvature;

ε_1^I, ε_2^I, $2\varepsilon_{12}^I$, $\varkappa_1^I$, $\varkappa_2^I$, $\varkappa_{12}^I$--elongation, shear, change of curvature, and twist of the surface σ^0 when transforming into surface σ^I.

When considering a single deformed state for a given load, one can omit the index "I" on the symbols $u_1^I, \ldots\ldots$ But if the equilibrium position σ^I is not stable and a change to a new position of equilibrium σ^* is possible, then the additional displacements, elongations, etc, which characterize this change are denoted by $u_1, \ldots, \varkappa_{12}$ respectively.

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* We give here merely a short list of some concepts and notations which will be dealt with fully in the corresponding parts of the book.

We adopt the following notations for static quantities:

T^I_{11}, T^I_{22}, $T^I_{12} = T^I_{21}$ —Tensile and shear stresses of the middle surface σ^I, per unit length of the relevant cross section.

T^I_1, T^I_2, T^I_{12} -the same quantities for lines of curvature;
M^I_{11}, M^I_{22}, $M^I_{12} = M^I_{21}$ or correspondingly M^I_1, M^I_2, M^I_{12}--bending and twisting moments per unit length of cross-section;
$T_1, \ldots, M_{12}$-additional forces and moments caused by the loss of the stability of equilibrium of the shell;
p -- density of normal external pressure on the shell;
p_1, p_2, τ--external tensile and shearing forces applied to the normal edge-sections of the shell. The positive directions of the forces and moments are shown in Figure 2.

General notations

E and ν --modulus of elasticity and Poisson's ratio of the material of the shell which are taken to be constant;

$K = \frac{Et}{1 - \nu^2}$-rigidity under compression;

$D = Et^3/12\,(1 - \nu^2)$--rigidity under bending, the so-called flexural rigidity;

$\overrightarrow{\underleftarrow{1,\ 2}}$--Symbol denoting that the other formulas result from the previous formulas, by permutation of the indices 1, 2, and of the letters u, v;
The symbol $\sim$ shows that the two quantities are of the same order of magnitude;

$(\ldots)_{,1} = \frac{\partial(\ldots)}{\partial\alpha_1}$, $(\ldots)_{,2} = \frac{\partial(\ldots)}{\partial\alpha_2}$ abbreviations for partial derivatives, used only in places where they cannot be misunderstood;
ε_p --relative elongation at the limit of proportionality of the material of the shell;
$[\bar{a}\,\bar{b}]$ and $\bar{a}\,\bar{b}$--respectively the vector and scalar products of vectors a and b.

We shall assume that elongations and shears are small in comparison with unity, although the displacements and changes of curvature are of finite and even of considerable magnitudes.

The bending of the shell is called medium when the deflection is comparable with the thickness of the shell, but is small compared with the other dimensions of the shell. It is called large when the displacement is of the same order as the length and width of the shell. In sections 2-19 of this monograph, the general non-linear theory of shells is dealt with, without restrictions on the degree of bending; the other sections deal with the case of medium bending.

Let L be the characteristic dimension of the shell (its width or its smallest radius of curvature). The shell is considered thin when

$$t/L \sim \varepsilon_p;$$

but if

$$t/L \sim \sqrt{\varepsilon_p},$$

then the shell is of medium thickness.

The subsequently described theory was in fact derived for thin shells for small deformations, by neglecting quantities of order ε_p in comparison with unity.

It can be used in many cases for shells of medium thickness but then the permissible error is of the order of $\sqrt{\varepsilon_p}$ in comparison with unity.

In the following, the Kirchhoff-Love hypothesis is assumed, in which the perpendicular to the middle surface before deformation remains perpendicular to it also after deformation, and at the same time normal stresses perpendicular to σ are considered to be small in comparison with the stresses tangential to the surfaces parallel to σ. As in the linear theory, this hypothesis leads to an error of at most t/L in comparison with unity*.

The non-linear theory of shells constructed without such a hypothesis was given in a tensor formulation in the work by Wei-Tsang Chien /0.19/ and in a linear formulation by N. A. Kil'chevskii /1.6/.

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* See the papers by Novozhilov and Finkelstein /1.1/ and the papers by Mushtari /1.2/.

Chapter I

THEORY OF SHELL DEFORMATION

§ 2. Some Considerations from Differential Geometry*

The Cartesian coordinates x, y, z of a surface without point or line discontinuities can be expressed in terms of independent parameters α_1 and α_2 in the form

$$x = f_1(\alpha_1, \alpha_2), \quad y = f_2(\alpha_1, \alpha_2), \quad z = f_3(\alpha_1, \alpha_2), \tag{2.1}$$

where f_1, f_2 and f_3 are continuous and single-valued functions of α_1 and α_2.

Let $\alpha_1 = \alpha_1^0$ = constant. Then equations (2.1) become the parametric equations of the curve α_1 = constant which lies on the surface (2.1).

Giving a series of values to the parameter α_2, we obtain on the surface a family of curves α_1 = constant on which the parameter α_2 varies. We shall call these α_2-lines. Similarly the equations α_2 = constant determine a second family of curves, the α_1-lines. With the above hypotheses regarding the functions f_1, f_2 and f_3 only one curve of each family will pass through any point of the surface (2.1). Therefore every point of the surface can be taken as the intersection of α_1 and α_2 lines and the position of a point can be specified by the values of α_1 and α_2. The parameters α_1 and α_2 are called curvilinear or Gaussian coordinates of the point on the surface, and the curves α_i = constant are called coordinate lines (or curves) on the surface.

Assume that $\bar{i}$, $\bar{j}$, $\bar{k}$ are unit vectors along the Cartesian coordinate axes and $\bar{r}$ is the radius vector of a point on the surface (2.1); then

$$\bar{r} = \bar{i}x + \bar{j}y + \bar{k}z$$

or, substituting for x, y, z from (2.1),

$$\bar{r} = \bar{i}f_1 + \bar{j}f_2 + \bar{k}f_3 = \bar{r}(\alpha_1, \alpha_2), \tag{2.2}$$

i.e., the radius vector of a point on the surface can be taken as a function of the parameters α_1 and α_2. The expression (2.2) is the vector equation of the surface. In the following we shall assume that the surface is specified by its vector equation (2.2). We adopt the following notations for the partial derivatives of $\bar{r}$ with respect to the coordinates α_1 and α_2:

$$\bar{r}_{,1} = \frac{\partial \bar{r}}{\partial \alpha_1}, \quad \bar{r}_{,2} = \frac{\partial \bar{r}}{\partial \alpha_2}. \tag{2.3}$$

From the definition, $\bar{r}_{,1}$ is the derivative of $\bar{r}$ with constant α_2; therefore, as α_1 varies, the vertex of the vector $\bar{r}$ will describe the curve α_1. Therefore the vector $\bar{r}_{,1}$ is tangent to the curve α_1 and the vector $\bar{r}_{,2}$ tangent to α_2. Thus, $\bar{r}_{,1}$ and $\bar{r}_{,2}$ lie at the given point of the surface, in the tangent plane. These vectors

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* In this section geometrical formulas which are essential for what follows are given, usually without derivations. Details on these matters can be found, for example, in P.K. Rashevskii's book /1.3/.

are called fundamental coordinate vectors of the surface (Figure 1), or simply, coordinate vectors.

We denote the moduli of coordinate vectors and their scalar products as follows

$$|\bar{r}_{,1}| = A_1, \quad |\bar{r}_{,2}| = A_2, \quad \bar{r}_{,1}\bar{r}_{,2} = A_1A_2\cos\chi, \tag{2.4}$$

where χ is the angle between the coordinate lines α_1 and α_2. The quantities A_1 and χ are functions of the coordinates α_1 and α_2. The unit vectors of the curvilinear coordinates $\bar{e}_1$ and $\bar{e}_2$ are:

$$\bar{e}_1 = \frac{\bar{r}_{,1}}{|\bar{r}_{,1}|} = \frac{\bar{r}_{,1}}{A_1}; \quad \bar{e}_2 = \frac{\bar{r}_{,2}}{A_2}. \tag{2.5}$$

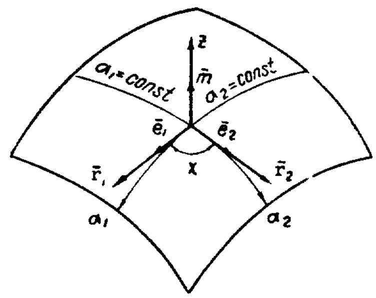

Figure 1

The square of the distance between two infinitesimally near points (x, y, z) and (x + dx, y + dy, z + dz) is

$$ds^2 = dx^2 + dy^2 + dz^2 = |d\bar{r}|^2.$$

If $\bar{r}$ is taken to be a function of α_1 and α_2 we obtain

$$d\bar{r} = \bar{r}_{,1}d\alpha_1 + \bar{r}_{,2}d\alpha_2.$$

Therefore,

$$ds^2 = A_1^2 d\alpha_1^2 + 2A_1A_2\cos\chi\, d\alpha_1 d\alpha_2 + A_2^2 d\alpha_2^2. \tag{2.6}$$

The expression on the right-hand side of the equation is called the "first principal quadratic form of the surface". This form determines the infinitesimal lengths, the angle between curves, and the area on the surface, i. e., it determines the intrinsic geometry of the surface.

To calculate the curvature of a curve which lies on the surface, we have to consider a "second principal quadratic form" of the surface. Let Γ be a certain curve on the surface, given by the vectorial equation $\bar{r} = \bar{r}(s)$ where s is the arc-length from a certain origin and $\bar{\tau}$ is the unit tangent to this curve:

$$\bar{\tau} = \frac{d\bar{r}}{ds} = \bar{r}_{,1}\frac{d\alpha_1}{ds} + \bar{r}_{,2}\frac{d\alpha_2}{ds}. \tag{2.7}$$

According to the Frenet formula, the derivative of this vector is

$$\frac{d\bar{\tau}}{ds}=\frac{\bar{v}}{\rho}, \tag{2.8}$$

where $1/\varrho$ is the curvature of the line Γ, and $\bar{v}$ is the unit vector of the principal normal to this curve.

Substituting for $\bar{\tau}$ from (2.7) in (2.8) we obtain

$$\frac{\bar{v}}{\rho}=\bar{r}_{,11}\left(\frac{da_1}{ds}\right)^2+2\bar{r}_{,12}\frac{da_1}{ds}\cdot\frac{da_2}{ds}+\bar{r}_{,22}\left(\frac{da_2}{ds}\right)^2+$$
$$+\bar{r}_{,1}\frac{d^2a_1}{ds^2}+\bar{r}_{,2}\frac{d^2a_2}{ds^2}. \tag{*}$$

Here and in the following $\bar{r}_{,ik}$ (i, k = 1, 2) are abbreviated expressions of the second order derivatives of $\bar{r}$:

$$\bar{r}_{,ik}=\frac{\partial\bar{r}_{,i}}{\partial a_k}=\frac{\partial\bar{r}_{,k}}{\partial a_i}=\frac{\partial^2\bar{r}}{\partial a_i\partial a_k}.$$

Let $\bar{m}$ be a unit normal to the surface forming a right-handed coordinate system with the fundamental vectors $\bar{r}_{,1}$ and $\bar{r}_{,2}$ i. e., the shortest rotation from $\bar{r}_{,1}$ to $\bar{r}_{,2}$ takes place anticlockwise, and let φ be the angle between $\bar{m}$ and $\bar{v}$ (Figure 2).

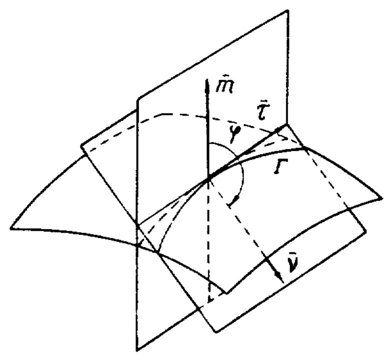

Figure 2

The vector $\bar{m}$ is perpendicular to the coordinate vectors:

$$\bar{m}\,\bar{r}_{,i}=0 \quad (i=1,\ 2). \tag{2.9}$$

If both sides of the expression (*) are scalar-multiplied by $\bar{m}$, we obtain

$$\frac{\cos\varphi}{\rho}=\frac{b_{11}da_1^2+2b_{12}da_1da_2+b_{22}da_2^2}{ds^2}, \tag{2.10}$$

where

$$b_{ik}=b_{ki}=\bar{m}\,\bar{r}_{,ik} \quad (i,\ k=1,\ 2). \tag{2.11}$$

The expression

$$b_{11}da_1^2+2b_{12}da_1da_2+b_{22}da_2^2$$

is called the second principal quadratic form of the surface and the quantities b_{ik}

are the coefficients of the form. Upon differentiating (2.9) with respect to α_k and taking into account (2.11) we obtain for the coefficients b_{ik} the expressions

$$b_{ik} = -\overline{m}_{,i}\overline{r}_{,k} = -\overline{m}_{,k}\overline{r}_{,i} \quad (i,\ k = 1,\ 2), \tag{2.12}$$

where

$$\overline{m}_{,i} = \partial\overline{m}/\partial\alpha_i \quad i = 1,\ 2. \tag{2.13}$$

Thus, one can see from (2.10) that the curvature of a curve on the surface depends on the ratio $d\alpha_1 : d\alpha_2$ i.e., on the sense of the curve. From (2.10) one can, in particular, obtain the curvature of the normal section. For this section, $\overline{m}$ and $\overline{v}$ are either parallel ($\varphi = 0$) or have opposite directions ($\varphi = \pi$). Since a 'plane' curve always leaves its tangent in the direction of vector $\overline{v}$ and one takes its outer normal as the positive normal to the surface, we have $\varphi = \pi$.

Thus we obtain from (2.10) the curvature $1/R$ of the normal section

$$-\frac{1}{R} = \frac{b_{11}d\alpha_1^2 + 2b_{12}d\alpha_1 d\alpha_2 + b_{22}d\alpha_2^2}{A_1^2 d\alpha_1^2 + 2A_1A_2\cos\chi\, d\alpha_1 d\alpha_2 + A_2^2 d\alpha_2^2}. \tag{2.14}$$

From this, by taking α_2 = const and α_1 = const, we obtain the curvatures of the coordinate lines α_1 and α_2:

$$k_{11} = -\frac{b_{11}}{A_1^2}; \quad k_{22} = -\frac{b_{22}}{A_2^2}. \tag{2.15}$$

Through every point of the surface there are two normal sections at which $1/R$ reaches a maximum and a minimum. These are called principal sections for the particular point. The directions of the tangents lying in these sections are called principal directions, and the corresponding curvatures

$$\frac{1}{R_{max}} = \frac{1}{R_1} \text{ and } \frac{1}{R_{min}} = \frac{1}{R_2}$$

are called principal curvatures of the surface. The principal directions are perpendicular to one another. The curves for which the tangents coincide at every point with the principal direction are called the lines of curvature of the surface. Through each point of the surface there pass two mutually orthogonal lines of curvature. If one takes these as coordinate lines, then $\chi = 90°$.

★ The expansion formulae for the second order derivatives of the radius vector $\overline{r}$ with respect to the axes of the principal trihedron $\{\overline{r}_{,1},\ \overline{r}_{,2},\ \overline{m}\}$ are:

$$\overline{r}_{,ik} = \Gamma^1_{ik}\overline{r}_{,1} + \Gamma^2_{ik}\overline{r}_{,2} + \overline{m}b_{ik} \quad (i,\ k = 1,\ 2), \tag{2.16}$$

where Γ^i_{ik} are Christoffel's symbols of the second kind,

$$\begin{gathered} A_i\Gamma^i_{ii} = A_{i,i} + A_i\gamma_i \operatorname{ctg}\chi, \quad A_2\gamma_1 \sin\chi = A_{1,2} - (A_2\cos\chi)_{,1}, \\ A_1\gamma_2\sin\chi = A_{2,1} - (A_1\cos\chi)_{,2}, \quad A_i\sin^2\chi\Gamma^i_{ik} = A_{i,k} - \cos\chi A_{k,i}, \\ A_i\sin\chi\Gamma^i_{kk} = -A_k\gamma_k \quad (i \neq k). \end{gathered} \tag{2.17}$$

Let us calculate the derivatives of the unit vectors of the orthogonal coordinates ($\chi = 90°$). If one substitutes in (2.16) the expression (2.5) for $\overline{r}_{,i}$ and uses the formulas (2.17), one obtains, for $\chi = 90°$

$$A_2\overline{e}_{1,1} = -\overline{e}_2A_{1,2} - A_1A_2k_{11}\overline{m}; \quad A_1\overline{e}_{1,2} = \overline{e}_2A_{2,1} - A_1A_2k_{12}\overline{m}. \quad \overrightarrow{\underleftarrow{1,2}} \tag{2.18}$$

★ where

$$k_{ij} = -b_{ij}/A_i A_j. \tag{2.19}$$

Here and in the following the symbol $\overrightarrow{\underleftarrow{1,\ 2}}$ shows that the formulas which are not fully written down derive from the permutation of indices 1, 2.

For the derivatives of the unit vector $\overline{m}$ we have

$$\overline{m}_{,i} = -b_i^1 \overline{r}_{,1} - b_i^2 \overline{r}_{,2}\,. \tag{2.20}$$

where b_i^k are the expansion coefficients

$$\begin{aligned} A_1 b_i^1 \sin^2\chi &= A_i (k_{i2}\cos\chi - k_{i1}), \\ A_2 b_i^2 \sin^2\chi &= A_i (k_{i1}\cos\chi - k_{i2}). \end{aligned} \tag{2.21}$$

For orthogonal coordinates we obtain from (2.19) and (2.20):

$$\overline{m}_{,i} = A_i \sum_{s=1}^{2} k_{is}\overline{e}_s\,, \quad b_i^j A_j = -A_i k_{ij}. \tag{2.22}$$

With respect to the lines of curvature the latter formulas are:

$$\overline{m}_{,i} = \frac{A_i}{R_i}\overline{e}_i\,, \quad b_i^i = -\frac{1}{R_i}, \quad b_2^1 = b_1^2 = 0. \tag{2.23}$$

The coefficients of the first and second quadratic form of the surface are not independent but satisfy the differential relations of Gauss and Codazzi. The Gauss formula expresses the total curvature of the surface

$$\frac{1}{R_1 R_2} = \frac{b_{11}b_{22} - b_{12}^2}{(A_1 A_2 \sin\chi)^2} \tag{2.24}$$

in terms of the coefficients of the first principal quadratic form of the surface and its derivatives:

$$\frac{b_{12}^2 - b_{11}b_{22}}{A_1 A_2 \sin\chi} = \frac{\partial^2\chi}{\partial\alpha_1\partial\alpha_2} + \frac{\partial}{\partial\alpha_1}\,\frac{A_{2,1} - \cos\chi\, A_{1,2}}{A_1 \sin\chi} + \frac{\partial}{\partial\alpha_2}\,\frac{A_{1,2} - \cos\chi\, A_{2,1}}{A_2 \sin\chi}; \tag{2.25}$$

the two Codazzi relations are, in an arbitrary coordinate system

$$b_{11,2} - b_{12,1} + b_{12}\left(\Gamma_{11}^1 - \Gamma_{12}^2\right) - b_{11}\Gamma_{12}^1 + \Gamma_{11}^2 b_{22} = 0 \quad \overrightarrow{\underleftarrow{1,\ 2}}. \tag{2.26}$$

where Γ_{ik}^j is given by (2.17).

In case of orthogonal coordinates the expressions (2.25) and (2.26) simplify as follows:

$$\begin{gathered} \frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\,\frac{\partial A_2}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\,\frac{\partial A_1}{\partial\alpha_2}\right) = A_1 A_2 (k_{12}^2 - k_{11}k_{22}); \\ (A_2 k_{12})_{,1} - (A_1 k_{11})_{,2} + k_{12}A_{2,1} + k_{22}A_{1,2} = 0 \quad \overrightarrow{\underleftarrow{1,\ 2}}. \bigstar \end{gathered} \tag{2.27}$$

Let us consider curvilinear coordinates in a three-dimensional space. The position of a point P can be specified by the curvilinear coordinates α_1 and α_2 on a surface σ, and a third coordinate z perpendicular to this surface. Let us take, as before, $\overline{r}$ as the radius vector of a point M on the surface from an origin 0. Let $\overline{\rho}$ be the radius vector, taken from the same origin, of a point P in space so that, as seen in Figure 3, one obtains

$$\overline{\rho} = \overline{r}(\alpha_1, \alpha_2) + z\overline{m}(\alpha_1, \alpha_2), \tag{2.28}$$

where $\overline{m}$ is the unit normal to the surface at the particular point.

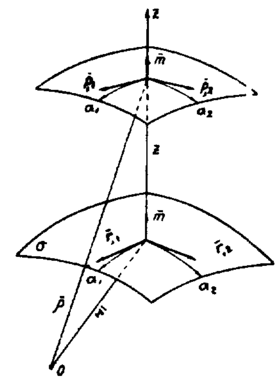

Figure 3

Differentiating with respect to α_i and z we obtain the coordinate vectors at the point P

$$\bar{\rho}_{,i} = \bar{r}_{,i} + \bar{m}_{,i} z, \quad \bar{\rho}_{,3} = \partial\bar{\rho}/\partial z = \bar{m} \quad (i = 1,\ 2).$$

Substituting for $\bar{m}_{,i}$ from (2.20) we obtain, in general coordinates

$$\begin{gathered}\bar{\rho}_{,1} = \bar{r}_{,1}(1 - b_1^1 z) - \bar{r}_{,2} b_1^2 z;\\ \bar{\rho}_{,2} = \bar{r}_{,2}(1 - b_2^2 z) - \bar{r}_{,1} b_2^1 z; \quad \bar{\rho}_{,3} = \bar{m}.\end{gathered} \tag{2.29}$$

By substituting for $\bar{m}_{,i}$ from (2.22) we find the coordinate vectors in orthogonal coordinates:

$$\begin{gathered}\bar{\rho}_{,1} = A_1(1 + k_{11} z)\bar{e}_1 + A_1 k_{12} z \bar{e}_2;\\ \bar{\rho}_{,2} = A_2(1 + k_{22} z)\bar{e}_2 + A_2 k_{12} z \bar{e}_1.\end{gathered} \tag{2.30}$$

For the lines of curvature the latter assume the form

$$\bar{\rho}_{,i} = \bar{r}_{,i}(1 + k_{ii} z). \tag{2.31}$$

In all these formulae $\bar{r}_{,i}$ are the coordinate vectors of the surface σ; therefore

$$\bar{\rho}_{,i}\bar{m} = 0. \tag{2.32}$$

We set

$$g_{ik} = \bar{\rho}_{,i}\bar{\rho}_{,k} \quad (i,\ k = 1,\ 2) \tag{2.33}$$

We denote the unit vectors of orthogonal coordinates by $\bar{e}_i^z$

$$\bar{e}_i^z = \bar{\rho}_{,i}/H_i, \quad H_i = |\bar{\rho}_{,i}|. \tag{2.34}$$

Substituting for $\bar{\rho}_{,i}$ from (2.30) we obtain

$$\begin{gathered}H_1\bar{e}_1^z = A_1(1 + k_{11} z)\bar{e}_1 + A_1 k_{12} z \bar{e}_2;\\ H_2\bar{e}_2^z = A_2(1 + k_{22} z)\bar{e}_2 + A_2 k_{12} z \bar{e}_1,\end{gathered}$$

where $\bar{e}_1$ and $\bar{e}_2$ are the unit vectors of the orthogonal coordinates on the surface σ.

From this we obtain the formulae for Lame's coefficients H_i

$$H_1 = A_1\sqrt{(1+k_{11}z)^2 + k_{12}^2 z^2} \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2,}}$$

or, by expanding these expressions in a power series in z and neglecting the squares and higher powers of the quantities $k_{ij}z$ we obtain:

$$H_i = A_i(1+k_{ii}z). \tag{2.35}$$

In this approximation we obtain, for the unit vector of the orthogonal coordinates,

$$\bar{e}_1{}^z = \bar{e}_1 + k_{12}z\bar{e}_2; \quad \bar{e}_2{}^z = \bar{e}_2 + k_{12}z\bar{e}_1. \tag{2.36}$$

In case of non-orthogonal coordinates α_1 and α_2 the square of the differential arc-length of a curve in space is:

$$ds^2 = |d\bar{\rho}|^2 = g_{11}d\alpha_1{}^2 + 2g_{12}d\alpha_1 d\alpha_2 + g_{22}d\alpha_2{}^2 + dz^2. \tag{2.37}$$

Substituting for $\bar{\rho}_{,i}$ from (2.29) we obtain the formulas

$$g_{ii} = A_i{}^2(1+2k_{ii}z) \quad i = 1,\ 2, \tag{2.38}$$

$$g_{12} = A_1A_2(\cos\chi + 2k_{12}z). \tag{2.39}$$

For the coordinate vectors of orthogonal coordinates (in case of an orthogonal system) we write

$$\begin{gathered} H_2[\bar{\rho}_{,1}\,\bar{m}] = -\bar{\rho}_{,2}H_1, \quad H_1[\bar{\rho}_{,2}\,\bar{m}] = \bar{\rho}_{,1}H_2, \\ [\bar{\rho}_{,1}\,\bar{\rho}_{,2}] = \bar{m}H_1H_2, \end{gathered} \tag{2.40}$$

and for the coordinate vectors of the surface

$$\begin{gathered} A_2[\bar{r}_{,1}\bar{m}] = -\bar{r}_{,2}A_1, \quad A_1[\bar{r}_{,2}\bar{m}] = \bar{r}_{,1}A_2, \\ [\bar{r}_{,1}\,\bar{r}_{,2}] = \bar{m}A_1A_2. \end{gathered} \tag{2.41}$$

From this we obtain for the unit vectors $\bar{e}_1$, $\bar{e}_2$, and $\bar{m}$ the formulas

$$\begin{gathered} [\bar{e}_1\,\bar{m}] = -\bar{e}_2, \quad [\bar{e}_2\,\bar{m}] = \bar{e}_1, \quad [\bar{e}_1\,\bar{e}_2] = \bar{m}, \\ \bar{e}_1\,[\bar{e}_2\,\bar{m}] = 1. \end{gathered} \tag{2.42}$$

§ 3. Deformations of a Surface

Let σ be an undeformed surface referred to orthogonal coordinates. We assume that the deformation of the points of this surface causes a displacement characterized by the vector $\bar{v} = \bar{v}(\alpha_1, \alpha_2)$. The surface σ turns into a new surface σ^* which will be called the deformed surface. We shall specify a point of this surface by the same values of the parameters α_1 and α_2 by which we specified the corresponding point on the undeformed surface, but in general the curvilinear coordinates of σ^* will not be orthogonal. The radius vector $\bar{r}^*$ of a point on the surface σ^* will be

$$\bar{r}^* = \bar{r} + \bar{v}, \tag{3.1}$$

where $\bar{r}$ is the radius vector of the point before deformation.

In the following, all quantities referring to the deformed surface will be marked with an asterisk. The projections of the displacement vector $\bar{v}$ on the (orthogonal) directions $\bar{e}_1$, $\bar{e}_2$, and $\bar{m}$ are:

$$u_1 = \bar{v}\,\bar{e}_1, \quad u_2 = \bar{v}\,\bar{e}_2, \quad w = \bar{v}\cdot\bar{m}. \tag{3.2}$$

Consequently, the displacement vector may be expressed as

$$\bar{v} = u_1\bar{e}_1 + u_2\bar{e}_2 + w\bar{m}. \tag{3.3}$$

Here u_1 and u_2 are tangential displacements and w is the normal displacement.

By differentiating (3.1) with respect to α_i and using the formulas (2.18) and (2.22) for the coordinate vectors on σ^*, we obtain

$$\begin{aligned} \bar{r}^*_{,1} &= A_1\{(1+e_{11})\bar{e}_1 + e_{12}\bar{e}_2 + \varkappa_1\bar{m}\}, \\ \bar{r}^*_{,2} &= A_2\{e_{21}\bar{e}_1 + (1+e_{22})\bar{e}_2 + \varkappa_2\bar{m}\}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} e_{11} &= \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + k_{11}w, \\ e_{12} &= \frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + k_{12}w, \quad \overrightarrow{\underleftarrow{1,2}} \\ \varkappa_1 &= \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - k_{11}u_1 - k_{12}u_2 \end{aligned} \tag{3.5}$$

The first principal quadratic form of the surface σ^* has the general form

$$(ds^*)^2 = (A_1^*)^2\,d\alpha_1^2 + 2A_1^*A_2^*\cos\chi^*\,d\alpha_1 d\alpha_2 + (A_2^*)^2\,d\alpha_2^2. \tag{3.6}$$

Here

$$A_i^* = |\bar{r}^*_{,i}|, \tag{3.7}$$

$$A_1^*A_2^*\cos\chi^* = \bar{r}^*_{,1}\,\bar{r}^*_{,2}, \tag{3.8}$$

where χ^* is the angle between the coordinate lines on σ^*. The differentials of the arc-lengths of coordinate lines before deformation are:

$$(ds)_1 = A_1 d\alpha_1, \quad (ds)_2 = A_2 d\alpha_2. \tag{3.9}$$

After deformation they will be respectively

$$(ds^*)_1 = A_1^* d\alpha_1, \quad (ds^*)_2 = A_2^* d\alpha_2. \tag{3.10}$$

Introducing (3.4) in (3.7) and (3.8) we obtain the formulas for the coefficients of the first quadratic form of σ^*;

$$(A_1^*)^2 = A_1^2(1+2\varepsilon_{11}), \quad (A_2^*)^2 = A_2^2(1+2\varepsilon_{22}), \tag{3.11}$$

$$\cos\chi^* = 2\varepsilon_{12}(1+2\varepsilon_{11})^{-1/2}(1+2\varepsilon_{22})^{-1/2}, \tag{3.12}$$

where we have introduced the new notations:

$$2\varepsilon_{ik} = e_{ik} + e_{ki} + \sum_{s=1}^{2} e_{is}e_{ks} + \omega_i\omega_k \quad (i, k = 1, 2). \tag{3.13}$$

The relative elongations ε_1 and ε_2 in the direction of the coordinate lines may be determined according to (3.9) and (3.10) by the formulas:

$$\varepsilon_1 = (A_1^* - A_1) : A_1 = (1+2\varepsilon_{11})^{1/2} - 1 = \varepsilon_{11} - \frac{1}{2}\varepsilon_{11}^2 + \ldots \overleftrightarrow{1, 2} \tag{3.14}$$

If γ is the change of the angle (initially 90°) between the coordinate lines, then by neglecting the squares of quantities small in comparison with unity, we have:

$$\cos\chi^* = \cos(90^\circ - \gamma) = \sin\gamma \approx \gamma \approx 2\varepsilon_{12}(1 - \varepsilon_{11} - \varepsilon_{22}). \tag{3.15}$$

It may be seen from (3.14) and (3.15) that the quantities ε_{11} and ε_{22} characterize the relative elongations in the direction of the coordinate lines, γ being the shear angle between them.

Let us consider the case of small deformations, i. e., of deformations for which one may neglect the elongations and shear in comparison with unity. According to (3.14) and (3.15) we have:

$$\varepsilon_1 \approx \varepsilon_{11}, \quad \varepsilon_2 \approx \varepsilon_{22}, \quad \gamma \approx 2\varepsilon_{12}.$$

Thus, for small deformations the quantities ε_{ii} are the relative elongations in the direction of the lines α_1 and α_2, and the quantity $2\varepsilon_{12}$ is the shear angle between them. We thus have:

$$A_1^* \approx A_1(1+\varepsilon_{11}), \quad A_2^* \approx A_2(1+\varepsilon_{22})$$

The quantities ε_{ii} and $2\varepsilon_{12}$ characterize the change in the dimensions of an element of the tangent plane. Hence, they are called the components of the tangential deformation of the surface.

In order to derive the expression of the components of the bending deformation of the surface, we express the unit vector of the normal to the deformed surface in terms of the displacement. The normal $\overline{m}^*$ to the deformed surface σ^* can be specified by the following formulas:

$$A_1 A_2 \overline{m}^* = [\vec{r}_{,1}\vec{r}_{,2}].$$

Here and in the following we ignore the elongation and shear in comparison with unity in calculating the bending of the middle surface. If one substitutes for $\bar{r}^*_{,i}$ from (3.4) and uses the vector products (2.42) one obtains for small deformations

$$\bar{m}^* = \bar{e}_1 E_1 + \bar{e}_2 E_2 + \bar{m} E_3, \tag{3.16}$$

where the following new notations have been introduced:

$$\begin{aligned} E_i &= e_{i1}\omega_1 + e_{i2}\omega_2 - (1 + e_{11} + e_{22})\,\omega_i, \\ E_3 &= (1 + e_{11})(1 + e_{22}) - e_{12}e_{21}. \end{aligned} \tag{3.17}$$

To clarify the meaning of the quantities e_{ik}, ω_i and E_i, we form the scalar products of (3.4) and (3.16) with $\bar{e}_i$ and $\bar{m}$. Then, for small deformations, taking the equalities $\bar{e}_1\bar{e}_1 = 1$, $\bar{e}_1\bar{e}_2 = 0$, $\bar{e}_1\bar{m} = 0$ into consideration we obtain

$$\cos(\bar{r}^*_{,i}\bar{e}_i) = 1 + e_{ii}, \quad \cos(\bar{r}^*_{,i}\bar{e}_k) = e_{ik} \quad (i \neq k), \quad i, k = 1, 2;$$
$$\cos(\bar{r}^*_{,i}\bar{m}) = \omega_i, \quad \cos(\bar{m}^*\bar{e}_i) = E_i, \quad \cos(\bar{m}^*\bar{m}) = E_3.$$

Therefore, the parameters e_{ik}, ω_i, E_i, E_3 characterize the angles of rotation of the coordinate vectors $\bar{r}_{,i}$ and $\bar{m}$ in the process of deformation. For the unit vectors $\bar{e}^*_1$ and $\bar{e}^*_2$ of the coordinates of the deformed middle surface σ^* we obtain the following formulas for small deformations (3.4):

$$\begin{aligned} \bar{e}_1^* &= (1 + e_{11})\bar{e}_1 + e_{12}\bar{e}_2 + \omega_1\bar{m}, \\ \bar{e}_2^* &= e_{21}\bar{e}_1 + (1 + e_{22})\bar{e}_2 + \omega_2\bar{m}, \end{aligned} \tag{3.18}$$

since

$$\bar{e}_1^* = \bar{r}^*_{,1}/A_1^* = \bar{r}^*_{,1}/A_1(1 + e_{11}) \approx \bar{r}^*_{,1}/A_1.$$

Let us express the unit vectors of the coordinates of the undeformed shell $\bar{e}_i$ and $\bar{m}$ in terms of the unit vectors $\bar{e}^*_i$ and $\bar{m}^*$. By vectorial multiplication of (3.16) and (3.18) by $\bar{e}_1$ we obtain:

$$\bar{m}^*\bar{e}_1 = E_1, \quad \bar{e}_1^*\bar{e}_1 = 1 + e_{11}, \quad \bar{e}_2^*\bar{e}_1 = e_{21}.$$

Setting here

$$\bar{e}_1 = \alpha\bar{e}_1^* + \beta\bar{e}_2^* + \gamma\bar{m}^*,$$

we obtain the coefficients

$$\alpha = 1 + e_{11}, \quad \beta = e_{21}, \quad \gamma = E_1,$$

since the vectors $\bar{e}^*_1$ and $\bar{e}^*_2$ are mutually orthogonal if we neglect the elongation and shear in comparison with unity; furthermore, according to the Kirchhoff-Love hypothesis $\bar{e}_i^* \perp \bar{m}^*$. We thus obtain the inverse relations

$$\begin{aligned} \bar{e}_1 &= (1 + e_{11})\bar{e}_1^* + e_{21}\bar{e}_2^* + E_1\bar{m}^*, \\ \bar{e}_2 &= e_{12}\bar{e}_1^* + (1 + e_{22})\bar{e}_2^* + E_2\bar{m}^*. \end{aligned} \tag{3.19}$$

By setting

$$\bar{m} = \alpha_1\bar{e}_1^* + \beta_1\bar{e}_2^* + \gamma_1\bar{m}^*.$$

we obtain for the coefficients

$$\alpha_1 = \bar{e}_1^*\bar{m}, \quad \beta_1 = \bar{e}_2^*\bar{m}, \quad \gamma_1 = \bar{m}^*\bar{m};$$

by substituting for $\bar{e}^*_i$ and $\bar{m}^*$ from (3.16) and (3.18) we find

$$\alpha_1 = \omega_1, \quad \beta_1 = \omega_2, \quad \gamma_1 = E_3.$$

Hence we obtain

$$\bar{m} = \bar{e}_1^*\omega_1 + \bar{e}_2^*\omega_2 + \bar{m}^*E_3. \tag{3.20}$$

★ Differentiating (3.16) with respect to α_i and using the relations (2.18) and (2.22), we obtain for the derivatives of $\bar{e}_i$ and $\bar{m}_{,i}$ the formulas:

$$\bar{m}^*_{,i} = A_i(\bar{e}_1E_{i1} + \bar{e}_2E_{i2} + \bar{m}E_{i3}). \tag{3.21}$$

where

$$E_{11} = \frac{1}{A_1}\frac{\partial E_1}{\partial \alpha_1} + \frac{E_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + k_{11}E_3, \quad E_{12} = \frac{1}{A_1}\frac{\partial E_2}{\partial \alpha_1} - \frac{E_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + k_{12}E_3 \quad \overrightarrow{\underleftarrow{1, 2}},$$

$$E_{13} = \frac{1}{A_1}\frac{\partial E_3}{\partial \alpha_1} - k_{11}E_1 - k_{12}E_2, \quad E_{23} = \frac{1}{A_2}\frac{\partial E_3}{\partial \alpha_2} - k_{22}E_2 - k_{12}E_1.$$

The coefficients of the second principal quadratic form of the deformed surface are expressed by formulas similar to (2.12):

$$b^*_{ij} = -\bar{m}^*_{,i}\bar{r}^*_{,j} = -\bar{m}^*_{,j}\bar{r}^*_{,i}. \tag{3.22}$$

By substituting for $\bar{m}^*_{,i}$ from (3.21) and for $\bar{r}^*_{,i}$ from (3.4) we obtain*, since the trihedron $\bar{e}_1\,\bar{e}_2\,\bar{m}$ is orthogonal

$$\begin{aligned} k^*_{i1} &= (1 + e_{11})E_{i1} + e_{12}E_{i2} + \omega_1E_{i3}, \\ k^*_{i2} &= e_{21}E_{i1} + (1 + e_{22})E_{i2} + \omega_2E_{i3} \end{aligned} \tag{3.23}$$

To simplify these formulas we note that the parameters e_{ik}, ω_k, E_k, E_3 satisfy the following algebraic identities:

$$E_1(1 + e_{11}) + e_{12}E_2 + E_3\omega_1 = 0, \quad E_2(1 + e_{22}) + e_{21}E_1 + E_3\omega_2 = 0, \tag{3.24}$$

which follow from the equation $\bar{r}^*_{,i}\,\bar{m}^* = 0$ on substituting for $\bar{r}^*_{,i}$ and $\bar{m}^*$ from formulas (3.4) and (3.16).

To calculate the changes in curvature we neglect the elongations and shear in comparison with unity, so that the following identities are valid

$$\begin{aligned} E_1e_{12} - E_2(1 + e_{11}) &\approx \omega_2; \quad E_3e_{12} - \omega_1E_2 \approx -e_{21} \quad \overrightarrow{\underleftarrow{1, 2}} \\ (1 + e_{11})E_3 - \omega_1E_1 &\approx 1 + e_{22} \quad \overrightarrow{\underleftarrow{1, 2}}, \end{aligned} \tag{3.25}$$

which can easily be verified by replacing E_1 and E_3 by their expressions (3.17) and then using the formulas (3.13). On substituting (3.21) in (3.23) we obtain

$$\begin{aligned} A_1k^*_{11} = {} & (1 + e_{11})E_{1,1} + e_{12}E_{2,1} + \omega_1E_{3,1} + A_2^{-1}[(1 + e_{11})E_2 - e_{12}E_1]A_{1,2} + \\ & + A_1[(1 + e_{11})E_3 - \omega_1E_1]k_{11} + (e_{12}E_3 - \omega_1E_2)A_1k_{12}; \end{aligned}$$

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* For general coordinates these formulas were obtained in /0.7/ and for the lines of curvature in the monograph of V. V. Novozhilov /0.15/. They were given earlier in a slightly different form by Kh. M. Mushtari /1.4/.

★ further replacing the brackets by their approximate expressions from (3.25) and by **again** neglecting the elongation and shear **in comparison with** unity, we have

$$A_1 k_{11}^{*} = (1 + e_{11}) E_{1,1} + e_{12} E_{2,1} + \omega_1 E_{3,1} - A^{-1}{}_2 A_{1,2} \omega_2 + \\ + k_{11} A_1 (1 + e_{22}) - A_1 k_{12} e_{21}.$$

By differentiating the first identity of (3. 24), we find:

$$(1 + e_{11}) E_{1,1} + e_{12} E_{2,1} + \omega_1 E_{3,1} = - E_1 e_{11,1} - E_2 e_{12,1} - E_3 \omega_{1,1}\,.$$

This gives for the coefficients of the second principal quadratic form the expressions

$$\begin{aligned} A_1 k_{11}^{*} &= - E_1 e_{11,1} - E e_{12,1} - E_3 \omega_{1,1} - A_2^{-1} A_{1,2} \omega_2 + A_1 k_{11} (1 + e_{22}) - \\ &\quad - A_1 k_{12} e_{21} \qquad \underleftarrow{\overrightarrow{1,2}}, \\ A_1 k_{12}^{*} &= - E_1 e_{21,1} - E_2 e_{22,1} - E_3 \omega_{2,1} + A_2^{-1} A_{1,2} \omega + A_1 k_{12} (1 + e_{11}) - \\ &\quad - A_1 k_{11} e_{21} \qquad \underleftarrow{\overrightarrow{1,2}}. \; ★ \end{aligned} \tag{3.26}$$

Let us now consider the components of the bending deformation. The quantities ε_{ik} describe the change in the dimensions of an element of the surface in deformation. These are, however, insufficient to define the form of the element owing to the possibility of twisting; we shall characterize the "twist" of the element by changes in the curvature of the coordinate lines α_1 and α_2 in the process of deformation (henceforth denoted by $\varkappa_{11}$ and $\varkappa_{12}$) and by changes of the torsion of the surface, $\varkappa_{12}$. The latter occurs through twisting of the coordinate lines in the tangent plane of the surface.

Therefore, we take the following quantities to be the components of the bending deformation:

$$\varkappa_{11} = \frac{1}{R^{*}_{\alpha_1}} - \frac{1}{R_{\alpha_1}}, \quad \varkappa_{22} = \frac{1}{R^{*}_{\alpha_2}} - \frac{1}{R_{\alpha_2}}, \quad \varkappa_{12} = \varkappa_{21} = k_{12}^{*} - k_{12}, \tag{3.27}$$

where $\frac{1}{R_{\alpha_1}}$, $\frac{1}{R_{\alpha_2}}$ and $\frac{1}{R^{*}_{\alpha_1}}$, $\frac{1}{R^{*}_{\alpha_2}}$ are the curvatures of the coordinate lines before and after deformation. For smaller deformations

$$\frac{1}{R^{*}_{\alpha_1}} \approx - \frac{b_{11}^{*}}{A_1^2} = k_{11}^{*}, \quad \frac{1}{R^{*}_{\alpha_2}} \approx - \frac{b_{22}^{*}}{A_2^2} = k_{22}^{*}. \tag{3.28}$$

On substituting these expressions into (3.27) and using (3.26), one obtains the formulas for the parameters of curvature

$$\begin{aligned} \varkappa_{11} &= k_{11} e_{22} - k_{12} e_{21} - \frac{1}{A_1} \left(E_1 \frac{\partial e_{11}}{\partial \alpha_1} + E_2 \frac{\partial e_{12}}{\partial \alpha_1} + E_3 \frac{\partial \omega_1}{\partial \alpha_1} \right) - \frac{\omega_2}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2}, \\ \varkappa_{12} &= k_{12} e_{11} - k_{11} e_{21} - \frac{1}{A_1} \left(E_1 \frac{\partial e_{21}}{\partial \alpha_1} + E_2 \frac{\partial e_{22}}{\partial \alpha_1} + E_3 \frac{\partial \omega_2}{\partial \alpha_1} \right) + \\ &\quad + \frac{\omega_1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \quad \underleftarrow{\overrightarrow{1,2}}. \end{aligned} \tag{3.29}$$

whereby $\varkappa_{12} = \varkappa_{21}$, **although there are different expressions for these corresponding** to the different expressions for b^{*}_{12} and b^{*}_{21}.

When deriving formulas (3.29), the quantities $b_{ik} \varepsilon_{jn}$ were neglected because they are small in comparison with unity.

The components of deformation of the surface ε_{ik} and χ_{ik} satisfy three differential relations which are called the conditions of continuity [conditions of compatability]. They are obtained by subtracting the Gauss-Codazzi relations for the undeformed surface from those for the deformed surface. If one substitutes the following into Gauss's formula (2.25) for the deformed surface,

$$\cos\chi^* \approx 2\varepsilon_{12}, \quad \sin\chi^* \approx 1, \quad A_i^* \approx A_i(1+\varepsilon_{ii}); \tag{3.30}$$

$$-\sin\chi^*\chi_{,1}^* \approx -\chi_{,1} \approx 2\varepsilon_{12,1}, \quad \overset{*}{b}_{ik} = b_{ik} - A_i A_k \varkappa_{ik}. \tag{3.31}$$

and then subtracts from it the formula (2.27) for the undeformed surface, one obtains, for small deformations*

$$\begin{gathered}\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left[\frac{\partial A_2\varepsilon_{22}}{\partial\alpha_1} - \frac{\partial A_1\varepsilon_{12}}{\partial\alpha_2} - \varepsilon_{11}\frac{\partial A_2}{\partial\alpha_1} - \varepsilon_{12}\frac{\partial A_1}{\partial\alpha_2}\right] + \\ + \frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left[\frac{\partial A_1\varepsilon_{11}}{\partial\alpha_2} - \frac{\partial A_2\varepsilon_{12}}{\partial\alpha_1} - \varepsilon_{22}\frac{\partial A_1}{\partial\alpha_2} - \varepsilon_{12}\frac{\partial A_2}{\partial\alpha_1}\right] = \\ = A_1A_2\left(\varkappa_{12}^2 - \varkappa_{11}\varkappa_{22} - \varkappa_{11}k_{22} - \varkappa_{22}k_{11} + 2\varkappa_{12}k_{12}\right).\end{gathered} \tag{3.32}$$

★ This is one of the conditions of continuity. We obtain two other conditions if we subtract from Codazzi's relations for a deformed surface

$$\overset{*}{b}_{11,2} - \overset{*}{b}_{12,1} + \overset{*}{b}_{12}\left(\overset{*}{\Gamma}{}^1_{11} - \overset{*}{\Gamma}{}^2_{12}\right) - \overset{*}{b}_{11}\overset{*}{\Gamma}{}^1_{12} + \overset{*}{\Gamma}{}^2_{11}\overset{*}{b}_{22} = 0, \quad \overleftrightarrow{1,2} \tag{3.33}$$

the corresponding conditions (2.27, second equation) for the undeformed surface. For this purpose, we first calculate Christoffel's symbols Γ_{ik}^l for the deformed surface. From (2.17) we obtain, using (3.30), the expressions

$$\begin{gathered}A_1A_2\overset{*}{\Gamma}{}^1_{11} = A_2A_{1,1} + A_1A_2\varepsilon_{11,1} + 2A_1\varepsilon_{12}A_{1,2}, \\ A_1\overset{*}{\Gamma}{}^1_{12} = A_{1,2} + A_1\varepsilon_{11,2} - 2\varepsilon_{12}A_{2,1}, \\ A_2^2\overset{*}{\Gamma}{}^2_{11} = -A_1A_{1,2} + 2A_1(A_2\varepsilon_{12})_{,1} + \varepsilon_{22}(A_1^2)_{,2} - (A_1^2\varepsilon_{11})_{,2} \quad \overrightarrow{1,2}.\end{gathered} \tag{3.34}$$

If we substitute (3.31) and (3.34) in (3.33) and use (2.27), two further conditions of continuity are obtained:

$$\begin{gathered}\frac{\partial A_1\varkappa_{11}}{\partial\alpha_2} - \frac{\partial A_2\varkappa_{12}}{\partial\alpha_1} - \varkappa_{22}\frac{\partial A_1}{\partial\alpha_2} - \varkappa_{12}\frac{\partial A_2}{\partial\alpha_1} + \\ + (k_{11}+\varkappa_{11})\left(2\varepsilon_{12}\frac{\partial A_2}{\partial\alpha_1} - A_1\frac{\partial\varepsilon_{11}}{\partial\alpha_2}\right) + \\ + (\varkappa_{12}+k_{12})\left[4\varepsilon_{12}\frac{\partial A_1}{\partial\alpha_2} + A_2\frac{\partial(\varepsilon_{11}-\varepsilon_{22})}{\partial\alpha_1}\right] + \\ + (k_{22}+\varkappa_{22})\left[2\frac{\partial A_2\varepsilon_{12}}{\partial\alpha_1} - 2(\varepsilon_{11}-\varepsilon_{22})\frac{\partial A_1}{\partial\alpha_2} - A_1\frac{\partial\varepsilon_{11}}{\partial\alpha_2}\right] = 0 \quad \overrightarrow{1,2}.\end{gathered} \tag{3.35}$$

It should be noted that in deriving (3.32), the products of the first derivatives of the elongations and shears were neglected in comparison with the second derivatives of the same. If on differentiating with respect to any coordinate, the deformations do not increase, products like $\varepsilon_{ij,k}\cdot\varepsilon_{rs,t}$ are small quantities of second order. But if the derivatives grow $\varepsilon_p^{-\lambda}$ times with respect to the deformation, as it happens in the zone of the edge effect, then:

$$\varepsilon_{ij,k} \sim \varepsilon_p\varepsilon_p^{-\lambda},$$

where ε_p is the maximum relative elongation in the limit of proportionality; the symbol $\sim$ shows that the compared quantities are of the same order of magnitude. The second derivatives will then be of order

$$\varepsilon_{ij,ks} \sim \varepsilon_p^{1-2\lambda},$$

whereas the products of the derivatives are

$$\varepsilon_{ij,k}\cdot\varepsilon_{rs,t} \sim \varepsilon_p^{2-2\lambda}.$$

Therefore, one can neglect the products of the derivatives in this case as well.★

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* The derivation of these relations in general coordinates is given in a paper by K. Z. Galimov /1.5/. See also N. A. Alumyae /0.1/.

§ 4. Deformations of a Shell

We shall refer the middle surface σ of the undeformed shell to the orthogonal coordinates α_1 and α_2. We specify the position of any point P of the shell by the same Gaussian coordinates α_1 and α_2 and by the coordinate z perpendicular to the middle surface. Then the radius vector of the point P is:

$$\bar{\rho} = \bar{r} + \bar{m}z, \tag{4.1}$$

where $\bar{r}$ is the radius vector of a point on the middle surface, $\bar{m}$ the unit normal to the plane.

The coordinate vectors of the point P on the undeformed shell given by formulas (2.30), and Lame's coefficients H_i by (2.35).

The square of the distance between two infinitesinally near points on the surface σ^z, parallel to the surface σ is:

$$(ds^z)^2 = H_1^2 d\alpha_1^2 + H_2^2 d\alpha_2^2 + dz^2. \tag{4.2}$$

As before, we fix the position of the point P_* of the deformed shell by the Gaussian coordinates α_1 and α_2 on the deformed middle surface σ^* and by the coordinate z^* perpendicular to it. For small deformations one can assume $z^* \approx z$ because that is equivalent to neglecting the elongation in comparison with unity:

$$z^* = z(1 + \varepsilon_{33}) \approx z,$$

where ε_{33} is the relative elongation in the direction of the normal $\bar{m}$.

The radius vector of the point P_* is:

$$\bar{\rho}_* = \bar{\rho} + \bar{u}, \tag{4.3}$$

where u is the displacement vector of the point P. In the following, the quantities relating directly to the deformed shell or to its middle surface σ^* will be marked by an asterisk. The differential arc-length at the point P_* on the deformed shell is given by the general formula (2.37):

$$(ds^z_*)^2 = g^*_{11} d\alpha_1^2 + 2g^*_{12} d\alpha_1 d\alpha_2 + g^*_{22} d\alpha_2^2 + dz^2. \tag{4.4}$$

Here

$$g^*_{ik} = \bar{\rho}^*_{,i}\,\bar{\rho}^*_{,k}, \tag{4.5}$$

and $\bar{\rho}_1{}^*$ and $\bar{\rho}_2{}^*$ are the coordinate vectors of the point P_* of the deformed shell which can be expressed by formulas similar to (2.20):

$$\begin{aligned} \bar{\rho}^*_{,1} &= \bar{r}^*_{,1}(1 - \overset{*}{b}{}_1^1 z) - \bar{r}^*_{,2}\,\overset{*}{b}{}_1^2 z, \\ \bar{\rho}^*_{,2} &= \bar{r}^*_{,2}(1 - \overset{*}{b}{}_2^2 z) - \bar{r}^*_{,1}\,\overset{*}{b}{}_2^1 z, \quad \bar{\rho}^*_{,3} = \bar{m}^*, \end{aligned} \tag{4.6}$$

Here $\bar{m}^*$ is the normal to the surface σ^*, $r^*_{,1}$ and $r^*_{,2}$ are coordinate vectors on σ^*, and $b^{*}{}_k^i$ are quantities which are calculated for the deformed surface according to formulas (2.21). Let ud note that in deriving these expressions, as well as in the following, the Kirchhoff-Love hypothesis is used (see § 1).

The arc-lengths of the coordinate lines at height z above the middle surface are before deformation:

$$(ds^z)_1 = H_1 d\alpha_1, \quad (ds^z)_2 = H_2 d\alpha_2, \tag{4.7}$$

and after deformation, according to (4.4):

$$(ds^z_*)_1 = \sqrt{g^*_{11}}\, d\alpha_1, \quad (ds^z_*)_2 = \sqrt{g^*_{22}}\, d\alpha_2. \tag{4.8}$$

Hence, the relative elongations ε_1^z and ε_2^z in the direction of the coordinate lines α_1 and α_2 at the point $P(\alpha_1, \alpha_2, z)$ are given by

$$\varepsilon_1{}^z = \frac{(ds^z_*)_1 - (ds^z)_1}{(ds^z)_1}, \quad \varepsilon_2{}^z = \frac{(ds^z_*)_2 - (ds^z)_2}{(ds^z)_2},$$

or, substituting from (4.7) and (4.8), by

$$\varepsilon_1{}^z = \left(\sqrt{g^*_{11}} - H_1\right) : H_1, \quad \varepsilon_2{}^z = \left(\sqrt{g^*_{22}} - H_2\right) : H_2. \tag{4.9}$$

The cosine of the angle χ_*^z between $\bar{\rho}^*_{,1}$ and $\bar{\rho}^*_{,2}$ is given by

$$\cos \chi^z_* = \frac{\vec{\rho}^*_{,1}\, \vec{\rho}^*_{,2}}{|\vec{\rho}^*_{,1}| \cdot |\vec{\rho}^*_{,2}|} = \frac{g^*_{12}}{\sqrt{g^*_{11} g^*_{22}}}, \tag{4.10}$$

where g^*_{ik} are given by approximate formulas of the type (2.38) for the deformed shell:

$$g^*_{ii} = (A^*_i)^2 (1 + 2zk^*_{ii}),\ g^*_{12} = A^*_1 A^*_2 (\cos \chi^* + 2zk^*_{12}), \tag{4.11}$$
$$k^*_{is} = -\, b^*_{is} / A^*_i A^*_s \approx -\, b^*_{is} / A_i A_s.$$

The angle of shear between the coordinate lines α_1 and α_2 at height z from the undeformed middle surface is denoted by $2\varepsilon_{12}^z$, i. e., we set $2\varepsilon_{12}^z = 90° - \chi_*^z$. Then, according to (4.10), we have:

$$\cos \chi^z_* = \sin 2\varepsilon^z_{12} = g^*_{12} / \sqrt{g^*_{11} g^*_{22}}.$$

Since for small deformations $\sin 2\varepsilon_{12}^z \approx 2\varepsilon_{12}^z$ one obtains, considering (4.9):

$$2\varepsilon^z_{12} \approx g^*_{12} / H_1 H_2 (1 + \varepsilon^z_1)(1 + \varepsilon^z_2) \approx g^*_{12} / H_1 H_2. \tag{4.12}$$

Thus, $2\varepsilon_{12}^z \approx \cos \chi_*^z$ is the angle of shear and the quantities ε_1^z ε_2^z and ε_{12}^z determine the deformation of a surface element of σ^* which is at a distance z from the middle surface σ and is parallel to it. The angles of shear ε_{13}^z and ε_{23}^z are zero, because by Kirchhoff's hypothesis $\bar{\rho}^*_{,i}\, \bar{m}^* = 0$.

We shall express the components of deformation by the characteristics of the deformation of the middle surface.

When calculating the relative elongations of the shell, one can neglect the angle of shear between the coordinate lines in comparison with unity, because we assume that the deformations are small. Therefore, we obtain by formulas (4.11):

$$\sqrt{g_{ii}^*} \approx A_i^*(1 + k_{ii}^* z) \approx A_i^* + A_i k_{ii}^* z. \qquad (4.13)$$

So that according to (4.9) by neglecting the elongations and the quantities $k_{ij}z$ in comparison with unity, we obtain

$$\varepsilon_i^z = \frac{A_i^* + A_i k_{ii}^* z - A_i (1 + k_{ii} z)}{A_i (1 + k_{ii} z)} \approx \varepsilon_{ii} + z\varkappa_{ii}. \qquad (4.14)$$

Similarly, according to (4.11) and (4.12), and assuming that $\cos\chi^* = 2\varepsilon_{12}$, $k_{12}^* = k_{12} + \varkappa_{12}$, we obtain the angle of shear between the coordinate lines at the surface σ^z:

$$\varepsilon_{12}^z = \varepsilon_{12} + z\,(k_{12} + \varkappa_{12}). \qquad (4.15)$$

And thus, for small deformations of a thin shell, the first approximation formulas of Love (4.14) and (4.15) (see /0.11/) are also valid for finite displacements.

★ To end this chapter, let us consider the displacements within the shell. The displacement vector is:

$$\bar{U} = \bar{\rho}^* - \bar{\rho}.$$

By substituting here $\bar{\rho}^* = \bar{r}^* + \bar{m}^* z$ and $\bar{\rho} = \bar{r} + \bar{m} z$, we obtain:

$$\bar{U} = \bar{v} + z\,(\bar{m}^* - \bar{m}), \qquad (4.16)$$

where $\bar{v} = \bar{r}^* - \bar{r}$ is the displacement vector of points of the middle surface of the shell.

The unit vectors of the coordinates $\bar{e}_1^z$ and $\bar{e}_2^z$ are given by (2.36). Denoting the projections of the displacement vectors along the coordinates by U_1^z, U_2^z and W^z, we obtain:

$$U_1^z = \bar{e}_1^z \cdot \bar{U}, \quad U_2^z = \bar{e}_2^z \cdot \bar{U}, \quad W^z = \bar{m} \cdot \bar{U}.$$

Substituting here for $\bar{U}$ from (4.16) and for $\bar{e}_1^z$ and $\bar{e}_2^z$ from (2.36):

$$U_1^z = u_1 + z\bar{m}_* \bar{e}_1 + k_{12} z\,(u_2 + z\bar{m}^* \bar{e}_2) \approx u_1 + z\,(\bar{m}^* \bar{e}_1 + k_{12} u_2),$$

$$U_2^z \approx u_2 + z\,(\bar{m}^* \bar{e}_2 + k_{12} u_1), \quad W^z = w + z\,(\bar{m}^* \bar{m} - 1).$$

Further on substituting for $\bar{m}^*$ from (3.16) in the above expressions, we obtain

$$U_1^z = u_1 + z\,(E_1 + k_{12} u_2), \; U_2^z = u_2 + z\,(E_2 + k_{12} u_1), \; W^z = w + z\,(E_3 - 1). \qquad (4.17)$$

Thus the Kirchhoff-Love hypothesis leads to a linear distribution of displacements along the thickness of the shell. ★

Chapter II

EQUILIBRIUM EQUATIONS OF THE THEORY OF THIN SHELLS FOR SMALL DEFORMATIONS AND ARBITRARY DEFLECTIONS*

§ 5. Equilibrium Equations of the Theory of Elasticity in Orthogonal Curvilinear Coordinates

The orthogonal curvilinear coordinates α_1 and α_2 will no longer be orthogonal after deformation. In the following, when deriving the static relations we shall assume that they are orthogonal by neglecting shear, small in comparison with unity. But we shall not neglect the elongations (in comparison with unity) because in many cases the derivatives of the elongations may be of the same order of magnitude as the twists e_{ik} and ω_i of the coordinate axes during deformation. Neglecting this fact can lead, as will be shown in Chapter V, to substantial errors.

Let us derive the conditions of equilibrium for an elementary parallelepiped, cut out of the deformed shell and bounded by the planes α_1 = const and $\alpha_1 + d\alpha_1$ = const, z = const and $z + dz$ = const.

Here the α_i have the same numerical values as before the deformation. We assume that $\bar{F}$ is the body force vector per unit volume of the deformed shell, and $\bar{p}_i$ and $\bar{p}_z$ the stress vectors, applied to the surfaces α_i = const and z = const of the deformed parallelepiped per unit area.

The deformed parallelepiped is in equilibrium as a result of the following:

1. Stresses on the faces α_i = const and $\alpha_i +\ \alpha_i$ = const:

$$-\bar{p}_i\, d\sigma^i \text{ and } \bar{p}_i\, d\sigma^i + (\bar{p}_i\, d\sigma^i)_{,i}\, d\alpha_i \quad (i = 1, 2),$$

where $d\sigma^i$ is the area of the face α_i = const (the comma in front of the i signifies partial differentiation with respect to α_i);

2. Stresses on the faces z = const and $z + dz$ = const:

$$-\bar{p}_z\, d\sigma^z \text{ and } \bar{p}_z\, d\sigma^z + (\bar{p}_z\, d\sigma^z)_{,z}\, dz,$$

where $d\sigma^z$ is the area of the face z = const, and $(\bar{p}_z d\sigma^z)_{,z} = \partial(\bar{p}_z d\sigma^z)/\partial z$;

3. The body force $\bar{F} d\Omega$, where $d\Omega$ is the volume of the element under consideration.

When the shell element under consideration is in equilibrium, the geometrical

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* The contents of §§ 5-7 for general coordinates were given in a tensorial form in A. I. Lurie's article /11.1/. The Eulerian and Lagrangian formulations of the theory of finite deformations in general coordinates can be found in articles by K. Z. Galimov, /11.2/ and /11.3/. The theory of finite deformations of continuous media is also dealt with in the papers /0.21/, /11.4/, /11.7/, etc.

sum of all these forces is equal to zero:

$$(\bar{p}_1\, d\sigma^1)_{,1}\, d\alpha_1 + (\bar{p}_2\, d\sigma^2)_{,2}\, d\alpha_2 + (\bar{p}^z d\sigma^z)_{,z}\, dz + \bar{F}\, d\Omega = 0.$$

The areas of the faces of the deformed element in orthogonal coordinates are respectively:

$$d\sigma^1 = H_2^*\, d\alpha_2 dz,\ d\sigma^2 = H_1^* d\alpha_1 dz,\ d\sigma^z = H_1^* H_2^* d\alpha_1 d\alpha_2,$$

and its volume is $d\Omega = H_1^* H_2^*\, d\alpha_1 d\alpha_2 dz$, where according to (2.35), $H_i^* \approx A_i^*(1 + k_{ii}^* z)$.

For a thin shell the quantities $k_{ii}^* z$ can be neglected, so that we can set:

$$H_i^* \approx A_i^* = A_i(1 + \varepsilon_{ii}) \qquad (i = 1, 2).$$

Hence,

$$\begin{gathered} d\sigma^1 = A_2^* d\alpha_2\, dz;\quad d\sigma^2 = A_1^* d\alpha_1 dz; \\ d\sigma^z = A_1^* A_2^*\, d\alpha_1\, d\alpha_2;\quad d\Omega = A_1^* A_2^*\, d\alpha_1 d\alpha_2 dz. \end{gathered} \tag{5.1}$$

If we substitute these expressions in the above condition of equilibrium and drop the common factor $d\alpha_1 d\alpha_2 dz$, we obtain the vector equation of equilibrium of the deformed parallelepiped:

$$(\bar{p}_1 A_2^*)_{,1} + (\bar{p}_2 A_1^*)_{,2} + (\bar{p}_z A_1^* A_2^*)_{,z} + \bar{F} A_1^* A_2^* = 0. \tag{5.2}$$

For equilibrium, it is necessary that the resultant moment of all forces acting on the parallelepiped should vanish. Let us derive the vector equation of the moments. We denote by $\bar{\rho}^*$ the radius vector of the vertex $P^*(\alpha_1, \alpha_2, z)$ of the parallelepiped; $\delta'\bar{\rho}^*$, $\delta''\bar{\rho}^*$ and $\delta'''\bar{\rho}^*$ increments that have to be added to $\bar{\rho}^*$ to determine the radii vectors of the other vertices of the parallelepiped, that is, the displacements from the point P^* to the other vertices

$$\delta'\bar{\rho}^* = \bar{\rho}_{,1}^*\, d\alpha_1,\quad \delta''\bar{\rho}^* = \bar{\rho}_{,2}^*\, d\alpha_2,\quad \delta'''\bar{\rho}^* = \bar{m}^*\, dz,$$

Here $\bar{m}^*$ is the unit vector of the z^* axis of the deformed shell. If we take the moment of the stresses acting on the faces α_1 = const and $\alpha_1 + d\alpha_2$ = const with respect to the center of the parallelepiped, we obtain

$$\frac{1}{2}[-\delta'\bar{\rho}_{,}^*\ -\bar{p}_1 d\sigma^1] + \frac{1}{2}[\delta'\bar{\rho}_{,}^*\ \bar{p}_1 d\sigma^1 + (\bar{p}_1 d\sigma^1)_{,1}\, d\alpha_1] = [\bar{\rho}_{,1}^* \bar{p}_1]\, d\sigma' d\alpha_1,$$

Here, the fourth order quantities in the coordinate differentials were neglected. The moments of the stresses acting on the other faces will be $[\bar{\rho}_{,2}^* \bar{p}_2]\, d\sigma^2 d\alpha_2$ and $[\bar{m}^* \bar{p}_z]\, d\sigma^z dz$; the moment of body force will be a fourth order infinitesimal, which we neglect. Thus, the equation of the moments of the surface forces, after cancelling $d\alpha_1 d\alpha_2 dz$, will be

$$[\bar{\rho}_{,1}^* \bar{p}_1] A_2^* + [\bar{\rho}_{,2}^* \bar{p}_2] A_1^* + [\bar{m}^* \bar{p}_z] A_1^* A_2^* = 0. \tag{5.3}$$

The stress vectors $\bar{p}_1$ and $\bar{p}_2$ can be resolved along the coordinate axes of the deformed shell*

$$\bar{e}_1^* = \bar{\rho}_{,1}^*/|\bar{\rho}_1^*| \approx \bar{\rho}_{,1}^*/A_1^*,\quad \bar{e}_2^* = \bar{\rho}_{,2}^*/A_2^*,\quad \bar{m}^* \tag{5.4}$$

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* Although $z^* \approx z$, z^* will sometimes enter instead of z. Then it means that the segment z is taken in the direction of m.

are expressed as follows (Figure 4):

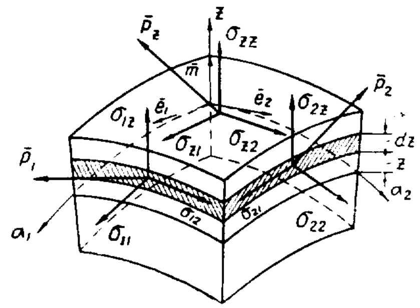

Figure 4

$$\begin{aligned}
\bar{p}_1 &= \sigma_{11}\bar{e}_1^{z*} + \sigma_{12}\bar{e}_2^{z*} + \sigma_{1z}\bar{m}^*, \\
\bar{p}_2 &= \sigma_{21}\bar{e}_1^{z*} + \sigma_{22}\bar{e}_2^{z*} + \sigma_{2z}\bar{m}^*, \\
\bar{p}_z &= \sigma_{z1}\bar{e}_1^{z*} + \sigma_{z2}\bar{e}_2^{z*} + \sigma_{zz}\bar{m}^*.
\end{aligned} \tag{5.5}$$

Let us prove that the components of the stresses are symmetrical:

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{1z} = \sigma_{z1}, \quad \sigma_{2z} = \sigma_{z2}. \tag{5.6}$$

On substituting (5.5) in (5.3) we obtain:

$$[\bar{e}_1^{z*}\bar{e}_2^{z*}](\sigma_{12} - \sigma_{21}) + [\bar{e}_1^{z*}\bar{m}^*](\sigma_{1z} - \sigma_{z1}) + [\bar{e}_2^{z*}\bar{m}^*](\sigma_{2z} - \sigma_{z2}) = 0,$$

from which equations (5. 6) follow, since the coefficnents of the vector products must vanish.

§ 6. Forces and Moments. Reduction of Stresses and External Forces to the Middle Surface of the Deformed Shell

In order to derive the static relations of the theory of shells, let us take an element of the deformed shell which is cut out by the normal cross-sections α_i = const, $\alpha_i + d\alpha_i$ = const and bounded by the surfaces $z = \pm t/2$ (Figure 5).

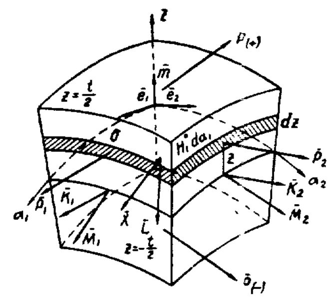

Figure 5

Here α_i are the Gaussian coordinates on the deformed middle surface of the shell and z the coordinate perpendicular to this surface. The same coordinates α_i and z give the position of the point P* (α_1, α_2, z) of the shell, but the unit vectors of the coordinates of the shell will be e_i^* and m*. We denote by $\overline{K}_1$ and $\overline{K}_2$ the resultants, (per unit length of the coordinates α_1 and α_2 of the middle surface) of all forces which act upon the surfaces α_2 = const and α_1 = const. $\overline{K}_1$ and $\overline{K}_2$ are the internal forces in the shell. To calculate these, let us assume $\bar{p}_1$ and $\bar{p}_2$ to be the stresses acting on strips of the faces α_1 = const and α_2 = const. As the areas of the strips (of height dz) at the distance z from the middle surface are respectively $A_2^* \, d\alpha_2 \, dz$ and $A_1^* \, d\alpha_1 \, dz$, the forces on the strips of the faces α_1 = const and α_2 = const will be $\bar{p}_1 A_2^* \, d\alpha_2 dz$ and $\bar{p}_2 A_1^* \, d\alpha_1 dz$ respectively.

Therefore, the resultants of all forces which are acting the faces α_1 = const and α_2 = const of the element of the shell are

$$\int_{-t/2}^{t/2} \bar{p}_1 A_2^* \, d\alpha_2 dz \quad \text{and} \quad \int_{-t/2}^{t/2} \bar{p}_2 A_1^* \, d\alpha_1 dz.$$

Dividing by the arc-lengths $A_2^* \, d\alpha_2$ and $A_1^* \, d\alpha_1$ of the coordinate lines α_1 and α_2 on the middle surface σ*, we obtain:

$$\overline{K}_1 = \int_{-t/2}^{t/2} \bar{p}_1 \, dz, \quad \overline{K}_2 = \int_{-t/2}^{t/2} \bar{p}_2 \, dz. \qquad (6.1)$$

Let us calculate the principal moments of internal forces about a point on the deformed middle surface. For the force acting on a strip of the face α_1 = const about a point on the middle surface it is the vector product $[\bar{m}^* z, \, \bar{p}_1 A_2^* d\alpha_2 dz]$, as the radius vector of this force is $\bar{m}^* z$. The principal moment of all internal forces acting on the face

α_1 = const is the integral $\int_{-t/2}^{t/2} [\overline{m}^* z, \overline{p}_1 A_2^* d\alpha_2]\, dz$. Dividing this by the arc-length $A_2^* d\alpha_2$ we obtain:

$$\int_{-t/2}^{t/2} [\overline{m}^* z,\ \overline{p}_1]\, dz.$$

Thus, the principal moments $\overline{M}_1$ and $\overline{M}_2$ of the internal forces on the faces α_1 = const and α_2 = const per unit length of the coordinate lines α_1 and α_2 on the middle surface σ^* are:

$$\overline{M}_1 = \int_{-t/2}^{t/2} [\overline{m}^* z,\ \overline{p}_1]\, dz, \quad \overline{M}_2 = \int_{-t/2}^{t/2} [\overline{m}^* z,\ \overline{p}_2]\, dz. \tag{6.2}$$

Thus, the internal forces acting on the lateral faces α_i = const of an element of the shell are statically equivalent to the force $\overline{K}_i$ and the principal moment $\overline{M}_i$ acting on the coordinate lines α_i = const on the deformed middle surface.

The external forces acting on the shell are also reduced to the middle surface. Let $\overline{p}_{(+)}$ and $\overline{p}_{(-)}$ be the external loads per unit area which act on the boundary surfaces $z = t/2$ and $z = -t/2$, and $\overline{F}$ the body force, per unit volume of the shell. Let us find the resulting principal force and principal moment of the external forces. Since

$$(\overline{p}_z)_{z=t/2} = \overline{p}_{(+)},\ (\overline{p}_z)_{z=-t/2} = -\overline{p}_{(-)},\ \overline{p}_{-z} = -\overline{p}_z,$$

the external loads on the surfaces $z = \pm\, t/2$, per unit area of the middle surface, are

$$\overline{p}_z \Big|_{z=-t/2}^{z=t/2} \tag{6.3}$$

The resultant of the body forces acting on the element of the shell is

$$\int_{-t/2}^{t/2} \overline{F}\, d\sigma^*\, dz = \int_{-t/2}^{t/2} \overline{F} A_1^* A_2^*\, d\alpha_1\, d\alpha_2\, dz.$$

On reduction to unit area of the middle surface, this becomes

$$\int_{-t/2}^{t/2} \overline{F}\, dz.$$

Adding this expression to (6.3) we obtain the vector:

$$\overline{X} = \overline{p}_z \Big|_{z=-t/2}^{z=t/2} + \int_{-t/2}^{t/2} \overline{F}\, dz. \tag{6.4}$$

Therefore, the vector $\overline{X}$ is the principal force of all given external forces reduced to unit area of the middle surface. $\overline{X}$ is called the external force acting on the shell.

The moments of the surface forces $\overline{p}_{(+)} A_1^* A_2^* d\alpha_1 d\alpha_2$ and $\overline{p}_{(-)} A_1^* A_2^* d\alpha_1 d\alpha_2$ about any point of the element of the shell will be

$$[\overline{m}^*\, t/2,\ \overline{p}_{(+)} A_1^* A_2^*\, d\alpha_1\, d\alpha_2] \quad \text{and} \quad [-\overline{m}^*\, t/2,\ \overline{p}_{(-)} A_1^* A_2^*\, d\alpha_1\, d\alpha_2].$$

Reducing their sum to unit area of the middle surface, we obtain:

$$[\bar{m}^* z,\ \bar{p}_z]\Big|_{z=-t/2}^{z=t/2}$$

The moment of the body force per unit area of the middle surface is

$$\int_{-t/2}^{t/2} [\bar{m}^* z,\ \bar{F}] A_1^* A_2^* d\alpha_1 d\alpha_2 dz : A_1^* A_2^* d\alpha_1 d\alpha_2 = \int_{-t/2}^{t/2} [\bar{m}^* z,\ \bar{F}]\, dz.$$

Therefore, the principal moment of all the forces which act upon the element of the shell reduced to unit area of the middle surface is

$$\bar{L} = [\bar{m}^* z,\ \bar{p}_z]\Big|_{z=-t/2}^{z=t/2} + \int_{-t/2}^{t/2} [\bar{m}^* z,\ \bar{F}]\, dz. \tag{6.5}$$

We term $\bar{L}$ the external moment acting on the shell. Thus the external forces acting on the element of the shell are statically equivalent to the principal force $\bar{X}$ and the principal moment $\bar{L}$. The point of application of these vectors can be any point on the middle surface of the element.

Let us express the forces and the moments in terms of the stress components. If we introduct (5.5) in (6.1) and (6.2), we obtain

$$\bar{K}_1 = \int_{-t/2}^{t/2} (\sigma_{11} \bar{e}_1^{z*} + \sigma_{12} \bar{e}_2^{z*} + \sigma_{1z} \bar{m}^*)\, dz,$$

$$\bar{M}_1 = \int_{-t/2}^{t/2} [\bar{m}^* z,\ \sigma_{11} \bar{e}_1^{z*} + \sigma_{12} \bar{e}_2^{z*}]\, dz.$$

Substituting for $\bar{e}_i^{z*}$ in the above from formulas of type (2.36)

$$\bar{e}_1^{z*} = \bar{e}_1^* + k_{12}^* z\, \bar{e}_2 \approx \bar{e}_1^*,\quad \bar{e}_2^{z*} \approx \bar{e}_2^*, \tag{6.6}$$

where $\bar{e}_i$ are the unit vectors of the orthogonal coordinates on the deformed surface, we obtain

$$\bar{K}_1 = \bar{e}_1^* \int_{-t/2}^{t/2} \sigma_{11}\, dz + \bar{e}_2^* \int_{-t/2}^{t/2} \sigma_{12}\, dz + \bar{m}^* \int_{-t/2}^{t/2} \sigma_{1z}\, dz,$$

$$\bar{M}_1 = [\bar{m}^* \bar{e}_1^*] \int_{-t/2}^{t/2} \sigma_{11} z\, dz + [\bar{m}^* \bar{e}_2^*] \int_{-t/2}^{t/2} \sigma_{12} z\, dz.$$

We proceed similarly with vectors $\bar{K}_2$ and $\bar{M}_2$ Taking into account the expression for vector products

$$[\bar{e}_1^* \bar{m}^*] = -\bar{e}_2^*,\ [\bar{e}_2^* \bar{m}^*] = \bar{e}_1^*\quad [\bar{e}_1^* \bar{e}_2^*] = \bar{m}^*, \tag{6.7}$$

we can write the force and moment vectors in the following form:

$$\bar{K}_1 = \bar{e}_1^* T_{11}^* + \bar{e}_2^* T_{12}^* + \bar{m}^* N_1^*, \tag{6.8}$$

$$\bar{K}_2 = \bar{e}_1^* T_{21}^* + \bar{e}_1^* T_{22}^* - \bar{m}^* N_2^*;$$

$$\bar{M}_1 = \bar{e}_2^* M_{11}^* - \bar{e}_1^* M_{12}^*,\quad \bar{M}_2 = \bar{e}_2^* M_{21}^* - \bar{e}_1^* M_{22}^*. \tag{6.9}$$

In the above we have introduced the following notations:

$$T_{ik}^* = \int_{-t/2}^{t/2} \sigma_{ik}\, dz,\ N_i^* = \int_{-t/2}^{t/2} \sigma_{iz}\, dz,\ M_{ik}^* = \int_{-t/2}^{t/2} \sigma_{ik} z\, dz, \tag{6.10}$$
$$(i, k = 1, 2)$$

Since the vectors $T_{i1}^* \bar{e}_1^* + T_{i2}^* \bar{e}_2^*$ lie in the tangent plane* of the deformed middle surface, the forces T_{ik}^* are called tangential forces: T^*_{11}, T^*_{22} and T^*_{12}, T^*_{21} are respectively the normal and the tangential forces acting on the cross sections α_1 = const. and α_2 = const; N^* are shearing forces in the surfaces of intersection α_i = const, M^*_{11} and M^*_{22} are bending moments and M^*_{12} and M^*_{21} are twisting moments. Their positive directions are shown in Figure 6.

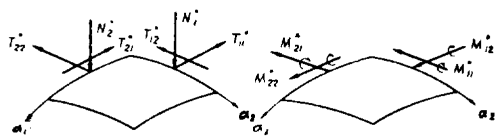

Figure 6

The external forces and moments may be written in terms of the projections of the external stresses σ_{1z} σ_{2z} and σ_{zz} and the body forces F_1, F_2, F_3 in the directions of the unit vectors of the coordinates of the shell:

$$\begin{aligned} p_z &= \sigma_{1z}\bar{e}_1^{*} + \sigma_{2z}\bar{e}_2^{*} + \sigma_{zz}\bar{m}^*, \\ \bar{F} &= F_1\bar{e}_1^{*} + F_2\bar{e}_2^{*} + F_3\bar{m}^*. \end{aligned} \tag{6.11}$$

If these expressions are introduced into (6.4) and (6.5), we obtain:

$$\bar{X} = X_1^*\bar{e}_1^* + X_2^*\bar{e}_2^* + X_3^*\bar{m}^*, \tag{6.12}$$

$$\bar{L} = L_1^*\bar{e}_2^* - L_2^*\bar{e}_1^*, \tag{6.13}$$

where we introduced the following notations:

$$\begin{aligned} X_i^* &= \sigma_{iz}\Big|_{z=-t/2}^{z=t/2} + \int_{-t/2}^{t/2} F_i\,dz, \quad X_3^* = \sigma_{zz}\Big|_{z=-t/2}^{z=t/2} + \int_{-t/2}^{t/2} F_3\,dz, \\ L_i^* &= z\sigma_{iz}\Big|_{z=-t/2}^{z=t/2} + \int_{-t/2}^{t/2} zF_i\,dz, \qquad (i=1,2). \end{aligned} \tag{6.14}$$

Here X_i^* and L_i^* are respectively the projections of the external force and moment in the directions of the unit vectors of the deformed middle surface. X_3^* is the projection of the external force on the normal to this surface.

§ 7. Equilibrium Equations for the Shell in Orthogonal Curvilinear Coordinates

If we multiply the vector equation of equilibrium of a three-dimensional body (5.2) by dz and integrate with respect to the thickness of the shell from $-t/2$ to $+t/2$ we obtain:

$$(A_2^*\bar{K}_1)_{,1} + (A_1^*\bar{K}_2)_{,2} + A_1^*A_2^*\bar{X} = 0, \tag{7.1}$$

where $\bar{K}_i$ and $\bar{X}$ can be expressed by (6.1) and (6.4). Further, we take the vector product of the equation (5.2) with $\bar{m}^* z$ and integrate over the thickness of the shell:

$$\int_{-t/2}^{t/2} [\bar{m}^* z\{(\bar{p}_1 A_2^*)_{,1} + (\bar{p}_2 A_1^*)_{,2} + (\bar{p}_z A_1^* A_2^*)_{,z} + \bar{F}A_1^*A_2^*\}]\,dz = 0. \tag{*}$$

---

* [The word tangential (kasatel'nyi) used by the author may sometimes mean 'osculating' - Translator]

Since

$$\int_{-t/2}^{t/2} [\overline{m}^* z, (\overline{p}_1 A_2^*)_{,1}]\, dz = \int_{-t/2}^{t/2} [\overline{m}^* z, \overline{p}_1 A_2^*]_{,1}\, dz - \int_{-t/2}^{t/2} [\overline{m}^*_{,1} z, \overline{p}_1 A_2^*]\, dz,$$

$$\int_{-t/2}^{t/2} [\overline{m}^* z, (\overline{p}_z A_1^* A_2^*)_{,z}]\, dz = [\overline{m}^* z, \overline{p}_z A_1^* A_2^*]\Big|_{z=-t/2}^{z=t/2} -$$

$$-\int_{-t/2}^{t/2} [\overline{m}^*, \overline{p}_z] A_1^* A_2^* dz,$$

$$\overline{m}^*_{,i} z = \overline{\rho}^*_{,i} - \overline{r}_{,i},$$

thus, on introducing the left hand side of these equations in (*) and taking into account the equation of moments (5.3) we obtain:

$$(A_2^* \overline{M}_1)_{,1} + (A_1^* \overline{M}_2)_{,2} + [\overline{r}^*_{,1} \overline{K}_1] A_2^* + [\overline{r}^*_{,2} \overline{K}_2] A_1^* + \overline{L} A_1^* A_2^* = 0, \qquad (7.2)$$

where $\overline{M}_i$ and $\overline{L}$ can be expressed by (6.2) and (6.5).

Thus, the equations of equilibrium (5.2) and (5.3) for an element of the shell considered as a three dimensional body are replaced by equations of equilibrium (7.1) and (7.2) of an element of the deformed middle surface of the shell. In these equations the required vectors $\overline{K}_i$ and $\overline{M}_i$ depend only on the two variables α_1 and α_2; therefore, our problem is now only a two dimensional one instead of a three-dimensional one.

Let us express the vector equations of equilibrium (7.1) and (7.2) in scalar form. For this purpose we substitute for $\overline{K}_i$ from (6.8) and for $\overline{X}_i$ from (6.12) in (7.1) and in the equation obtained we replace $\overline{e}^*_{i,k}$ and $\overline{m}^*_{,i}$ by their expressions (2.18) and (2.22) for the deformed surface. Then we equate the coefficients of $\overline{e}^*_i$ and $\overline{m}^*$ to zero.

If, in considering the equilibrium of the shell element, we can neglect the displacement as small compared with unity, we can use the formulas for orthogonal coordinates:

$$\begin{aligned} A_2^* \overline{e}^*_{1,1} &= -A^*_{1,2} \overline{e}^*_2 - \overline{m}^* A_1^* A_2^* k^*_{11}; \quad A_1^* \overline{e}^*_{1,2} = A^*_{2,1} \overline{e}^*_2 - \\ &\quad - \overline{m}^* A_1^* A_2^* k^*_{12}, \quad \overleftrightarrow{1, 2}; \\ \overline{m}^*_{,i} &= A_i^* (k^*_{i1} \overline{e}_1^* + k^*_{i2} \overline{e}_2^*), \quad (i = 1, 2). \end{aligned} \qquad (7.3)$$

With the help of these we obtain:

$$\begin{aligned} &(A_2^* T^*_{11})_{,1} + (A_1^* T^*_{21})_{,2} + T^*_{12} A^*_{1,2} - T^*_{22} A^*_{2,1} + A_1 A_2 (N_1^* k^*_{11} + \\ &\quad + N_2^* k^*_{12} + X_1^*) = 0; \\ &(A_1^* T^*_{22})_{,2} + (A_2^* T^*_{12})_{,1} + T^*_{21} A^*_{2,1} - T^*_{11} A^*_{1,2} + A_1 A_2 (N_2^* k^*_{22} + \\ &\quad + N_1^* k^*_{12} + X_2^*) = 0; \\ &(A_2 N_1^*)_{,1} + (A_1 N_2^*)_{,2} - (T^*_{11} k^*_{11} + T^*_{22} k^*_{22} + k^*_{12} T^*_{12} + \\ &\quad + k^*_{12} T^*_{21} - X_3^*) A_1 A_2 = 0. \end{aligned} \qquad (7.4)$$

Thus, the vector equation (7.1) is equivalent to the three scalar ones (7.4).

In order to write the vector equation of moments (7.2) in expanded form, we insert $\overline{M}_i$ and $\overline{L}$ as given by (6.9) and (6.13). If we consider (6.8) we obtain in a similar fashion the scalar equations:

$$(A_2M^*_{11})_{,1} + (A_1M^*_{21})_{,2} + M^*_{12}A_{1,2} - M^*_{22}A_{2,1} + \\ + A_1A_2(\bar{L}_1 - N^*_1) = 0 \qquad \overleftrightarrow{1, 2;} \tag{7.5}$$

$$T^*_{12} - T^*_{21} + M^*_{12}k^*_{11} - M^*_{21}k^*_{22} - k^*_{12}(M^*_{11} - M^*_{22}) = 0. \tag{7.6}$$

Here we assumed $A^*_i \approx A_i$, because in calculating the variations of curvature we neglected not only the shear but also the elongation as small (compared with unity).

The sixth equation of equilibrium (7.6 is identically satisfied within the assumed limits of accuracy. This ensures that the tangential stresses are conjugate: $\sigma_{12} = \sigma_{21}$. Thus, for six unknown forces $\overline{T}^*_{ik}$ and $\overline{N}^*_i$ and four unknown moments M^*_{ik}, we have a system of five differential equations (7. 4) and (7. 5).

In order to ensure that the tangential stresses are conjugate with respect to the thickness of the shell $\sigma_{1z} = \sigma_{z1}$, $\sigma_{2z} = \sigma_{z2}$, one must add one more differential equation to the six equations of equilibrium, which in orthogonal coordinates will be as follows /0.8/:

$$t^2 \sum_{i=1}^{2} (T^*_{1i}k^*_{i2} - T^*_{2i}k^*_{i1}) - 3(M^*_{12} - M^*_{21}) = 0. \tag{7.7}$$

The equations of equilibrium (7.1) and (7.2) hold also for a shell of varying thickness which is symmetrical about the middle surface and for which the equations of the boundary surfaces are as follows:

$$z = f(\alpha_1\alpha_2), \quad z = -f(\alpha_1\alpha_2).$$

The principal forces and moments are defined by the equations

$$\overline{K}_i = \int_{-f}^{f} \overline{p}_i \, dz, \quad \overline{M}_i = \int_{-f}^{f} [\overline{m}^* z, \overline{p}_i] \, dz. \tag{7.8}$$

§ 8. Boundary Conditions

Let us consider first the static boundary conditions.

The locus of the boundary points of the middle surface of a deformed open shell shall be called the boundary contour C^*. The ruled surface Σ^* which is formed by all the normals from C^* to the middle surface will be called the boundary section of the shell. We denote by n^* the unit normal to C^* in the plane tangent to the middle surface σ^*; $\overline{\tau}^*$ is the unit tangent to the contour C^*; $\overline{m}^*$ is the unit vector of the normal to σ^* at the points of the contour C^*. We consider the trihedron $\{\overline{n}^*, \overline{\tau}^*, \overline{m}^*\}$ to be equilateral (Figure 7). We further denote by $\overline{K}_n$ and $\overline{M}_n$ the elastic force and the elastic moment at the contour, per unit length of the contour C^*, and by T^*, S^*, N^*, H^* and G^* their projections on the axes of trihedron:

$$\overline{K}_n = \overline{n}^*T^* + \overline{\tau}^*S^* + \overline{m}^*N^*; \tag{8.1}$$

$$M_n = n^*H^* + \tau^*G^*. \tag{8.2}$$

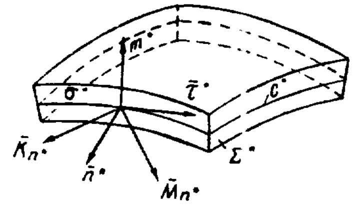

Figure 7

Here T* and S* are a normal and an tangential force, N* a shearing force, G* and H* are bending and twisting moments at the contour. It is not difficult to express these quantities in terms of components of [principal] forces and moments. Actually, using formulas from the theory of elasticity which express the stress on an inclined plane with normal $\bar n^*$ in terms of the stresses on areas perpendicular to the axes of the orthogonal coordinates and taking into account that in our case $\bar n^* \bar m^* = 0$, we obtain:

$$\bar K_n = \bar K_1 n_1^* + \bar K_2 n_2^*, \tag{8.3}$$

where n^*_i is the projection of the normal $\bar n^*$ along the unit vectors $\bar e^*$.

★ Let us express n^*_i in terms of projections τ_i^* of the vector $\bar\tau_i^*$ along $\bar e_i^*$. If ds* is an element of arc of the contour C* then we have, by definition

$$\bar\tau^* = \frac{d\bar r^*}{ds^*} = \bar r_{,1}^* \frac{d\alpha_1}{ds^*} + \bar r_{,2}^* \frac{d\alpha_2}{ds^*} = \bar e_1^* \tau_1^* + \bar e_2^* \tau_2^*,$$

$$\bar n^* = [\bar\tau^*, \bar m^*] = [\bar e_1^* \tau_1^* + \bar e_2^* \tau_2^*, \bar m^*] = \bar e_1^* \tau_2^* - \bar e_2^* \tau_1^*,$$

Here we have used (2.41).

Since $\bar r_{,i}^* = A_i^* \bar e_i^*$, it follows that

$$\tau_i^* = A_i^* \frac{d\alpha_i}{ds^*} \qquad (i = 1, 2) \tag{8.4}$$

When one further resolves $\bar n^*$ in the directions e_i^*, then

$$n_1^* \bar e_1^* + n_2^* \bar e_2^* = \bar e_1^* \tau_2^* - \bar e_2^* \tau_1^*.$$

Now taking (8.4) into account we obtain the relations

$$n_1^* = \tau_2^* = A_2^* \frac{d\alpha_2}{ds^*}, \quad n_2^* = -\tau_1^* = -A_1^* \frac{d\alpha_1}{ds^*}. \tag{8.5}$$

We denote the unit vectors of the undeformed contour C of the shell by $\bar\tau$ and $\bar n$ and their projections along $\bar e_i$ by τ_i and n_i. Let us prove that for small deformations, $\bar n^* \neq \bar n$, $\bar\tau^* \neq \bar\tau$ although $n_i^* \approx n_i$, $\tau_i^* \approx \tau_i$. If ds is an element of arc of the contour C, then it follows from (8. 5) that:

$$\tau_i = A_i \frac{d\alpha_i}{ds}, \quad n_1 = A_2 \frac{d\alpha_2}{ds}, \quad n = -A_1 \frac{d\alpha_1}{ds},$$

but when neglecting elongations, small compared to unity, it is clear that $A_i^* \approx A_i$ and $ds^* \approx ds$.

Thus for small deformations, the following relations are valid:

$$n_i^* = n_i, \quad \tau_i^* = \tau_i; \tag{8.6}$$

$$\bar{n}^* = \bar{e}_1^* n_1 + \bar{e}_2^* n_2, \quad \bar{\tau}^* = \bar{e}_1^* \tau_1 + \bar{e}_2^* \tau_2. \tag{8.7}$$

We have the formula

$$\bar{M}_n = \bar{M}_1 n_1^* + \bar{M}_2 n_2^* \tag{8.8}$$

for determining the moment at the contour. If we substitute in (8.3) and (8.8) the expressions (6.8) and (6.9) for $\bar{K}_i$ and $\bar{M}_i$ we obtain

$$\bar{K}_n = \sum_{i=1}^{2}\Big(\sum_{k=1}^{2} T_{ik}^* \bar{e}_k^* + \bar{m}^* N_i^*\Big) n_i^*, \tag{8.9}$$

$$\bar{M}_n = \sum_{i,k} [\bar{m}^* \bar{e}_k^*] M_{ik}^* n_i^* = \sum_{i=1}^{2} (\bar{e}_2^* M_{i1}^* - \bar{e}_1^* M_{i2}^*) n_i^*. \tag{8.10}$$

Let us find the normal, tangential, and shear forces at the contour. By (8.1)

$$T^* = \bar{K}_n \cdot \bar{n}^*, \quad S^* = \bar{K}_n \cdot \bar{\tau}^*, \quad N^* = \bar{K}_n \cdot \bar{m}^*,$$

or, considering (8.9),

$$T^* = \sum_{i,k} T_{ik}^* n_i^* n_k^*, \quad S^* = \sum_{i,k} T_{ik}^* n_i^* \tau_k^*, \quad N^* = \sum_{i=1}^{2} N_i^* n_i^*, \tag{8.11}$$

where $\tau_i^* = \bar{\tau}^* \bar{e}_i^*$. Multiplying (8.2) by $\bar{\tau}^*$ and $\bar{n}^*$, we obtain the following expressions for the bending and twisting moments at the contour:

$$G^* = \bar{M}_n \cdot \bar{\tau}^*, \quad H^* = \bar{M}_n \cdot \bar{n}^*.$$

If for $\bar{M}_n$ one substitutes here from (8.10) and uses (8.5), one obtains:

$$\begin{aligned} G^* &= \sum_{i,k} [\bar{m}^*, \bar{e}_k^*] M_{ik}^* n_i^* \bar{\tau}^* = \sum_{i,k} M_{ik}^* n_i^* n_k^*, \\ H^* &= \sum_{i,k} [\bar{m}^* \bar{e}_k^*] M_{ik}^* n_i^* \bar{n}^* = -\sum_{i,k} M_{ik}^* n_i^* \tau_k^*. \end{aligned} \tag{8.12}$$

Thus, integrating over the thickness of the shell, we have replaced the stresses acting on the contour of a thin shell by three equivalent forces T*, S*, N*, and two moments G* and H* per unit length of the contour of the shell (based on Saint-Venant's principle). So it seems that there should be five static boundary conditions at the contour of the shell; these were formulated for the first time by Poisson:

$$T^* = \tilde{T}, \quad S^* = \tilde{S}, \quad N^* = \tilde{N}, \quad G^* = \tilde{G}, \quad H^* = \tilde{H},$$

where the right-hand sides are the forces and moments given at the contour. However, it was shown by Kirchhoff and later by Thomson and Tait that the number of static boundary conditions may be reduced to four. They proceeded from the assumption that the actual distribution of the stresses at the boundary which gives the twisting moment is of no great importance. Therefore the twisting moment at the contour of a thin shell may be replaced by a distributed force of the type of $\frac{\partial H^* \bar{m}^*}{\partial s^*}$ per unit length of the contour C*. Thereby a certain redistribution of the stresses near the boundaries of the shell is admitted, but, according to Saint-Venant's principle, this replacement has an effect only in the immediate neighborhood of the boundary of the shell.

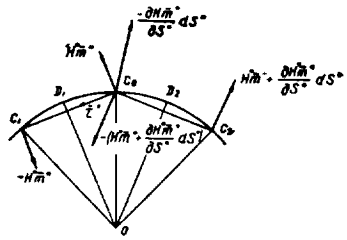

Figure 8

★ Let us consider a portion of the contour C* in the neighborhood of the point C_0. We shall replace it by a broken line consisting of a great number of equal segments (Figure 8). The length of each segment of the broken line may be assumed equal (to a degree of accuracy of infinitesimals of higher order) to the length of the subtended arc $(C_0C_2 = C_0C_1 = ds^*,\ \overline{C_0C_1} = \overset{\frown}{C_0C_1})$; the unit vectors of the segment and of the tangent to the point C_0 coincide in the limit.

Let us place the points D_1 and D_2 half-way between the points C_1, C_0, and C_0, C_2, so that $D_1D_2 = ds^*$. Let furthermore $H^*\bar{n}^*$ be the vector of the twisting moment at the point D_1, where $\bar{n}^*$ is the normal to the contour C* at the relevant point (the direction of the vector $\bar{n}^*$ is perpendicular to the plane of the figure, taken towards the reader).

The resultant twisting moment $H^*\bar{n}^*ds^*$ may be replaced, on the part C_1C_0 of the contour, by a couple $(-H^*\bar{m}^*,\ +H^*\bar{m}^*)$, parallel to the normal $\bar{m}^*$ to the middle surface at the point D_1 (parallel to O_1D_1). In fact, the moment of that couple with respect to the point D_1 is:

$$[\bar{\tau}^*ds^*,\ H^*\bar{m}^*] = [\bar{\tau}^*,\ \bar{m}^*]\,H^*ds^* = H^*\bar{n}^*ds^*,$$

where $\bar{\tau}^*ds^* = \overline{C_0C_1}$ is the arm of the couple.

In the same manner the vector of the twisting moment in the adjacent portion C_0C_2 may be replaced by a couple

$$\left\{-\left(H^*\bar{m}^* + \frac{\partial H^*\bar{m}^*}{\partial s^*}ds^*\right),\ \left(H^*\bar{m}^* + \frac{\partial H^*\bar{m}^*}{\partial s^*}ds^*\right)\right\},$$

applied at C_0 and C_2 respectively, and parallel to the normal $\bar{m}^*$ to the middle surface at the point D_2 (parallel to O_1D_2).

As a result of the transformation of the twisting moment, two forces are applied at the point C_0:

$$H^*\bar{m}^* \text{ and } -\left(H^*\bar{m}^* + \frac{\partial H^*\bar{m}^*}{\partial s^*}ds^*\right).$$

Their geometrical sum is $-\frac{\partial H^*\bar{m}^*}{\partial s^*}ds^*$. Hence the twisting moment per unit length of the contour is statically equivalent to a force of a linear density $-\frac{\partial H^*\bar{m}^*}{\partial s^*}$. ★

In this manner, the sum of the force per unit length $\bar{K}_n$ and of the twisting moment $\bar{M}_n$ at the contour of the shell is statically equivalent to the force

$$\overline{\Phi}=\overline{K}_n-\frac{\partial H^*\overline{m}^*}{\partial s^*}\tag{8.13}$$

and to the bending moment:

$$G^*=\sum_{i,k} M_{ik}^* n_i^* n_k^*.\tag{8.14}$$

If the contour has the corners A and B, then in addition to the distributed force on the contour $-\frac{\partial H^*\overline{m}^*}{\partial s^*}$ localized forces will appear at the relevant points:

$$(-H^*\overline{m}^*)_A \quad \text{and} \quad (H^*\overline{m}^*)_B.$$

Therefore the twisting moment $H^*\overline{n}^*$ is statically equivalent to a distributed force of the type $-\frac{\partial H^*\overline{m}^*}{\partial s^*}$ and to two forces $(-H^*\overline{m}^*)_A$ and $(H^*\overline{m}^*)_B$ acting at the corner points.

The vectorial relation (8.13) enables one to express the static boundary conditions in any system of coordinates. We shall now consider this in an orthogonal system of coordinates. Let $\overline{\Phi}$ be the vector of the external load applied to the contour C* of the deformed shell, referred to unit length along this contour, and

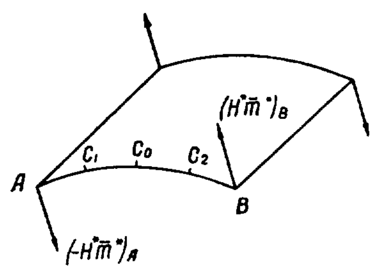

Figure 9

G be the external bending moment, applied to the contour, also per unit length of the contour. Then the vector expressions of the static boundary conditions will be:

$$\overline{\Phi}=\overline{K}_n-\frac{\partial H^*\overline{m}^*}{\partial s^*};\tag{8.15}$$

$$\overline{G}=\sum_{i,k} M_{ik}^* n_i^* n_k^*,\tag{8.16}$$

where $\overline{K}_n$ may be calculated from (8.9) and H* from (8.12). Let us now express (8.15) in scalar form. If Φ_i^* and Φ_3^* are the projections of the contour load on the axes $\overline{e}_i^*$ and $\overline{m}^*$ then

$$\overline{\Phi}=\Phi_1^*\overline{e}_1^*+\Phi_2^*\overline{e}_2^*+\Phi_3^*\overline{m}^*.\tag{8.17}$$

★ Substituting this expression in (8.15) and taking into account (8.9), we find:

$$\Phi_1^*\overline{e}_1^*+\Phi_2^*\overline{e}_2^*+\Phi_3^*\overline{m}=\sum_i\Big(\sum_k T_{ik}^*\overline{e}_k^*+N_i^*\overline{m}^*\Big)n_i^*-$$
$$-\overline{m}^*\frac{\partial H^*}{\partial s^*}-H^*\frac{\partial \overline{m}^*}{\partial s^*}.$$

★ But since according to (2.22)

$$\frac{\partial \bar{m}^*}{\partial s^*} = \sum_{i=1}^{2} \bar{m}^*_{,i} \frac{d\alpha_i}{ds^*} = \sum_{i,j} A_i^* k_{ij}^* \bar{e}_j^* \frac{d\alpha_i}{ds^*}, \quad \frac{d\alpha_i}{ds^*} = \tau_i^*/A_i^*, \tag{8.18}$$

by substituting this expression in the preceding equation and comparing the coefficients of $\bar{e}_j^*$ and $\bar{m}^*$, we find the scalar form of the boundary conditions:

$$\Phi_i^* = \sum_j \left(T_{ji}^* n_j^* + k_{ij}^* H^* \tau_j^*\right), \quad \Phi_3^* = N_1^* n_1^* + N_2^* n_2^* - \frac{\partial H^*}{\partial s^*};$$
$$\tilde{G} = \sum_{i,k} M_{ik}^* n_i^* n_k^* \quad (i, j, k = 1, 2). \tag{8.19}$$

Here we can neglect the elongations which are small in comparison with unity, without affecting the degree of accuracy, assuming $n_i^* \simeq n_i$, $\tau_i^* \simeq \tau_i$, $ds^* \simeq ds$; analogously, we can neglect the terms due to bending n the expressions Φ_i^*. Hence, the boundary conditions may be expressed in their final form as follows:

$$\Phi_k^* = \sum_{i=1}^{2} T_{ik}^* n_i, \quad \Phi_3^* = \sum_{i=1}^{2} N_i^* n_i - \frac{\partial H^*}{\partial s}, \quad \tilde{G} = \sum_{i,k} M_{ik}^* n_i n_k. \tag{8.20}$$

The vector $\bar{\Phi}$ may also be projected on the axes of the trihedron $\{\bar{n}^*, \bar{\tau}^*, \bar{m}^*\}$

$$\bar{\Phi} = \Phi_{n^*} \cdot \bar{n}^* + \Phi_{\tau^*} \cdot \bar{\tau}^* + \Phi_3^* \bar{m}^* = T^* \bar{n}^* + S^* \bar{\tau}^* + N^* \bar{m} - \frac{\partial \bar{m}^* H^*}{\partial s^*}.$$

Hen[illegible] iplying scalarly by $\bar{n}^*$, $\bar{\tau}^*$, $\bar{m}^*$ we obtain:

$$\Phi_{n^*} = T_* - \bar{n}^* \frac{\partial \bar{m}^* H^*}{\partial s^*}, \quad \Phi_{\tau^*} = S^* - \bar{\tau}^* \frac{\partial \bar{m}^* H^*}{\partial s^*}, \quad \Phi_3^* = N^* - \bar{m}^* \frac{\partial \bar{m}^* H^*}{\partial s^*}$$

or, [illegible] 8.18), and the equations

$$\bar{n}^* \bar{m}^* = \bar{\tau}^* \bar{m}^* = \bar{m}^* \partial \bar{m}^* / \partial s^* = 0,$$
$$n_i^* \Rightarrow n_i, \quad \tau_i^* \Rightarrow \tau_i, \quad ds^* \Rightarrow ds,$$

w

$$\Phi_{n^*} = T^* - H^* \sum_{i,j} k_{ij}^* \tau_i n_j; \quad \Phi_{\tau^*} = S - H^* \sum_{i,j} k_{ij}^* \tau_i \tau_j;$$
$$\Phi_3^* = N^* - \frac{\partial H^*}{\partial s}. \tag{8.21}$$

, S, N* are given by (8.11). If φ is the angle between the positive n the α_1 axis and the vector $\bar{\tau}$, we can introduce into the boundary conditions e projections of the vectors $\bar{\tau}$ and $\bar{n}$ of the non-deformed contour of the

$$\tau_1 = -n_2 = \cos\varphi; \quad \tau_2 = n_1 = \sin\varphi. \star \tag{8.22}$$

n addition to the static boundary conditions, there may also be geometrical iscellaneous boundary conditions. We shall list here some possible variants boundary conditions, and for simplicity we shall assume them to be homous.

1. A hinged, immovably supported edge: n this case, the following conditions to be fulfilled at the boundary of the shell:

$$u_1 = u_2 = w = 0, \quad G^* = 0, \tag{8.23}$$

re G* is the bending moment at the contour or at its elements.

2. A hinged edge, free in the normal direction. The following conditions have to be fulfilled at the boundary:

$$G^*=0,\quad N^*-\frac{\partial H^*}{\partial s}=0,\quad u_1=u_2=0. \tag{8.24}$$

3. A free edge:

$$\Phi_{n^*}=\Phi_{\tau^*}=\Phi_3^*=G^*=0. \tag{8.25}$$

4. In the case of a clamped edge, the condition $u_1 = u_2 = w = 0$ must be fulfilled and the angle or rotation at the contour has to be zero $\bar{n}\bar{m}^* = 0$. By substituting for $\bar{m}^*$ from (3.16) and putting $\bar{n} = \bar{e}_1 n_1 + \bar{e}_2 n_2$ we obtain $n_1 E_1 + n_2 E_2 = 0$. That condition will be fulfilled if

$$\frac{\partial w}{\partial n}=n_1\partial w/A_1\partial\alpha_1+n_2\partial w/A_2\partial\alpha_2=0$$

at the contour. Hence, in the case of clamped edges, and of a large deflection, the following ordinary boundary conditions of the linear theory have to be fulfilled:

$$u_1=u_2=w=\frac{\partial w}{\partial n}=0. \tag{8.26}$$

Besides these main cases of fixed edges, there may be cases of elastically fixed edges. Some examples of this may be found in the book by S. P. Timoshenko /0.26/

In solving problems of stability of thin shells and other problems with several successive states of stress, it is sometimes useful to start from the equations of equilibrium with respect to the undeformed state of the shell, i. e., with respect to the system of coordinates of the undeformed middle surface*. The differential equations of equilibrium (7.5) that are satisfied by the moments and the shearing forces do not differ from the corresponding equations of the linear theory of shells. But the first three equations of equilibrium (7.4) differ substantially from the corresponding equations of the linear theory, because the former contain the coefficients of the second principal quadratic form of the deformed surface, and the projections of the external force on the coordinate axes after deformation. Therefore, we shall now project the vector equation of equilibrium (7.1) on the coordinate axes of the undeformed shell. Let X^H_1, X^H_2 and X^H_3 be the projections of the external forces on the axes

$$\bar{X}=X_1^H\bar{e}_1+X_2^H\bar{e}_2+X_3^H\bar{m}, \tag{8.27}$$

where $\bar{e}_1$ and $\bar{e}_2$ are the unit vectors of the coordinates at the undeformed middle surface and $\bar{m}$ is the normal to this surface. This decomposition will be convenient if the external forces are given in the system of coordinates for the undeformed state. We shall resolve the vectors of the internal forces in the same directions

$$\bar{K}_i=T_{i1}^H\bar{e}_1+T_{i2}^H\bar{e}_2+N_i^H\bar{m}. \tag{8.28}$$

where T^H_{ik} are the tangential forces in the system of coordinates of the undeformed surface, and N^H_i are shearing forces normal to that surface, whereas $T^H_{21} \neq T^H_{21}$. We shall substitute (8.27) and (8.28) into the vector equation of equilibrium (7.1). We can assume $A_i^* \approx A_i$, because the rotation of coordinate axes was taken into account. Therefore, using the formulas (2.18) and (2.22) for differentiation of the unit vectors we find:

$$\begin{gathered}(A_2T_{11}^H)_{,1}+(A_1T_{21}^H)_{,2}+T_{12}^HA_{1,2}-T_{22}^HA_{2,1}+\\+A_1A_2(k_{11}N_1^H+k_{12}N_2^H+X_1^H)=0\qquad \underleftarrow{\overrightarrow{1,2;}}\\(A_2N_1^H)_{,1}+(A_1N_2^H)_{,2}-A_1A_2[k_{11}T_{11}^H+k_{22}T_{22}^H+\\+k_{12}(T_{12}^H+T_{21}^H)-X_3^H]=0.\end{gathered} \tag{8.29}$$

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* See the author's paper /0.7/.

★ In the above-mentioned equations k_{ij} refer to the undeformed middle surface of the shell.

Let us express the forces T^H_{ik} and N^H_i in terms of the forces T^*_{ik} and N^*_i. According to the definition

$$T^H_{i1} = \bar{e}_1 \bar{K}_i, \quad T^H_{i2} = \bar{e}_2 \bar{K}_i, \quad N^H_i = \bar{m} \bar{K}_i.$$

Substituting here for K_i from (6.8) we find

$$T^H_{11} = T^*_{11}\bar{e}_1\bar{e}_1^* + T^*_{12}\bar{e}_1\bar{e}_2^* + N_1^*\bar{e}_1\bar{m}^*,$$
$$N^H_1 = T^*_{11}\bar{m}\bar{e}_1^* + T^*_{12}\bar{m}\bar{e}_2^* + N_1^*\bar{m}\bar{m}^*.$$

Since

$$\bar{e}_1\bar{e}_1^* = 1 + e_{11}, \quad \bar{e}_1\bar{e}_2^* = e_{21}, \quad \bar{e}_1\bar{m}^* = E_1, \quad \bar{m}\bar{m}^* = E_3,$$

therefore

$$T^H_{11} = T^*_{11}(1 + e_{11}) + T^*_{12}e_{21} + N_1^*E_1,$$
$$T^H_{12} = T_{11}e_{12} + T^*_{12}(1 + e_{22}) + N_1^*E_2, \quad N^H_1 = T^*_{11}\omega_1 + T^*_{12}\omega_2 + E_3N_1^* \quad \overleftarrow{\overrightarrow{1, 2}}. \tag{8.30}$$

Let us derive the static boundary conditions.

After substituting for K_n from (8.8) the expression for the force acting on the contour (8.15) will be

$$\bar{\Phi} = \bar{K}_1 n_1 + \bar{K}_2 n_2 - \frac{\partial \bar{m}^* H^*}{\partial s}. \tag{8.31}$$

Here, n_1, n_2, ∂s relate to the undeformed contour owing to the smallness of the deformation. Let Φ^H_1 and Φ^H_2 be the projections of the external load on the contour on the coordinate lines of the undeformed surface, and Φ^H_3 the projection on the normal to this surface. Then we have

$$\bar{\Phi} = \Phi^H_1\bar{e}_1 + \Phi^H_2\bar{e}_2 + \Phi^H_3\bar{m} = \bar{K}_1 n_1 + \bar{K}_2 n_2 - \frac{\partial \bar{m}^* H^*}{\partial s}.$$

from which we obtain

$$\Phi_1 = \bar{e}_1\left(\bar{K}_1 n_1 + \bar{K}_2 n_2 - \bar{m}^*\frac{\partial H^*}{\partial s} - H^*\frac{\partial \bar{m}^*}{\partial s}\right) = T^H_{11} n_1 +$$
$$+ T^H_{21} n_2 - E_1\frac{\partial H^*}{\partial s} - \bar{e}_1 H^*\frac{\partial \bar{m}^*}{\partial s},$$
$$\Phi_3 = \bar{m}\left(\bar{K}_1 n_1 + \bar{K}_2 n_2 - \bar{m}^*\frac{\partial H^*}{\partial s} - H^*\frac{\partial \bar{m}^*}{\partial s}\right) = N^H_1 n_1 +$$
$$+ N^H_2 n_2 - E_3\frac{\partial H^*}{\partial s} - \bar{m}H^*\frac{\partial \bar{m}^*}{\partial s}$$

According to (8.18)

$$\frac{\partial \bar{m}^*}{\partial s} = \sum_{i,k} k^*_{ij}\bar{e}_j^*\tau_i.$$

The quantity $H^* k^*_{rs} \sim t^3 x_{ij} k^*_{sr}$ may be neglected in comparison with the other terms of the expression for Φ^H_i. In fact, if the rotations are of the order of magnitude unity, then according to (8.30), N^H_i will be of the same order of magnitude as the membrane forces, being large in comparison with $\bar{m}\frac{\partial \bar{m}^*}{\partial s}H^*$. If the rotations are small, $\bar{m}\frac{\partial \bar{m}^*}{\partial s}H^*$ will be small in comparison with $E_3\frac{\partial H^*}{\partial s}$ because $E_3 \approx 1$.

★ Therefore, the static boundary conditions are:

$$\Phi_1^H = T_{11}^H n_1 + T_{21}^H n_2 - E_1 \frac{\partial H^*}{\partial s}, \quad \Phi_2^H = T_{12}^H n_1 + T_{22}^H n_2 - E_2 \frac{\partial H^*}{\partial s}. \qquad (8.32)$$
$$\Phi_3^H = N_1^H n_1 + N_2^H n_2 - E_3 \frac{\partial H^*}{\partial s}, \quad \widehat{G} = \sum_{i,k} M_{ik}^* n_i n_k.$$

where the twisting moment H* is to be introduced from (8.12).

Let T^H, S^H, N^H be respectively the projections of the vector $\overline{K}_n$ on the normal $\bar{n}$, the tangent $\bar{\tau}$ to the contour C, and the normal $\overline{m}$ to the undeformed middle surface:

$$T^H = \bar{n}\,\overline{K}_n = (\overline{K}_1 n_1 + \overline{K}_2 n_2)\,\bar{n}, \quad S^H = \overline{K}_n\,\bar{\tau}, \quad N^H = \overline{K}_n\,\overline{m}.$$

Substituting for $\overline{K}_i$ from (8.28) we find:

$$T^H = \sum_{i,k} T_{ik}^H n_i n_k, \quad S^H = \sum_{i,k} T_{ik}^H n_i \tau_k, \quad N^H = \sum_i N_i^H n_i. \qquad (8.33)$$

Since

$$\bar{\tau} = \tau_1 \bar{e}_1 + \tau_2 \bar{e}_2, \quad \bar{n} = n_1 \bar{e}_1 + n_2 \bar{e}_2,$$

we have for the scalar products

$$\bar{n}\,\overline{m}^* = E_1 n_1 + E_2 n_2, \quad \bar{\tau}\,\overline{m}^* = E_1 \tau_1 + E_2 \tau_2. \qquad (8.34)$$

Projecting the vector (8.31) along the unit vectors $\bar{n}$, $\bar{\tau}$, $\overline{m}$, and using (8.33) and (8.34), we find the boundary conditions in another form

$$\Phi_n^H = T^H - (E_1 n_1 + E_2 n_2)\frac{\partial H^*}{\partial s}, \quad \Phi_\tau^H = S^H - (E_1 \tau_1 + E_2 \tau_2)\frac{\partial H^*}{\partial s}. \qquad (8.35)$$
$$\Phi_3^H = N^H - E_3 \frac{\partial H^*}{\partial s},$$

where Φ_n^H, Φ_τ^H, Φ_3^H are respectively the projections of the contour load on the normal and the tangent to the undeformed contour of the shell and on the normal to the undeformed middle surface. ★

Chapter III

ELASTICITY RELATIONS. VARIATIONAL EQUATIONS OF THE NON-LINEAR THEORY OF SHELLS

§ 9. Relations between Stresses, Moments, and Deformations of the Middle Surface

The principal geometrical and static equations of the theory of thin shells, which have been obtained in Chapters I and II, were derived under the assumption of small deformations and arbitrary bending; therefore, to determine the state of stress of the shell, we shall use Hooke's linear law for a homogenous and isotropic body:

$$\sigma_{11}=\frac{E}{1+\nu}\left(\varepsilon^z_{11}+\frac{\nu}{1-2\nu}\theta\right);\quad \sigma_{22}=\frac{E}{1+\nu}\left(\varepsilon^z_{22}+\frac{\nu}{1-2\nu}\theta\right);$$
$$\sigma_{12}=\frac{E}{1+\nu}\varepsilon^z_{12};\quad \sigma_{zz}=\frac{E}{1+\nu}\left(\varepsilon^z_{33}+\frac{\nu}{1-2\nu}\theta\right);\quad \sigma_{1z}=\frac{E}{1+\nu}\varepsilon^z_{13};\tag{9.1}$$
$$\sigma_{2z}=\frac{E}{1+\nu}\varepsilon^z_{23};\quad \theta=\varepsilon^z_{11}+\varepsilon^z_{22}+\varepsilon^z_{33},$$

where E is the modulus of elasticity and ν is Poisson's ratio.

The above-mentioned relations are, however, not sufficient to determine the relation between forces, moments, and the deformations of the middle surface. In addition, one should also know the law of the variation of the deformation ε^z_{ik} or of the stress along the thickness of the shell.

The ε^z_{ik} were determined in the analysis of deformations by means of the geometrical hypothesis of Kirchhoff and Love, assuming that $\varepsilon^z_{13} = \varepsilon^z_{23} = 0$. Neglecting the shear ε^z_{13} and ε^z_{23} is equivalent to neglecting of the tangential stresses σ_{1z} and σ_{2z}, and therefore of the shearing forces N^*_1 and N^*_2. For thin shells, although σ_{1z} and σ_{2z} are small in comparison with the stresses σ_{23}, σ_{12}, and the corresponding shearing forces are small in comparison with the tangential ones, they cannot be neglected, because that would be in contradiction with the conditions of equilibrium inside the shell and at its edge. Therefore, the shearing forces N^*_1 and N^*_2 which depend on σ_{1z} and σ_{2z} have to be determined from the equations of the moments (7.5).

For the determination of the stress components σ_{11}, σ_{12}, and σ_{22} we shall rely on the further assumption of the Kirchhoff-Love hypothesis (see § 1). Therefore, we shall assume that the normal stress σ_z on surfaces parallel to the middle surface is negligibly small in comparison with the other stresses. The stress σ_{zz} may then be determined from the third equation of equilibrium for a three-dimensional body, by integrating it with respect to z. From the existing solutions of particular problems it may be concluded that $\sigma_{zz} \sim Et\,\varepsilon_p/R$, i. e., they may be neglected when considering small deformations of a thin shell. One can use this fact for the determination of the relative elongation ε^z_{33} in direction of the normal to the middle surface. Assuming $\sigma_{zz} = 0$, we find from (9.1) the following expressions for ε^z_{33} and θ:

$$\varepsilon^z_{33}=-\frac{\nu}{1-\nu}\left(\varepsilon^z_{11}+\varepsilon^z_{22}\right);\quad \theta=\frac{1-2\nu}{1-\nu}\left(\varepsilon^z_{11}+\varepsilon^z_{22}\right).\tag{9.2}$$

By introducing this expression for θ into (9.1) we obtain Hooke's law for shells:

$$\sigma_{11}=\frac{E}{1-\nu^2}\left(\varepsilon_{11}^z+\nu\varepsilon_{22}^z\right),\quad \sigma_{22}=\frac{E}{1-\nu^2}\left(\varepsilon_{22}^z+\nu\varepsilon_{11}^z\right), \tag{9.3}$$

$$\sigma_{12}=\frac{E}{1+\nu}\varepsilon_{12}^z.$$

The tangential stresses σ_{1z} and σ_{2z} may be determined on the basis of the known shearing forces N_1^* and N_2^*.

The relations (9.3) enable one to express the tangential forces and the moments in terms of the deformation of the surface. Substituting in these the expressions for deformation from the first approximation formulae (4.14) we find

$$\sigma_{11}=\frac{E}{1-\nu^2}\left[\varepsilon_{11}+\nu\varepsilon_{22}+z\left(\varkappa_{11}+\nu\varkappa_{22}\right)\right];\quad \sigma_{12}=\frac{E}{1+\nu}\left(\varepsilon_{12}+z\varkappa_{12}\right); \tag{9.4}$$

$$\sigma_{22}=\frac{E}{1-\nu^2}\left[\varepsilon_{22}+\nu\varepsilon_{11}+z\left(\varkappa_{22}+\nu\varkappa_{11}\right)\right].$$

Introducing these expressions in the formulas (6.10) for the forces and moments and integrating over the thickness of the shell from $-t/2$ to $t/2$, we obtain Love's formulas in the first approximation /0.11/:

$$T_{11}^*=K(\varepsilon_{11}+\nu\varepsilon_{22}),\quad T_{12}^*=T_{21}^*=K(1-\nu)\,\varepsilon_{12},\quad T_{22}^*=K(\varepsilon_{22}+\nu\varepsilon_{11});$$

$$M_{11}^*=D(\varkappa_{11}+\nu\varkappa_{22}),\quad M_{12}^*=M_{21}^*=D(1-\nu)\,\varkappa_{12}, \tag{9.5}$$

$$M_{22}^*=D(\varkappa_{22}+\nu\varkappa_{11}),$$

where $K=Et/(1-\nu^2)$ is the tensile-compressional and $D=Et^3/12(1-\nu^2)$ the flexural rigidity of the shell.

From (9.5) we obtain the inverse relations:

$$\varepsilon_{11}=K'\left(T_{11}^*-\nu T_{22}^*\right),\quad \varepsilon_{22}=K'\left(T_{22}^*-\nu T_{11}^*\right),\quad \varepsilon_{12}=K'\left(1+\nu\right)T_{12}^*; \tag{9.6}$$

$$\varkappa_{11}=D'\left(M_{11}^*-\nu M_{22}^*\right),\quad \varkappa_{22}=D'\left(M_{22}^*-\nu M_{11}^*\right),$$

$$\varkappa_{12}=D'\left(1+\nu\right)M_{12}^*,$$

$$\text{where } K'=\frac{1}{Et},\quad D'=12/Et^3.$$

Formulas (9.5) represent the simplest variants of elasticity relations and coincide with the corresponding formulas of the theory of plates. The order of magnitude of the error is t/R in comparison with unity, i.e., it corresponds to the error of the initial hypotheses. The above-mentioned formulas satisfy the sixth equation of equilibrium (7.6) with the same degree of accuracy. By adding to the right-hand sides of (9.5) secondary terms which contain the error in Kirchhoff's hypothesis, they can be made to satisfy the equation (7.6) exactly and also the requirements of the general theorems of the theory of elasticity. But without deviating from that hypothesis it is impossible to make them more precise.

For a thin shell $(t\sim\varepsilon_p R)$ the relations (9.6) are sufficiently precise. For a shell of medium thickness $(t\sim R\sqrt{\varepsilon_p})$, for which the error in Kirchhoff's hypothsis is of the order $\sqrt{t/R}$ /III. 1/, it may be necessary to make them more precise. After the addition of secondary terms the general relations of the theory of shells become symmetric, being more useful for theoretical research. The elasticity relations with additional secondary terms have been dealt with in the monographs by V. V. Novozhilov /0.15/ and A. L. Gol'denvaizer /0.8/. When using the components of bending deformation in the form (3.31), taking finite displacements into account, the type of elasticity relation given in § 12 of the present chapter is the most convenient variant. The problems of non-linear elasticity relations and of strain hardening have been briefly considered in the paper /III. 3/.

§ 10. Principle of Virtual Displacements. Deformation Energy of the Shell. The Ritz Method

Let us consider a shell in equilibrium under action of the body force $\bar{F}$ and the stresses $\bar{p}_z$. Let $\delta u_1^z, \delta u_2^z, \delta w^z$ be infinitesimal variations of the elongations, according to the constraints imposed on the shell. Then the work done the external forces acting on the shell, in the variation of elongations, will be:

$$\delta A = \iiint\limits_{(\Omega)} \bar{F}\delta\bar{u}d\Omega + \iint\limits_{(\Pi)} \bar{p}_\nu \delta\bar{u}d\Pi, \tag{10.1}$$

where $d\Omega = A_1^* A_2^* d\alpha_1 d\alpha_2 dz$ is an element of volume of the shell, and Π a surface composed of the boundary surfaces $z = t/2$ and $z = -t/2$, and the surface Σ of the boundary section of the shell. According to (4.16) the variation of the displacement $\bar{u}$ is:

$$\delta\bar{u} = \delta\bar{v} + z\delta\bar{m}^*, \tag{10.2}$$

where $\delta\bar{v}$ is the variation of the displacement vector of the points of the middle surface and $\delta\bar{m}^*$ the variation of the normal to σ^*. Let us express the virtual work of the external forces in terms of the deformation energy of the shell. The stress vector p_ν acting on an area which has a normal $\bar{\nu}$ may be expressed in terms of the stress vectors $\bar{p}_1$, $\bar{p}_2$ and $\bar{p}_z$ which act on the areas taken on the coordinate surfaces α_1 = const, α_2 = const, and z = const. From the theory of elasticity we have the formula

$$\bar{p}_\nu = \bar{p}_1 \cos(\bar{\nu}, \alpha_1) + \bar{p}_2 \cos(\bar{\nu}, \alpha_2) + \bar{p}_z \cos(\bar{\nu}, z).$$

Introducing this into (10.1) we obtain

$$\delta A = \iiint\limits_{(\Omega)} \bar{F}\delta\bar{u}d\Omega + \iint\limits_{(\Pi)} \{\bar{p}_1 \cos(\bar{\nu}, \alpha_1) + \\ + \bar{p}_2 \cos(\bar{\nu}, \alpha_2) + \bar{p}_z \cos(\bar{\nu}, z)\} \delta\bar{u}d\Pi.$$

★Tran rming the surface integral into a volume integral by the following formula (Gaus strogradskii theorem in orthogonal curvilinear coordinates)

$$\iint\limits_{(\Pi)} \{\bar{f}_1 \cos(\bar{\nu}, \alpha_1) + \bar{f}_2 \cos(\bar{\nu}, \alpha_2) + \bar{f}_z \cos(\bar{\nu}, z)\} d\Pi = \\ = \iiint\limits_{(\Omega)} \left(\frac{\partial \bar{f}_1 H_2 H_3}{\partial \alpha_1} + \frac{\partial \bar{f}_2 H_1 H_3}{\partial \alpha_2} + \frac{\partial \bar{f}_z H_1 H_2}{\partial z} \right) d\alpha_1 d\alpha_2 dz,$$

where $\bar{f}_1$, $\bar{f}_2$, $\bar{f}_z$ are arbitrary vectors gives, with the further assumptions $H_3 = 1$, $H_1 \approx A_1^*$, $H_2 \approx A_2^*$, the result

$$\delta A = \iiint\limits_{(\Omega)} \{\bar{F}\delta\bar{u}A_1^* A_2^* + (\bar{p}_1 A_2^* \delta\bar{u})_{,1} + \\ + (\bar{p}_2 A_1^* \delta\bar{u})_{,2} + (\bar{p}_z A_1^* A_2^* \delta\bar{u})_{,z}\} d\alpha_1 d\alpha_2 dz$$

or, by using the equation of equilibrium (5.2),

$$\delta A = \iiint\limits_{(\Omega)} \{\bar{p}_1 A_2^* (\delta\bar{u})_{,1} + \bar{p}_2 A_1^* (\delta\bar{u})_{,2} + \\ + \bar{p}_z A_1^* A_2^* (\delta\bar{u})_{,z}\} d\alpha_1 d\alpha_2 dz. \tag{10.3}$$

★ This is precisely the expression for the principle of virtual displacements for a shell considered as a three-dimensional body. The triple integral expresses the work done by the deformation of the shell. We shall express the external forces and internal stresses, which appear in the variation equation (10.3), in terms of forces and moments in the same manner as we did in Chapter II for deriving the equations of equilibrium. First we shall transform the right-hand side of the equation (10.3).

As, according to (10.2)

$$(\delta\overline{u})_{,i} = (\delta\overline{v})_{,i} + z\delta\overline{m}^*_{,i} = \delta\overline{r}^*_{,i} + z\delta\overline{m}^*_{,i} \qquad (\delta\overline{u})_{,z} = \delta\overline{m}^*,$$

we may write (10.3) as follows:

$$\delta A = \iint\limits_{(\Omega)} \int \Big\{ \overline{p}_1 A_2^* \delta\overline{r}^*_{,1} + \overline{p}_2 A_1^* \delta\overline{r}^*_{,2} + z\left(\overline{p}_1 A_2^* \delta\overline{m}^*_{,1} + \overline{p}_2 A_1^* \delta\overline{m}^*_{,2}\right) + $$
$$ + \overline{p}_z A_1^* A_2^* \delta\overline{m}^* \Big\} d\alpha_1 d\alpha_2 dz. \tag{a}$$

To transform the integrand in the right-hand side of this equation, we shall write the stress-vectors (5.5) as follows

$$\overline{p}_i = \overline{q}_i + \sigma_{iz}\overline{m}^*,$$

where

$$\overline{q}_i = \sigma_{i1}\frac{\overline{r}^*_{,1}}{A_1^*} + \sigma_{i2}\frac{\overline{r}^*_{,2}}{A_2^*}, \quad \overline{p}_z = \sigma_{1z}\frac{\overline{r}^*_{,1}}{A_1^*} + \sigma_{2z}\frac{\overline{r}^*_{,2}}{A_2^*} + \sigma_z\overline{m}^*.$$

Using these expressions and the fact that

$$\overline{p}_{,i} = \overline{r}^*_{,i} + \overline{m}^*_{,i}z, \quad \overline{m}^{*2} = 1, \quad \overline{m}^*\delta\overline{m}^* = 0, \qquad \overline{m}^*\delta\overline{m}^*_{,i} + \overline{m}^*_{,i}\delta\overline{m}^* = 0,$$

we obtain

$$z\left(\overline{p}_1 A_2^* \delta\overline{m}^*_{,1} + \overline{p}_2 A_1^* \delta\overline{m}^*_{,2}\right) + \overline{p}_z A_1^* A_2^* \delta\overline{m}^* =$$
$$= z\left(\overline{q}_1 A_2^* \delta\overline{m}^*_{,1} + \overline{q}_2 A_1^* \delta\overline{m}^*_{,2}\right) + \left(\sigma_{1z}\overline{r}^*_{,1} A_2^* + \sigma_{2z}\overline{r}^*_{,2} A_1^*\right)\delta\overline{m}^*.$$

From this equation, taking into account the expressions for the forces in the shell

$$\overline{K}_i = \int\limits_{-t/2}^{t/2} \overline{p}_i dz, \qquad N_i^* = \int\limits_{-t/2}^{t/2} \sigma_{iz} dz,$$

the equation (a) may be transformed into

$$\delta A = \iint\limits_{(\sigma^*)} \Big\{ \overline{K}_1 A_2^* \delta\overline{r}^*_{,1} + \overline{K}_2 A_1^* \delta\overline{r}^*_{,2} + \left(N_1^* A_2^* \overline{r}^*_{,1} + N_2^* A_1^* \overline{r}^*_{,2}\right)\delta\overline{m}^* +$$
$$+ \int\limits_{-t/2}^{t/2} z\left(\overline{q}_1 A_2^* \delta\overline{m}^*_{,1} + \overline{q}_2 A_1^* \delta\overline{m}^*_{,2}\right) dz \Big\} d\alpha_1 d\alpha_2. \tag{b}$$

Here σ^* is the entire area of the deformed middle surface of the shell, which, owing to the smallness of the deformation, is equal to the undeformed area. For a further transformation of the integrand of expression (b) we note that the moment vectors $\overline{M}_i$ given by (6.2), may be written as:

$$\overline{M}_i = \int\limits_{-t/2}^{t/2} z\left[\overline{m}^*\overline{p}_i\right] dz = \int\limits_{-t/2}^{t/2} z\left[\overline{m}^*\overline{q}_i\right] dz,$$

where $\overline{q}_i$ is a vector perpendicular to $\overline{m}^*$. Hence, multiplying vectorially by $\overline{m}^*$ and using the formula for the vector triple product $[\overline{a}[\overline{b}\,\overline{c}]] = \overline{b}(\overline{a}\,\overline{c}) - \overline{c}(\overline{a}\,\overline{b})$, we find

★ $$[\overline{M}_i \cdot \overline{m}^*] = \int_{-t/2}^{t/2} z\,\overline{q}_i\,dz.$$

As a result, the equation (b) becomes

$$\delta A = \int\int \Big\{ \overline{K}_1 A_2^* \delta \overline{r}^*_{,1} + \overline{K}_2 A_1^* \delta \overline{r}^*_{,2} + (N_1^* A_2^* \overline{r}^*_{,1} + N_2^* A_1^* \overline{r}^*_{,2}) \delta \overline{m}^* +$$
$$+ A_2^* [\overline{M}_1 \cdot \overline{m}^*] \delta \overline{m}^*_{,1} + A_1^* [\overline{M}_2 \cdot \overline{m}^*] \delta \overline{m}^*_{,2} \Big\} d\alpha_1 d\alpha_2 \qquad (10.4)$$

We shall introduce in this the projections of the forces and moments. Substituting for K_i from (6.8) we obtain

$$\overline{K}_1 A_2^* \delta \overline{r}^*_{,1} + \overline{K}_2 A_1^* \delta \overline{r}^*_{,2} + (N_1^* A_2^* \overline{r}^*_{,1} + N_2^* A_1^* \overline{r}^*_{,2}) \delta \overline{m}^* =$$
$$= \sum_{i,j} \frac{A_1^* A_2^*}{A_i^*} T_{ij}^* \overline{e}_j^* \delta \overline{r}^*_{,i}. \qquad (c)$$

Further, by substituting for $\overline{M}_i$ from (6.9), we find

$$A_2^* [\overline{M}_1 \cdot \overline{m}^*] \delta \overline{m}^*_{,1} + A_1^* [\overline{M}_2 \cdot \overline{m}^*] \delta \overline{m}^*_{,2} =$$
$$= \sum_{i,j} \frac{A_1^* A_2^*}{A_i^*} M_{ij}^* \overline{e}_j^* \delta \overline{m}^*_{,i}. \qquad (d)$$

We shall adopt the following notations for the scalar products in the right-hand side of the equation

$$\overline{e}_j^* \delta \overline{r}^*_{,i} = \overline{r}^*_{,j} \delta \overline{r}^*_{,i} / A_j^* = A_i^* \delta \tilde{\varepsilon}_{ji} \qquad (10.5)$$

and shall transform the right-hand side of (d).

To that end, we note that the identity

$$\overline{r}^*_{,j} \delta \overline{m}^*_{,i} = \delta (\overline{r}^*_{,j} \overline{m}^*_{,i}) - \overline{m}^*_{,i} \delta \overline{r}^*_{,j}$$

upon substitution of the expressions

$$\overline{r}^*_{,j} \overline{m}^*_{,i} = A_i^* A_j^* k_{ij}^*, \quad \overline{m}^*_{,i} = \sum_{s=1}^{2} A_i^* k_{is}^* \overline{e}_s^*$$

becomes

$$\overline{r}^*_{,j} \delta \overline{m}^*_{,i} = \delta (A_i^* A_j^* \varkappa_{ij}) - \sum_{s=1}^{2} A_i^* A_j^* k_{is}^* \delta \tilde{\varepsilon}_{sj}.$$

After taking $A_i^* A_j^*$ outside the variation sign, because the deformation is small, equation (d) becomes

$$\sum_{i,j} \frac{A_1^* A_2^*}{A_i^*} M_{ij}^* \overline{e}_j^* \delta \overline{m}^*_{,i} = A_1^* A_2^* \sum_{i,j} \left(\delta \varkappa_{ij} - \sum_{s=1}^{2} k_{is}^* \delta \tilde{\varepsilon}_{sj} \right) M_{ij}^*.$$

Introducing the right-hand sides of this equation and of equation (c) into (10.4) we obtain

$$\delta A = \int\int_{(\sigma)} \delta W d\sigma \quad (d\sigma = A_1 A_2 d\alpha_1 d\alpha_2). \qquad (10.6)$$

where the integrand

$$\delta W = \sum_{ij} \left\{ T_{ij}^* \delta \tilde{\varepsilon}_{ji} + M_{ij}^* \left(\delta \varkappa_{ij} - \sum_{s=1}^{2} k_{is}^* \delta \tilde{\varepsilon}_{sj} \right) \right\} \qquad (10.7)$$

★ is the variation of deformation energy of the shell per unit area of the middle surface. On using the simplest variant of elasticity relations (9.5) **and eliminating the quantities** that are of the same order as $k_{is}\varepsilon_{ik}$, (10.7) simplifies and becomes, in view of the equality $T^*_{ij} = T^*_{ji}$,

$$\delta W = \sum_{i,j} \left(T^*_{ij}\delta\varepsilon_{ij} + M^*_{ij}\delta\varkappa_{ij} \right), \qquad (10.8)$$

where ε_{ij} and $\varkappa_{ij}$ may be expressed by (3.13) and (3.29). In fact, according to (10.5) we have

$$\delta\widetilde{\varepsilon}_{11} = \bar{r}^*_{,1}\delta\bar{r}^*_{,1}/A_1^{*2},\ \delta\widetilde{\varepsilon}_{12} + \delta\widetilde{\varepsilon}_{21} = \left(\bar{r}^*_{,1}\delta\bar{r}^*_{,2} + \bar{r}^*_{,2}\delta\bar{r}^*_{,1}\right)/A_1^*A_2^*,$$

and by varying the equations

$$\left(\bar{r}^*_{,i}\right)^2 = A_i^{*2} = A_i^2(1+2\varepsilon_{ii}),\quad \bar{r}^*_{,i}\bar{r}^*_{,j} = 2A_iA_j\varepsilon_{ij},\quad i\neq j$$

we find

$$\bar{r}^*_{,i}\delta\bar{r}^*_{,j} + \bar{r}^*_{,j}\delta\bar{r}^*_{,i} = 2A_iA_j\delta\varepsilon_{ij} \qquad (i,\ j = 1,\ 2).$$

Therefore

$$\delta\widetilde{\varepsilon}_{ii} = \delta\varepsilon_{ii},\ \delta\widetilde{\varepsilon}_{12} + \delta\widetilde{\varepsilon}_{21} = 2\delta\varepsilon_{12}. \text{ ★}$$

Thus, the variation of the deformation energy of the shell is composed of the variations of the energies of elongation and shear

$$\delta W_1 = T^*_{11}\delta\varepsilon_{11} + T^*_{22}\delta\varepsilon_{22} + 2T^*_{12}\delta\varepsilon_{12} \qquad (10.9)$$

together with the variations of the energies of bending and torsion:

$$\delta W_2 = M^*_{11}\delta\varkappa_{11} + M^*_{22}\delta\varkappa_{22} + 2M^*_{12}\delta\varkappa_{12}. \qquad (10.10)$$

By introducing the expressions (9.5) for the forces and moments in (10.8) and integrating the resulting expression over the components of deformation between the state of zero deformation and the state that has the deformations ε_{ik} and $\varkappa_{ik}$, we obtain the expression for the specific work of deformations of the shell,

$$\begin{aligned} 2W = K\left[(\varepsilon_{11}+\varepsilon_{22})^2 - 2(1-\nu)\,(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)\right] + \\ + D\left[(\varkappa_{11}+\varkappa_{22})^2 - 2(1-\nu)\,(\varkappa_{11}\varkappa_{22} - \varkappa_{12}^2)\right], \end{aligned} \qquad (10.11)$$

where ε_{ik} and $\varkappa_{ik}$ may be expressed in terms of the displacements u_1, u_2, and w, according to (3.13) and (3.29). The formula (10.11) is similar to the formula for the deformation energy of a plate. Use of this formula for a thin shell involves an error of the order of magnitude t/R in comparison with unity.

★ We shall now express δA the left-hand side of (10.3) in terms of the external forces and moments. Substituting for $\delta\bar{u}$ from (10.2) into (10.1), we obtain:

$$\delta A = \iint_{(\sigma)} \left(\delta\bar{v}\int_{-t/2}^{t/2}\bar{F}\,dz + \delta\bar{m}\int_{-t/2}^{t/2}\bar{F}z\,dz\right) A_1^*A_2^*\,d\alpha_1\,d\alpha_2 + \\ + \iint_{(\Pi)}\left(\bar{p}_\nu\delta\bar{v} + z\bar{p}_\nu\delta\bar{m}^*\right)d\Pi,$$

where Π is the surface composed of the boundary surfaces $\Pi_{(+)}$ and $\Pi_{(-)}$ with the areal elements

$$d\Pi_{(+)} = A_1^*A_2^*\,d\alpha_1\,d\alpha_2,\quad d\Pi_{(-)} = A_1^*A_2^*\,d\alpha_1\,d\alpha_2.$$

★ The stress vectors on these surfaces are $\bar{p}_v = \bar{p}_z$ and $\bar{p}_v = -\bar{p}_z$ respectively. Furthermore, the surface Π includes the surface Σ of the boundary section of the shell, with the areal element $d\Sigma$. The stress vector at the point (α_1, α_2, z) of this section is:

$$\bar{p}_{n^z} = \bar{p}_1 n_1{}^z + \bar{p}_2 n_2{}^z.$$

where n^z are the projections of the vector normal to Σ on the directions of the unit vectors $\bar{e}_i^{*z}$ of the deformed shell at the distance z from the middle surface σ^*.

In the notations adopted above, we obtain for δA the following equation:

$$\delta A = \iint_{(\sigma)} \left\{ \delta\bar{v} \left(\bar{p}_z \Big|_{z=-t/2}^{z=t/2} + \int_{-t/2}^{t/2} \bar{F} dz \right) + \delta\bar{m}^* \left(z\bar{p}_z \Big|_{z=-t/2}^{z=t/2} + \right.\right.$$
$$\left.\left. + \int_{-t/2}^{t/2} z\bar{F}\, dz \right) \right\} A_1 A_2\, d\alpha_1 d\alpha_2 + \iint_{(\Sigma)} (\bar{p}_{n_z} \delta\bar{v} + z\bar{p}_{n_z} \delta\bar{m}^*)\, d\Sigma.$$

Furthermore, according to formulas of the type (8.5), the projections of the normal vectors to Σ at the points z = 0 and z = z will be written as

$$n_1{}^* = A_2{}^* \frac{d\alpha_2}{ds^*},\quad n_2{}^* = -A_1{}^* \frac{d\alpha_1}{ds^*},$$
$$n_1{}^z = H_2{}^* \frac{d\alpha_2}{ds^z},\quad n_2{}^z = -H_1{}^* \frac{d\alpha_1}{ds^z}. \qquad (*)$$

where ds^z is an element of the arc of intersection of Σ and the surface parallel to the middle surface at a distance z from it. Evidently, an element of the area is $d\Sigma = ds^z dz$. Therefore, using the equations (*) we find that $n_i{}^z d\Sigma = n_i{}^* ds^* dz$. Thus,

$$\bar{p}_{n_z} d\Sigma = (\bar{p}_1 n_1{}^z + \bar{p}_2 n_2{}^z)\, d\Sigma = (\bar{p}_1 n_1{}^* + \bar{p}_2 n_2{}^*)\, ds^* dz.$$

With this, and using (6.4) and (8.8), we may write the previous expression for δA as follows:

$$\delta A = \iint_{(\sigma)} (\bar{X}\delta\bar{v} + \bar{Y}\delta\bar{m}^*)\, d\sigma + \int_{C^*} \bar{K}_n \delta\bar{v}\, ds^* +$$
$$+ \int_{C^*} \int_{-t/2}^{t/2} z(\bar{p}_1 n_1{}^* + \bar{p}_2 n_2{}^*)\, ds^* dz.$$

where $d\sigma = A_1 A_2 d\alpha_1 d\alpha_2$; C is the contour of the non-deformed shell; ds is an element of arc of this contour (because of the small deformation); the vector $\bar{Y}$ is

$$\bar{Y} = z\bar{p}_z \Big|_{z=-t/2}^{z=t/2} + \int_{-t/2}^{t/2} z\bar{F} dz.$$

We can find $\bar{Y}$ in the following manner. Let the tangential and normal components of $\bar{Y}$ be $\bar{Y}_1$ and $\bar{Y}_3$ respectively: $\bar{Y} = \bar{Y}_1 + \bar{Y}_3$. Multiplying vectorially the external moment (6.5) by $\bar{m}^*$, we find for $\bar{Y}_1$ the expression

$$\bar{Y}_1 = [\bar{L}, \bar{m}^*].$$

Therefore,

$$\bar{Y}\delta\bar{m}^* = \bar{Y}_1 \delta\bar{m}^* = [\bar{L}, \bar{m}^*]\, \delta\bar{m}^*.$$

In the same manner, by multiplying vectorially the moment at the contour (8.8) by $\bar{m}^*$, we obtain

★
$$\int_{-t/2}^{t/2} z(\bar{p}_1 n_1^* + \bar{p}_2 n_2^*)\,dz = [\bar{M}_n, \bar{m}^*].$$

Taking the above into account, we obtain for δA the expression

$$\delta A = \iint_{(\sigma)} (\bar{X}\delta\bar{v} + [\bar{L}, \bar{m}^*]\delta\bar{m}^*)\,d\sigma + \int_C (\bar{K}_r\delta\bar{v} + [\bar{M}_n, \bar{m}^*]\delta\bar{m}^*)\,ds. \tag{10.12}$$

Here we have still to transform the integral along the contour. Using the expression (8.2) for $\bar{M}_r$ and the formulas $[\tau^*, \bar{m}^*] = \bar{n}^*$, $[\bar{m}^*, \bar{n}^*] = \tau^*$, we obtain:

$$[\bar{M}_n, \bar{m}^*] = G\bar{n}^* - H^*\bar{\tau}^*.$$

Furthermore, we have

$$\bar{K}_n\delta\bar{v} - H^*\bar{\tau}^*\delta\bar{m}^* = \bar{K}_n\delta\bar{v} - H\left(\bar{r}_{,1}^*\frac{da_1}{ds^*} + \bar{r}_{,2}^*\frac{da_2}{ds^*}\right)\delta\bar{m}^* =$$
$$= \bar{K}_n\delta\bar{v} + H^*\bar{m}^*\left(\delta\bar{r}_{,1}^*\frac{da_1}{ds^*} + \delta\bar{r}_{,2}^*\frac{da_2}{ds^*}\right) = \bar{K}_n\delta\bar{v} + H^*\bar{m}^*\frac{d\delta\bar{r}^*}{ds^*} =$$
$$= \bar{K}_n\delta\bar{v} + H^*\bar{m}^*\frac{d}{ds^*}\delta\bar{v} = \left(\bar{K}_n - \frac{dH^*\bar{m}^*}{ds^*}\right)\delta\bar{v} + \frac{d}{ds^*}(H^*\bar{m}^*\delta\bar{v}). \text{ ★}$$

Using these and the preceding equations we finally obtain:

$$\delta A = \iint_{(\sigma)} (\bar{X}\delta\bar{v} + [\bar{L}, \bar{m}^*]\delta\bar{m}^*)\,d\sigma +$$
$$+ \int_C (\bar{\Phi}\delta\bar{v} + \tilde{G}\bar{n}^*\delta\bar{m}^*)\,ds + \tilde{H}^*\bar{m}^*\delta v\Big|_C, \tag{10.13}$$

where $\bar{\Phi}$ is the vector of the external force on the contour, and $\tilde{G}^*$ and $\tilde{H}$ are the external bending and twisting moments at the contour of the deformed shell.

The surface integral in (10.13) represents the work of the external forces in infinitesimal variations of the displacements and the work of the external moments in infinitesimal variations of the twisting angles, since

$$[\bar{L}, \bar{m}^*]\delta\bar{m}^* = \sum_{i=1}^{2} L_i\bar{e}_i^*\delta\bar{n}^*,$$

where $\bar{e}_i^*\delta\bar{m}^*$ are the variations of the twisting angle.

The integral along the contour in (10.13) is essentially the work of the external forces and of the moments respectively in variations of the displacements and of the twisting angle, because $\bar{n}^*\delta\bar{m}^*$ is the variation of the twisting angle about the tangent to the contour.

The term outside the integral $\tilde{H}^*\bar{m}^*\delta\bar{v}|_C$ represents the work of the localized contour forces on the displacements. When the edges of the shell are hinged or clamped, this term vanishes. It also vanishes when the contour has no singular points and $\tilde{H}^*$ or $\bar{v}$ have no discontinuities. If the contour of the shell has singular points, localized forces of the type $\tilde{H}^*\bar{m}^*$ can appear at the singularities as localized reaction forces.

Thus in the non-linear theory of shells, the variational equation of the principle of virtual displacements is expressed by the relation

$$\delta A = \iint_{(\sigma)} \delta W A_1 A_2\,da_1\,da_2, \tag{10.14}$$

where δW is given by (10.7) or (10.8) and δA by (10.13). It should be noted that the equation (10.14) is also valid for the general non-linear theory of shells, where the displacements and deformations are considered to be arbitrary /III. 3/. The variational equation can be interpreted as follows:

Let $\mathcal{E}_1$ be the potential deformation energy of the shell and $\delta\mathcal{E}_1$ its total variation for isothermal or adiabatic deformation:

$$\mathcal{E}_1 = \iint\limits_{(\sigma)} W\,d\sigma, \quad \delta\mathcal{E}_1 = \iint\limits_{(\sigma)} \delta W\,d\sigma.$$

Let, furthermore, $\delta\mathcal{E}_2 = -\delta A$ be the variation of the potential energy of the load. Then we can write (10.14) as

$$\delta\mathcal{E} = \delta\mathcal{E}_1 + \delta\mathcal{E}_2 = 0, \tag{10.15}$$

where $\mathcal{E}$ is the total potential energy of the system.

Thus, the state of equilibrium of the shell differs from the adjacent geometrically possible states by the fact that for arbitrary virtual infinitesimal displacements of the system from equilibrium, the increment in total potential energy equals zero. This is the variational principle of Lagrange. We shall use the term "geometrically possible states" for any states for which the displacement variations do not disturb the geometrical constraints imposed on the shell. The following are geometrical constraints:

1. geometrical boundary conditions;

2. the deformations ε_{ik} and $\varkappa_{ik}$ allowed by the variational principle of Lagrange should be continuous deformations satisfying the conditions of continuity (3.32) and (3.35). These conditions will be fulfilled by expressing the ε_{ik} and $\varkappa_{ik}$ in terms of the displacements u_i and w according to (3.13) and (3.29).

The increment δA represents in the work of the external forces and moments the total variation only in some particular cases. Let us consider, for instance, the case when the external forces may be taken to be independent of the deformations and the parameters e_{ik} are small, i. e., $e_{ik} \sim \varepsilon_p$. Neglecting them as small in comparison with unity, we obtain from the formulas in § 3:

$$\bar{e}_i^*\,\delta\bar{m}^* \approx -\delta\omega_i,\quad \bar{n}^*\,\delta\bar{m}^* \approx -\delta\omega_n, \text{ where } \omega_n = n_1\omega_1 + n_2\omega_2$$

(n_i is the projection of the external normal on the undeformed contour of the shell). Then we obtain from (10.13):

$$A = \iint\limits_{(\sigma)} (\bar{X}\bar{v} - M_1\omega_1 - M_2\omega_2)\,d\sigma + \int\limits_{C_1} \bar{\Phi}\,\bar{v}\,ds - \int\limits_{C_2} \bar{G}\,\omega_n\,ds. \tag{10.16}$$

Here C_1 is that part of the contour where the vector of the contour force $\bar{\Phi}$ is given; C_2 the part where the bending moment is given. The variational equation (10.15) may be written as:

$$\delta(\mathcal{E}_1 + \mathcal{E}_2) = \delta\mathcal{E} = 0. \tag{10.17}$$

This variational equation is also valid for finite bending, provided the edges of the shell are hinged or clamped and the external forces are conservative:

$$X_1 = \frac{\partial f}{\partial u_1},\ X_2 = \frac{\partial f}{\partial u_2},\ X_3 = \frac{\partial f}{\partial w},\ (M_i = 0).$$

Therefore, (10.17) may be interpreted as follows: among all virtual displacements **compatible with the geometrical constraints imposed on the shell, only those** can occur in reality for which the potential energy of the system $\mathcal{E}$ assumes a **stationary** value (i. e., $\delta\mathcal{E} = 0$).

From the variational equation (10.14) one can derive the equations of equilibrium (7.4) and (7.5) and also the static boundary conditions (8.15) and (8.16). Let us now prove that.

★ Using (c) and (d) we write (10.4) as follows:

$$\delta A = \iint_{(\sigma)} \sum_i \sum_j A_j^{*-1} (T_{ji}^* \bar{e}_i^* \, \delta \bar{r}_{,j}^* + M_{ji}^* \bar{e}_i^* \, \delta \bar{m}_{,j}^*) \, d\sigma \; (i, j = 1, 2), \tag{10.18}$$

$$d\sigma = A_1^* A_2^* \, d\alpha_1 \, d\alpha_2.$$

where δA is given by (10.13).

Integrating by parts the right-hand side of (10.18) we obtain

$$\delta A = \int_{C^*} \sum_i \sum_j (T_{ji}^* \bar{e}_i^* \, \delta \bar{v} + M_{ji}^* \bar{e}_i^* \, \delta \bar{m}^*) \, n_j^* \, ds^* -$$

$$- \iint_{(\sigma)} \sum_i \sum_j \{\delta \bar{v} \, (T_{ji}^* \bar{e}_i^* \, A_1^* A_2^* A_j^{*-1})_{,j} + \delta \bar{m}^* \, (M_{ji}^* \bar{e}_i^* \, A_1^* A_2^* A_j^{*-1})_{,j}\} \, d\alpha_1 \, d\alpha_2;$$

for which we used the following formula for transforming the surface integral into a contour integral:

$$\iint_{(\sigma)} \sum_i f \varphi_{,i} \, d\sigma = \int_C \sum_i f \varphi \, A_i n_i \, dC - \iint_{(\sigma)} \sum_i \varphi \, (f A_1 A_2)_{,i} \, d\alpha_1 \, d\alpha_2. \tag{10.20}$$

By substituting for δA from (10.13) and introducing the quantities N_i^* given by the equation

$$\sum \sum_j \{(M_{ji}^* \bar{e}_i^* \, A_1^* A_2^* A_j^{*-1})_{,j} - A_1^* A_2^* M_{ji}^* \, k_{ij} \, m^* \} = \sum_{i=1} (N_i^* - M_i^*) \, A_1^* A_2^* \bar{e}_i^*, \tag{10.20}$$

we find

$$\begin{aligned} \int_{C^*} \Big\{ \Big(\bar{\Phi} - \sum_i \sum_j T_{ji}^* \bar{e}_i^* \, n_j \Big) \delta \bar{v} + \Big(\bar{G}^* n^* - \sum_i \sum_j M_{ji}^* \bar{e}_i^* \, n_j^* \Big) \delta \bar{m}^* \Big\} ds^* + \\ + H^* \bar{m}^* \delta \bar{v} \Big|_{C^*} = - \iint_{(\sigma)} \Big\{ \sum_i \sum_j (T_{ji}^* e_i \, A_1^* A_2^* A_j^{*-1})_{,j} \, \delta \bar{v} + \\ + \sum_{i=1}^{2} A_1^* A_2^* N_i^* \bar{e}_i^* \, \delta \bar{m}^* + A_1^* A_2^* \, \bar{\varphi} \, \delta \bar{v} \Big\} \, d\alpha_1 \, d\alpha_2. \end{aligned} \tag{10.20a}$$

After differentiating according to (2.18) and using the equality $\bar{m}^* \delta \bar{m}^* = 0$, the equation (10.20) becomes the vector equation of moments, equivalent to the two scalar equations (7.5). Therefore N_i^* are the shearing forces. Since

$$\bar{e}_i^* \, \delta \bar{m}^* = \bar{r}_{,i}^* \, A_i^{*-1} \, \delta \bar{m}^* = - A_i^{*-1} \, \bar{m}^* (\delta \bar{v})_{,i}, \quad \bar{m}_{,i}^* = \sum_j A_i^* k_{ij}^* \bar{e}_j^*,$$

we obtain, after integrating once more by parts:

★ $$\iint_{(s^*)}\sum_i N_i^* \bar{e}_i^* \delta \bar{m}^* d\sigma^* = -\iint_{(s^*)}\sum_i (N_i^* A_i^{*-1}\bar{m}^*)(\delta\bar{v})_{,i}\, d\sigma^* =$$

$$= -\int_C \sum_i N_i^* n_i^* \bar{m}^* \delta\bar{v}\, ds + \iint_{(s^*)}\sum_i (N_i^* A_1^* A_2^* A_i^{*-1}\bar{m}^*)_{,i}\, \delta\bar{v}\, d\alpha_1 d\alpha_2 =$$

$$= -\int_C \sum_i N_i^* n_i^* \bar{m}^* \delta\bar{v}\, ds + \iint_{(s^*)}\Big\{\sum_i (N_i^* A_1^* A_2^* A_i^{*-1})_{,i}\, \bar{m}^* \delta\bar{v} +$$

$$+ \sum_{i,j} A_1^* A_2^* N_i^* k_{ij}^* \bar{e}_j^* \delta\bar{v}\Big\} d\alpha_1 d\alpha_2.$$

Introducing this expression in (10.20a) and recalling the notations for the contour forces (8.9) we obtain the relation

$$\int_{C^*}\Big\{(\bar{\Phi} - \bar{K}_n)\delta\bar{v} + (\bar{G}^* \bar{n}^* - \sum_i\sum_j M_{ji}^* \bar{e}_i^* n_j^*)\delta\bar{m}^*\Big\} ds^* + \bar{H}^* \bar{m}^* \delta\bar{v}\Big|_{C^*} =$$
$$= -\iint_{(s^*)}\{(A_2^* \bar{K}_1)_{,1} + (A_1^* \bar{K}_2)_{,2} + A_1^* A_2^* \bar{X}\} d\alpha_1 d\alpha_2, \tag{10.20b}$$

since

$$\sum_i\sum_j\{(M_{ji}^* \bar{e}_i^* A_1^* A_2^* A_j^{*-1})_{,j} + A_1^* A_2^* N_i^* k_{ij}^* \bar{e}_j^*\} +$$
$$+ \sum_i (N_i^* A_1^* A_2^* A_i^{*-1})_{,i}\, \bar{m}^* = \sum_i (\bar{K}_i A_1^* A_2^* A_i^{*-1})_{,i}\,. \tag{10.21}$$

It is possible to check the validity of this equation by differentiating both sides by equations (2.18) for the deformed surface. While deriving (10.13), we proved that:

$$\frac{d}{ds^*}(H^* \bar{m}^* \delta\bar{v}) = \delta\bar{v}\frac{dH^* \bar{m}^*}{ds^*} - H^* \bar{\tau}^* \delta\bar{m}^*;$$
$$H^* \bar{m}^* \delta\bar{v}\Big|_{C^*} = \int_{C^*}\Big(\delta\bar{v}\frac{dH^* \bar{m}^*}{ds^*} - H^* \bar{\tau}^* \delta\bar{m}^*\Big) ds^*. \tag{10.22}$$

Using the latter equality, we write the variational equation (10.20b) in the following form:

$$\int_{C^*}\Big\{\Big(\bar{\Phi} - \bar{K}_n + \frac{dH^* \bar{m}^*}{ds^*}\Big)\delta\bar{v} + \Big(\bar{G}\bar{n}^* - H^* \bar{\tau}^* - \sum_i\sum_j M_{ji}^* \bar{e}_i^* n_j^*\Big)\delta\bar{m}^*\Big\} ds^* +$$
$$+ (\bar{H}^* - H^*)\bar{m}^* \delta\bar{v}\Big|_{C^*} + \iint_{(s^*)}\{(A_2^* \bar{K}_1)_{,1} + (A_1^* \bar{K}_2)_{,2} + A_1^* A_2^* \bar{X}\}\delta\bar{v}\, d\alpha_1 d\alpha_2 = 0.$$

In order to obtain the final result, we transform here the contour integral containing the term $\delta\bar{m}^*$. Since

$$\bar{M}_n = H^* \bar{n}^* + G^* \bar{\tau}^*, \; [\bar{M}_n, \bar{m}^*] = G^* \bar{n}^* - H^* \bar{\tau}^*, \tag{10.23}$$

we obtain, substituting for $\bar{M}_n$ from (8.10),

$$[\bar{M}_n, \bar{m}^*] = \sum_i\sum_j M_{ji}^* \bar{e}_i^* n_j^*. \tag{10.24}$$

Therefore

$$H^* \bar{\tau}^* + \sum_i\sum_j M_{ji}^* \bar{e}_i^* n_j^* = G^* \bar{n}^*. \bigstar \tag{10.25}$$

Hence the expression (10.20b) finally becomes:

$$\int_{C^*} \left\{ \left(\bar{\Phi} - \bar{K}_n + \frac{dH^* \bar{m}^*}{ds^*} \right) \delta \bar{v} + (\tilde{G}^* - G^*) \bar{n}^* \delta \bar{m}^* \right\} ds^* +$$
$$+ (\tilde{H}^* - H^*) \bar{m}^* \delta \bar{v} \Big|_{C^*} + \iint_{(\sigma)} \{ (A_2^* \bar{K}_1)_{,1} + \tag{10.26}$$
$$+ (A_1^* \bar{K}_2)_{,2} + A_1^* A_2^* \bar{X} \} \delta \bar{v} d\alpha_1 d\alpha_2 = 0.$$

Here $\bar{\Phi}$ is the vector of the external load at the contour, $\tilde{G}^*$ and $\tilde{H}^*$ are the external bending and twisting moments, and G^* and H^* are given by (8.12).

Since the variations of the displacement vector and of the twisting angle are arbitrary, being independent, the static boundary conditions and the equations of equilibrium are derived from the variational equation (10.26) under special assumptions on their separate variation.

Let us consider those virtual displacements for which the variations of the displacement vector and of the twisting angle vanish on the contour: $\delta \bar{v} = 0$ and $\bar{n}^* \delta \bar{m}^* = 0$. According to the fundamental theorem of the calculus of variations, we obtain from (10.26) the vector equation of equilibrium, because of the arbitrariness of $\delta \bar{v}$ inside the shell. As far as the vector equation of moments is concerned, it has no independent role, serving only for the determination of the shearing forces. We obtained this equation in the above-mentioned expression (10.20) while we derived the equation of variation (10.26).

We shall assume that at the contour $\delta \bar{v} = 0$, but that the variation of the twisting angle is $\bar{n}^* \delta \bar{m}^* \neq 0$. Then from (10.26) we obtain

$$\int_{C} (\tilde{G}^* - G^*) \bar{n}^* \delta \bar{m}^* = 0 \tag{10.27}$$

Since $\bar{n}^* \delta \bar{m}^*$ is arbitrary, we obtain static boundary condition (8.16), $\tilde{G} = G_1$. If the state with $\delta \bar{v} = 0$ at the contour is a geometrically possible state, we can derive from the variational equation (10.26) the vectorial form (8.15) of the static boundary condition.

Let us note that at the end of the 19th century, the fundamental equations of the theory of shells were obtained on the basis of the principle of virtual displacements.

The well-known Ritz approximation method for determining the strength and stability of shells is based on the variational principle for virtual displacements.

The essence of this method is as follows: it was shown above that the variational equation (10.14) contains the equation of equilibrium and the static boundary conditions. Therefore by satisfying the variational equation we satisfy the static conditions inside the shell and at the contour. The higher the degrees of approximation with which the problem is solved, the higher the degree of accuracy with which the static conditions are fulfilled. In this case, the geometrical boundary conditions are essential, i.e., they should be satisfied in advance. Therefore, in approximate solutions of actual problems by means of the variational equation (10.14) we shall use approximation [trial] functions like the following ones:

$$u_1 = \sum_{k=1}^{n} A_k f_k(\alpha_1, \alpha_2), \qquad u_2 = \sum_{k=1}^{n} B_k \varphi_k(\alpha_1, \alpha_2), \tag{10.28}$$
$$w = \sum_{k=1}^{n} C_k \psi_k(\alpha_1, \alpha_2),$$

where A_k, B_k, and C_k, are constants to be determined and f_k, φ_k, and ψ_k are given functions chosen in such a manner that the displacements u_1, u_2, and w should satisfy the geometrical boundary conditions. Introducing then (10.28) in (10.14) and comparing the coefficients of the variations δA_k, δB_k, δC_k we obtain a system of algebraic equations from which we can calculate the constants A_k, B_k, and C_k. In general, the derived system of algebraic equations will be non-linear. It will become linear only in the linear problems of the theory of shells. In actual cases, besides the difficulties of choosing the trial functions (10.28), it is very difficult to solve the non-linear system from a purely algebraic point of view. But in spite of that, the Ritz method is the most widely used and the most reliable one. The convergence of Ritz approximations has been proved in the monograph by S. G. Mikhlin /III. 4/.

§ 11. Equations of the Bubnov-Galerkin Method

The variational equation (10.26) represents the equations of the Bubnov-Galerkin method in vectorial form. On projecting the vector $\delta\bar{v}$ along the unit vectors of the coordinates of the deformed shell

$$\delta\bar{v} = \bar{e}_1^* (\delta\bar{v})_1 + \bar{e}_2^* (\delta\bar{v})_2 + \bar{m}^* (\delta\bar{v})_3, \tag{11.1}$$

the expression (10.26) becomes

$$\begin{gathered}\int\limits_{C^*}\left\{\left(\bar{\Phi} - \bar{K}_n + \frac{dH^*\bar{m}^*}{ds^*}\right)\delta\bar{v} + (\tilde{G}^* - G^*)\,\bar{n}^*\delta\bar{m}^*\right\} ds^* + \\ + (\tilde{H}^* - H^*)\,\bar{m}^*\delta\bar{v}\,|_{C^*} + \int\limits_{\sigma^*}\int\left\{(7.4)_1(\delta\bar{v})_1 + (7.4)_2(\delta\bar{v})_2 + (7.4)_3(\delta\bar{v})_3\right\} d\alpha_1 d\alpha_2 = 0.\end{gathered} \tag{11.2}$$

Here the notations $(7.4)_1$, $(7.4)_2$, $(7.4)_3$ represent the left-hand sides of the equations of equilibrium (7.4) and in addition

$$(\delta\bar{v})_i = \delta u_i + \frac{\delta\bar{v}}{A_i}\frac{\partial\bar{v}}{\partial\alpha_i},\ (\delta\bar{v})_3 = \delta w + (\bar{m}^* - \bar{m})\,\delta\bar{v}\ (i = 1, 2), \tag{11.3}$$

furthermore, $\bar{n}^*\delta\bar{m}^*$ is the variation of the twisting angle about the tangent to the contour. For small deformations this is

$$\begin{aligned}\bar{n}^*\delta\bar{m}^* = (\bar{e}_1^* n_1 + \bar{e}_2^* n_2)\,\delta\bar{m}^* &= -\bar{m}^*(n_1\delta\bar{e}_1^* + n_2\delta\bar{e}_2^*) = \\ &= -n_1(E_1\delta e_{11} + E_2\delta e_{12} + E_3\delta\omega_1) - \\ &\quad - n_2(E_1\delta e_{21} + E_2\delta e_{22} + E_3\delta\omega_2).\end{aligned} \tag{11.4}$$

Hence, when the displacements are small, neglecting the parameters e_{ik} in comparison with unity, we find

$$\bar{n}^*\delta\bar{m}^* = -n_1\delta\omega_1 - n_2\delta\omega_2 = -\delta\omega_n. \tag{11.5}$$

This is the variational equation of the Bubnov-Galerkin method in the system of coordinates of the deformed shell. In this form, these equations are suited for theoretical research, because they are related to the energy functional which is positive for small deformations and finite displacements. Another variant can be obtained by projecting the vector $\delta\bar{v}$ along the unit vectors of the coordinates of the undeformed shell:

$$\delta\bar{v} = \bar{e}_1\delta u_1 + \bar{e}_2\delta u_2 + \bar{m}\,\delta w. \tag{11.6}$$

Projecting the vector equation of equilibrium (7.1) along the same directions we obtain

$$\begin{gathered}\int\limits_{C}\left\{\left(\bar{\Phi} - \bar{K}_n + \frac{dH^*\bar{m}^*}{ds}\right)(\bar{e}_1\delta u_1 + \bar{e}_2\delta u_2 + \bar{m}\delta w) + \right. \\ \left. + (\tilde{G}^* - G^*)\,\bar{n}^*\delta\bar{m}^*\right\} ds + (\tilde{H}^* - H^*)\,\bar{m}^*\delta\bar{v}\,|_C + \\ + \int\limits_{\sigma}\int\left\{(8.29)_1\delta u_1 + (8.29)_2\delta u_2 + (8.29)_3\delta w\right\} d\alpha_1 d\alpha_2 = 0,\end{gathered} \tag{11.7}$$

where the integration is performed over the contour and the middle surface of the undeformed shell; $(8.29)_1$, $(8.29)_2$, and $(8.29)_3$ represent the right-hand sides of the system (8.29) in the above-mentioned order.

★ Let us now derive an integral relation which will subsequently be needed. Let us consider the integral

$$I=\iint_{\sigma}\left\{(7.4)_1\cdot f_1+(7.4)_2\cdot f_2+(7.4)_3\cdot f_3\right\}d\alpha_1 d\alpha_2,$$

where f_1, f_2 and f_3 are arbitrary functions. Integrating by parts (formula (10.19) we obtain:

$$I=\int_{C^*}\sum_{i=1}^{2}\left(\sum_{j=1}^{2}T_{ij}^*f_j+N_i^*f_3\right)n_i^*\,ds^*-\iint_{\sigma^*}\left[\sum_{i=1}^{2}\left(\sum_{j=1}^{2}T_{ij}\Omega_{ij}+N_i^*\Omega_i-X_i^*f_i\right)-X_3^*f_3\right]d\sigma^*. \tag{11.8}$$

Here we used the identity (10.21) and set

$$A_i^*\Omega_{ij}=\bar{e}_j^*\sum_{s=1}^{2}(\bar{e}_s^*f_s)_{,i}+A_i^*k_{ij}^*f_3,$$

$$\Omega_iA_i^*=f_{3,i}-\sum_{s=1}^{2}A_i^*k_{is}^*f_s. \tag{11.9}$$

Substituting for N_i^* from (10.20) we obtain

$$\iint_{\sigma^*}\sum_{s=1}^{2}N_s^*\Omega_s d\sigma^*=\iint_{\sigma^*}\sum_{s=1}^{2}\left\{\sum_{i,k}(M_{ik}^*\bar{e}_k^*A_{3-i}^*)_{,i}\bar{e}_s^*\Omega_s+A_1^*A_2^*M_s\Omega_s\right\}d\alpha_1d\alpha_2$$

$$(A_{3-i}=A_2 \text{ for } i=1;\ A_{3-i}=A_1 \text{ for } i=2),$$

whence, using (10.19) we find

$$I_1=\iint_{\sigma^*}\sum_{s}N_s^*\Omega_s d\sigma^*=\int_{C^*}\sum_{i,k}M_{ik}^*\Omega_kn_i^*ds^*+$$

$$+\iint_{\sigma^*}\sum_{k=1}^{2}\left(\sum_{i=1}^{2}M_{ik}^*\tilde{\varkappa}_{ik}+M_k\Omega_k\right)d\sigma^*, \tag{11.10}$$

where we introduced the new notations

$$A_i\tilde{\varkappa}_{ik}=-\bar{e}_k^*\sum_{s=1}^{2}(\bar{e}_s^*\Omega_s)_{,i}. \tag{11.11}$$

Using this newly-found expression for I_1 and taking into account that $I = 0$, we obtain the integral relation

$$\iint_{\sigma^*}\left[\sum_{i=1}^{2}(X_i^*f_i-L_i^*\Omega_i)+X_3^*f_3\right]d\sigma^*+I_{C^*}=\iint_{\sigma^*}\sum_{i,k}(T_{ik}^*\Omega_{ik}+M_{ik}^*\tilde{\varkappa}_{ik})\,d\sigma^*. \tag{11.12}$$

where I_{C^*} represents the contour integral

$$I_{C^*}=\int_{C^*}\left[\sum_{i,k}(T_{ik}^*f_k-M_{ik}^*\Omega_k)n_i^*+\sum_{i=1}^{2}N_i^*n_i^*f_3\right]ds^*.$$

We now transform this contour integral. By scalar multiplication of (10.25) by $\bar{e}_k^*$ we obtain

$$\sum_{i=1}^{2}M_{ik}^*n_i^*=G^*n_k^*-H^*\tau_k^*.$$

Hence

$$\int_{C^*}\sum_{i,k}M_{ik}^*\Omega_kn_i^*ds^*=\int_{C^*}\sum_{i}(G^*n_i^*-H^*\tau_i^*)\Omega_ids^*=$$

$$=\int_{C^*}G^*\Omega_nds^*-\int_{C^*}H^*\sum_{i=1}^{2}\tau_i^*\left(\frac{1}{A_i}\frac{\partial f_3}{\partial\alpha_i}-\sum_{j}k_{ij}^*f_j\right)ds^*=$$

$$=\int_{C^*}G^*\Omega_nds^*-\int_{C^*}H^*df_3+\int_{C^*}H^*\sum_{i,j}k_{ij}\tau_i^*f_jds^*,\quad\left(\Omega_n=\sum_{i=1}^{2}n_i^*\Omega_i\right).$$

★ By introducing the right-hand side of this equality in the expression for I_{C^*}, we obtain:

$$I_{C^*} = \int_{C^*} (\Phi_1^* f_1 + \Phi_2^* f_2 + \Phi_3^* f_3 - \bar{G}^* \Omega_n)\, ds^* + \bar{H}^* f_3 \Big|_{C^*}. \tag{11.13}$$

where we introduced the contour forces Φ_i^* and Φ_3^* and the bending moment at the contour $\bar{G}^*$ according to formula (8.19).

This, if the equations of equilibrium are satisfied, the integral relation (11.12) is valid for arbitrary functions f_i and f_3 which are sufficiently differentiable. On assuming $f_i = u_i$, $f_3 = w$ and neglecting the second and higher powers of the displacements and of their derivatives, the equation (11.12) expresses Clapeyron's theorem for the non-linear theory of shells. After introducing (11.9) and (11.11) in the right-hand side of (11.12) and integrating by parts, one can obtain the expression:

$$\begin{aligned} &\int_{C^*} \left\{ \left(\bar{\Phi} - \bar{K}_n + \frac{dH^* \bar{m}^*}{ds^*} \right) \left(\sum_i \Phi_i^* f_i + \bar{m}^* f_3 \right) - (\bar{G}^* - G)\Omega_n \right\} ds^* + \\ &+ (\bar{H}^* - H^*) f_3 \big|_C + \iint_{\sigma} \{ (7.4)_1 \cdot f_1 + (7.4)_2 \cdot f_2 + (7.4)_3 \cdot f_3 \}\, d\alpha_1 d\alpha_2 = 0. \end{aligned} \tag{11.14}$$

It may be derived in precisely the same manner as (10.26) from (10.18). By setting $f_i = (\delta \bar{v})_i$, $f_3 = (\delta \bar{v})_3$ in (11.14) and taking into account the equality $\bar{e}_i^* \delta \bar{m}^* = -\bar{m}^* \delta \bar{e}_i^*$ we obtain (11.2). We shall set

$$f_i = \frac{\delta \bar{v}}{A_i} \frac{\partial \bar{v}}{\partial \alpha_i}, \quad f_3 = (\bar{m}^* - \bar{m})\, \delta \bar{v} \tag{11.15}$$

in (11.14) and subtract the result from (11.2). We thus obtain the third variant of the equation of the Bubnov-Galerkin method:

$$\begin{aligned} &\int_C \left\{ \left(\bar{\Phi} - \bar{K}_n + \frac{dH^* \bar{m}^*}{ds} \right) (\bar{e}_1^* \delta u_1 + \bar{e}_2^* \delta u_2 + \bar{m}^* \delta w) + \right. \\ &\left. + \sum_i (\bar{G}^* - G^*)\, n_i \delta \tilde{\omega}_i \right\} ds + (\bar{H}^* - H^*)\, \delta w \Big|_C + \iint_{\sigma} \{ (7.4)_1 \delta u_1 + (7.4)_2 \delta u_2 + (7.4)_3 \delta w \}\, d\alpha_1 d\alpha_2 = 0, \end{aligned} \tag{11.16}$$

where the integration is performed over the contour and the middle surface of the undeformed shell.

Here we used the notation

$$\delta \tilde{\omega}_i = \delta \omega_i - \sum_s \varkappa_{is} \delta u_s. \tag{11.17}$$

These quantities are obtained as follows:

$$\bar{n}^* \delta \bar{m}^* + \sum_i n_i \Omega_i = \bar{n}^* \delta \bar{m}^* + \sum_{i=1}^{2} n_i \left\{ \frac{\partial (\bar{m}^* - \bar{m})\, \delta \bar{v}}{A_i \partial \alpha_i} - \sum_{s=1}^{2} \frac{k_{is}^*}{A_s} \frac{\partial \bar{v}}{\partial \alpha_s} \delta \bar{v} \right\}.$$

from which, taking into account the equations

$$\bar{m}_{,i}^* = A_i^* \sum_s k_{is}^* \bar{r}_{,s}^* / A_s^* = A_i^* \sum_s k_{is}^* \left(\bar{e}_s + \frac{1}{A_s} \frac{\partial \bar{v}}{\partial \alpha_s} \right),$$

$$\sum_i n_i^* \frac{\bar{m}^*}{A_i^*} \frac{\partial \delta \bar{v}}{\partial \alpha_i} + \bar{n}^* \delta \bar{m}^* = \sum_i (n_i^* \bar{m}^* \delta \bar{e}_i^* + n_i^* \bar{e}_i^* \delta \bar{m}^*) = 0,$$

$$\frac{\bar{m}}{A_i} \cdot \frac{\partial \delta \bar{v}}{\partial \alpha_i} = \delta \omega_i; \quad \sum_i k_{ij}^* \bar{e}_j \delta \bar{v} = \sum_j k_{ij}^* \delta u_j; \quad \bar{m}_{,i} \cdot \frac{\delta \bar{v}}{A_i} = \sum_j k_{ij} \delta u_j,$$

we obtain (11.17).

★ The last two variants (11.14) and (11.16) of the equations of the Bubnov-Galerkin method are suited to actual calculations.

In the equation (11.16) the part with the non-linear terms has been transferred from the right-hand side to the left-hand side because the external forces are projected on the directions of the coordinates of the deformed shell, whereas in equations (11.2) and (11.7) they were projected on the direction of the coordinates of the undeformed shell. Denoting those projections by the respective letters without asterisks, we obtain

$$\begin{aligned} X_1^* &= \bar{e}_1^*\bar{X} = X_1(1+e_{11}) + X_2e_{12} + X_3\omega_1, \\ X_2^* &= X_1e_{21} + X_2(1+e_{22}) + X_3\omega_2, \\ X_3^* &= \bar{m}^*\bar{X} = X_1E_1 + X_2E_2 + X_3F_3. \end{aligned} \tag{11.18}$$

We can write similar formulas for the vector $\bar{\Phi}$:

$$\begin{aligned} \Phi_1^* &= \Phi_1(1+e_{11}) + \Phi_2e_{12} + \Phi_3\omega_1, \\ \Phi_2^* &= \Phi_1e_{21} + \Phi_2(1+e_{22}) + \Phi_3\omega_2, \\ \Phi_3^* &= \Phi_1E_1 + \Phi_2E_2 + \Phi_3E_3, \end{aligned} \tag{11.19}$$

where Φ_i and Φ_3 are the projections of the vector of the contour load on the directions of the coordinates of the undeformed shell. Formulas (11.18) and (11.19) take into account the twist of the external forces in the deformation.

If in (11.16) the contour integral

$$I_C = \int_C \left\{\left(\bar{\Phi} - \bar{K}_n + \frac{dH^*\bar{m}^*}{ds}\right)(\bar{e}_1^*\delta u_1 + \bar{e}_2^*\delta u_2 + \bar{m}^*\delta w) + \sum_i (\tilde{G}^* - G^*)\, n_i \delta\, \tilde{\omega}_i\right\} ds + (\tilde{H}^* - H^*)\,\delta w\Big|_C \tag{11.20}$$

vanishes, we obtain the following three equations because the variations δu_i and δw are independent:

$$\iint (7.4)_1 \delta u_1\, d\alpha_1 d\alpha_2 = 0, \quad \iint (7.4)_2 \delta u_2\, d\alpha_1 d\alpha_2 = 0, \quad \iint (7.4)_3 \delta w\, d\alpha_1 d\alpha_2 = 0, \tag{11.21}$$

These three equations are called "equations of the Bubnov-Galerkin method" because these authors /III. 7/ used variational equations of the kind (11.21) for the first time. As distinct from the Ritz method, expounded in the preceding section, the Bubnov-Galerkin method may be used, under certain conditions, for any differential equation.

The contour integral (11.20) vanishes, for instance, in the following cases:

1. When the static boundary conditions are fulfilled on the entire contour;

2. when the contour is clamped $\left(\bar{v} = 0,\ \sum n_i \delta\tilde{\omega}_i = 0\right)$;

3. when the contour is hinged $(\bar{v} = 0\ \ \tilde{G}^* = 0)$;

4. when the contour is freely supported $(w = \Phi_i^* = \Phi_2^* = 0)$, or for mixed boundary conditions, containing the above-mentioned cases.

When solving actual problems by the Bubnov-Galerkin method, one takes the functions (10.28) as trial functions for u_i and w, as in the Ritz method. By introducing (10.28) in (11.21) we obtain the required number of equations for determining the constants. The algebraic system obtained will be non-linear; therefore the difficulties arising in the solution of a non-linear system remain. In general, the reduction of equation (11.16) to (11.21) is not obligatory, because if the static

★ boundary conditions are not exactly fulfilled, one must retain the contour integral and thereby take into account the work of the non-equilibrium forces.

The expression $(\widetilde{G}^* - G^*)\delta\widetilde{\omega}_i$ for the composite state of stress may be simplified. In fact, by substituting for $\delta\omega_i$ from (11.17) we obtain:

$$(\widetilde{G}^* - G^*)\,\delta\widetilde{\omega}_i = (\widetilde{G}^* - G^*)\,\delta\omega_i - (\widetilde{G}^* - G^*)\sum_{k=1}^{2} \varkappa_{ik}\delta u_k.$$

As for small deformations and for $t\varkappa_{ik} \sim \varepsilon_p$

$$(\widetilde{G}^* - G^*)\sum_{k} \varkappa_{ik}\delta u_k \sim M_{ik}\varkappa_{sk}\delta u_s \sim Et^3\varkappa_{ik}^2\delta u_s \sim Et\varepsilon_p^2\delta u_s,$$

the quantities $(\widetilde{G}^* - G^*)\sum_i \varkappa_{ik}\,\delta u_k$ are negligibly small in comparison with $\Phi_i^*\,\delta u_s \sim Et\,\varepsilon_p\,\delta u_s$; hence:

$$(\widetilde{G}^* - G^*)\sum_{i} n_i\delta\widetilde{\omega}_i = (\widetilde{G}^* - G^*)\sum_{i=1}^{2} n_i\delta\omega_i. \quad ★ \tag{11.22}$$

The question of convergence of the Bubnov-Galerkin method for linear problems was examined in the book by S. G. Mikhlin /III. 4/ and for non-linear problems of the theory of shells in the book by I. I. Vorovich /III. 12/. It should be noted that the convergence of the method will increase if the functions (10. 28) satisfy all the boundary conditions. The convergence is usually examined in each concrete case. The theoretical foundation of different variational methods and their application to a series of problems of the linear theory of elasticity has been given in the monograph by L. S. Leibenzon /III. 5/. The question of the interrelations between the different variational methods has been examined in the book by Ya. A. Pratusevich /III. 6/. The same book also gives applications of the variational methods to many problems. The principal types of linear problems which can be solved by the Bubnov-Galerkin method are given in Mikhlin's book /III. 4/. Some indications of the application of this method to the approximate solution of non-linear differential equations are given in the paper by A. R. Rzhanitsyn /III. 7/.

§ 12. Introduction of Symmetric Components of Forces and Moments. Stress Functions

★ Let us introduce new components S_{ij} of the tangential forces by the equations

$$T_{ij}^{*} = S_{ij} + \sum_{s=1}^{2} k_{js}^{*} M_{si}^{*}. \tag{12.1}$$

Introducing this in the sixth equation of equilibrium (7.6) we obtain $S_{ij} = S_{ij}$, i. e., the forces S_{ij} are symmetric. We introduce also symmetric components of the moment M_{ij} by the equations

$$M_{ij}^{*} = M_{ij} + Q_{ij}, \tag{12.2}$$

where we assumed that

$$2M_{ij} = M_{ij}^{*} + M_{ji}^{*}; \quad 2Q_{ij} = M_{ij}^{*} - M_{ji}^{*}. \tag{12.3}$$

The quantities $Q_{ij} = -Q_{ji}$ may be expressed in terms of S_{ij} and M_{ij} after introducing (12.1) and (12.2) in the additional non-differential relation of the type (7.7). This relation is sufficient, since $Q_{ii} = 0$, $Q_{12} = -Q_{21}$.

We shall now write the equation of equilibrium and the static boundary conditions for symmetric components of forces and moments. Introducing (12.1) and (12.2) in the variational equation (10.5) we find, after the cancellation of similar terms

$$\delta A = \iint_{\sigma} \sum_{i} \sum_{j} (S_{ij} \delta\varepsilon_{ij} + M_{ij} \delta k_{ij}^{*})\, d\sigma, \tag{12.4}$$

where δA may be expressed by (10.13); $\delta\varepsilon_{ij}$ is the variation of the components of tangential deformation; $\delta k_{ij}^{*} = \delta \varkappa_{ij}$ is the variation of the components of the bending deformation. Therefore, the variation of deformation energy for new forces and moments is also of the kind (10.7) provided $T_{ij}^{*} \neq T_{ji}^{*}$, $M_{ij}^{*} \neq M_{ji}^{*}$.

The above-mentioned fact is one of the advantages of the new tensors of forces and moments. Another advantage is that by introducing these forces and moments, the sixth equation of equilibrium is identically satisfied.

Taking into account that

$$A_i^{*} A_j^{*} \delta k_{ij}^{*} = -\delta b_{ij}^{*} = \delta\left(\vec{r}_{,i}^{\,*}\, \vec{m}_{,j}^{\,*}\right) = \vec{m}_{,j}^{\,*}\, \delta\vec{r}_{,i}^{\,*} + \vec{r}_{,i}^{\,*}\, \delta\vec{m}_{,j}^{\,*}$$

and that the equality

$$\sum_{i} \sum_{j} S_{ij} \delta\varepsilon_{ij} A_i^{*} A_j^{*} = \sum_{i} \sum_{j} S_{ij} \vec{r}_{,j}^{\,*}\, \delta\vec{r}_{,i}^{\,*}$$

holds for symmetric S_{ij}, the relation (12.4) may also be expressed in the form

$$\delta A = \iint_{(\sigma^{*})} \sum_{i} \sum_{j} \left\{ \left(S_{ij} \vec{r}_{,j}^{\,*} + \vec{m}_{,j}^{\,*} M_{ij}\right) \delta\vec{r}_{,i}^{\,*} + M_{ij} \vec{r}_{,i}^{\,*}\, \delta\vec{m}_{,j}^{\,*} \right\} d\alpha_1 d\alpha_2.$$

Further, by substituting for $\overline{m}_{ij}^{*}$ for the deformed surface from (2.22) we obtain

★ $$\sum_i\sum_j M_{ij}\bar{m}^*_{,j}\,\delta\bar{r}^*_{,i} = \sum_i\sum_j\sum_s M_{ij}A_j^* k^*_{js}\bar{e}_s^*\,\delta\bar{r}^*_{,i}.$$

We thus find:

$$\delta A = \iint\limits_{(\sigma^*)} \sum_i\sum_j \frac{1}{A_i^*}\left(\hat{T}_{ij}\bar{e}_j^*\,\delta\bar{r}^*_{,i} + M_{ij}\bar{e}_j^*\,\delta\bar{m}^*_{,i}\right) d\sigma^* \tag{12.5}$$

where we set

$$\hat{T}_{ij} = S_{ij} + \sum_s M_{is}k^*_{sj}. \tag{12.6}$$

Formally, equation (12.5) is identical with (10.18). Since (10.26) is derived from the latter, a similar equation may be analogously obtained from (12.5):

$$\int\limits_{C^*}\left\{\left(\bar{\Phi} - \bar{g}_n + \frac{d\hat{H}\bar{m}^*}{ds^*}\right)\delta\bar{v} + \left(\bar{G}^* - \hat{G}\right)\bar{n}^*\delta\bar{m}^*\right\} ds + \iint\limits_{\sigma}\left(\frac{\partial A_2^*\bar{g}_1}{\partial\alpha_1} + \frac{\partial A_1^*\bar{g}_2}{\partial\alpha_2} + A_1^*A_2^*\bar{X}\right)\delta\bar{v}\,d\alpha_1 d\alpha_2 = 0, \tag{12.7}$$

where we set

$$\bar{g}_i = \sum_{j=1}^{2}\hat{T}_{ij}\bar{e}_j^* + \bar{m}^*Q_i; \tag{12.8}$$

$$\bar{g}_n = \bar{g}_1 n_1 + \bar{g}_2 n_2 = \sum_i\sum_j \hat{T}_{ij}\bar{e}_j^* n_i + \bar{m}^*\sum_i Q_i n_i; \tag{12.9}$$

$$\hat{G} = \sum_i\sum_j M_{ij}n_i n_j; \tag{12.10}$$

$$\hat{H} = -\sum_i\sum_j M_{ij}n_i\tau_j. \tag{12.11}$$

Here Q_i is the analog of the shearing forces connected with the symmetric moments M_{ij} by equations of the form (10.20)

$$A_1^*A_2^*Q_i = A_1^*A_2^*M_i^* + \sum_k\sum_j \bar{e}_i^*\left(M_{kj}\bar{e}_k^*A_1^*A_2^*A_j^{*-1}\right)_{,j}. \tag{12.12}$$

Thus, the variational equation (12.4) leads to the following equations of equilibrium for symmetric components of the forces and moments

$$(A_2^*\bar{g}_1)_{,1} + (A_1^*\bar{g}_2)_{,2} + A_1^*A_2^*\bar{X} = 0; \tag{12.13}$$

$$(A_2^*\bar{M}_1')_{,1} + (A_1^*\bar{M}_2')_{,2} + A_2^*\left[\bar{r}^*_{,1}\bar{g}_1\right] + A_1^*\left[\bar{r}^*_{,2}\bar{g}_2\right] + A_1^*A_2^*\bar{L} = 0, \tag{12.14}$$

$$\bar{M}_1' = \bar{e}_2^*M_{11} - \bar{e}_1^*M_{12}, \quad \bar{M}_2' = \bar{e}_2^*M_{21} - \bar{e}_1^*M_{22} \tag{12.15}$$

and the static boundary conditions

$$\bar{\Phi} = \bar{g}_n - \frac{d\hat{H}\bar{m}^*}{ds}, \quad \bar{G} = \sum_i\sum_j M_{ij}n_i n_j. \tag{12.16}$$

By replacing T_{ij} by $\hat{T}_{ij}$ and H by $\hat{H}$, the boundary conditions may be obtained in scalar form from (8.19). The scalar form of the equations (12.13) and (12.14) is identical with the corresponding equations (7.3), (7.4), and (7.5), after replacing in the latter T^*_{ij} by $\hat{T}^*_{ij}$, N^*_i by Q_i, and M^*_{ij} by M_{ij}. Therefore, using (12.6) we obtain:

$$(A_2^*\hat{T}_{11})_{,1} + (A_1^*\hat{T}_{21})_{,2} + \hat{T}_{12}A^*_{1,2} - \hat{T}_{22}A^*_{2,1} + A_1A_2(k^*_{11}Q_1 + k^*_{12}Q_2 + X_1^*) = 0 \quad \overrightarrow{\underleftarrow{1,2}}; \tag{12.17}$$

$$(A_2Q_1)_{,1} + (A_1Q_2)_{,2} + \sum_i\sum_j \hat{T}_{ij}k_{ij} + X_3^* = 0; \tag{12.18}$$

$$(A_2M_{11})_{,1} + (A_1M_{12})_{,2} + M_{12}A_{1,2} - M_{22}A_{2,1} + A_1A_2(M_1 - Q_1) = 0 \quad \overrightarrow{\underleftarrow{1,2}} \tag{12.19}$$

★ By permutation of the indices 1, 2, another form of equations **(12.17)** and **(12.19)** may be obtained.

Since the right-hand side of (12.4) is a total differential, we may write:

$$S_{ij}=\frac{\partial W}{\partial \varepsilon_{ij}}, \quad M_{ij}=\frac{\partial W}{\partial \varkappa_{ij}}, \tag{12.20}$$

where W is the deformation energy density of the shell. Therefore, S_{ij} and M_{ij} may be called the energy components of forces and moments, corresponding to the components of the deformation of the surface ε_{ij} and $\varkappa_{ij}$. The symmetrical forces and moments are convenient for theoretical work in which the sixth condition of equilibrium should be exactly satisfied. The symmetrical components of forces and moments for any deformation have been examined in the author's work /III. 3/. It was shown there that for arbitrary deformations, the energy components of forces and moments may be determined by Castigliano's formulas. For small deformations they have the usual form:

$$\varepsilon_{ij}=\frac{\partial W}{\partial S_{ij}}, \quad \varkappa_{ij}=\frac{\partial W}{\partial M_{ij}}. \tag{12.21}$$

It was further shown in /III. 3/ that for any deformations, the energy components of forces and moments may be expressed by three stress functions instead of the usual four, provided there are no external surface forces and moments. We shall prove that for small deformations. If $\overline{X} = \overline{L} = 0$, the homogeneous equations of equilibrium

$$\begin{gathered}(A_2{}^*\overline{g}_1)_{,1}+(A_1{}^*\overline{g}_2)_{,2}=0, \quad (A_2{}^*\overline{M}_1{}')_{,1}+(A_1{}^*\overline{M}_2{}')_{,2}+\\ +A_2{}^*\left[\overline{r}^*_{,1}\overline{g}_1\right]+A_1{}^*\left[\overline{r}^*_{,2}\overline{g}_2\right]=0\end{gathered} \tag{12.22}$$

may be satisfied by substituting

$$A_2{}^*\overline{g}_1=\overline{\varphi}_{,2}, \quad A_1{}^*\overline{g}_2=-\overline{\varphi}_{,1}; \tag{12.23}$$

$$A_2{}^*\overline{M}_1{}'-\left[\overline{r}^*_{,2}\overline{\varphi}\right]=\overline{\psi}_{,2}; \quad A_1{}^*\overline{M}_2{}'+\left[\overline{r}^*_{,1}\overline{\varphi}\right]=-\overline{\psi}_{,1}, \tag{12.24}$$

where $\overline{\varphi}$ and $\overline{\psi}$ are two arbitrary vectors:

$$\overline{\varphi}=\overline{e}_1{}^*\varphi_1+\overline{e}_2{}^*\varphi_2+\overline{m}^*\varphi, \quad \overline{\psi}=\overline{e}_1{}^*\psi_1+\overline{e}_2{}^*\psi_2+\overline{m}^*\psi. \tag{12.25}$$

φ will be the single independent component of the vector because after multiplying scalarly **(12.24)** by $\overline{m}^*$ and taking into account that $\overline{M}_i'\overline{m}^* = 0$ we obtain:

$$A_2{}^*\varphi_1=\overline{m}^*\overline{\psi}_{,2}; \quad A_1{}^*\varphi_2=-\overline{m}^*\overline{\psi}_{,1}. \tag{12.26}$$

Here $\overline{\varphi}_{,i}$ and $\overline{\psi}_{,i}$ may be found by differentiating **(12.25)** by means of (2.18) and (2.22) as follows

$$\begin{gathered}\overline{\varphi}_{,1}=A_1{}^*(\overline{e}_1{}^*\Phi_{11}+\overline{e}_2{}^*\Phi_{12}+\overline{m}^*\Phi_1), \quad \overline{\psi}_{,1}=A_1{}^*(\overline{e}_1{}^*\Psi_{11}+\\ +\overline{e}_2{}^*\Psi_{12}+\overline{m}^*\Psi_1). \qquad \overrightarrow{\underleftarrow{1,2}}\end{gathered} \tag{12.27}$$

where we set

$$\begin{aligned}\Psi_{11}&=\frac{1}{A_1{}^*}\,\frac{\partial\psi_1}{\partial\alpha_1}+\frac{\psi_2}{A_1{}^*A_2{}^*}\cdot\frac{\partial A_1{}^*}{\partial\alpha_2}+k^*_{11}\psi,\\ \Psi_{12}&=\frac{1}{A_1{}^*}\,\frac{\partial\psi_2}{\partial\alpha_1}-\frac{\psi_1}{A_1{}^*A_2{}^*}\cdot\frac{\partial A_1{}^*}{\partial\alpha_2}+k^*_{12}\psi,\end{aligned} \tag{12.28}$$

$$\Psi_1=\frac{1}{A_1{}^*}\,\frac{\partial\psi}{\partial\alpha_1}-k^*_{11}\psi_1-k^*_{12}\psi_2; \qquad \overrightarrow{\underleftarrow{1,2}} \tag{12.29}$$

from this, $\Phi_{11}, \Phi_{12}, \Phi_1, \ldots$ may be determined by replacing ψ_1, ψ_2, ψ by $\varphi_1, \varphi_2, \varphi$.

★ Introducing (11.27) into (12.26) we find for φ_1 and φ_2

$$
\begin{aligned}
\varphi_1 = \Psi_2 &= \frac{1}{A_2^*} \cdot \frac{\partial \psi}{\partial \alpha_2} - k_{12}^* \psi_1 - k_{22}^* \psi_2, \\
\varphi_2 = -\Psi_1 &= -\frac{1}{A_1^*} \cdot \frac{\partial \psi}{\partial \alpha_1} + k_1^* \psi + k_{12}^* \psi_2.
\end{aligned}
\tag{12.30}
$$

Introducing the vectors $\bar{g}_i$ and $\overline{M}'_i$ in (12.23) and (12.24) according to the formulas

$$
\bar{g}_i = \sum_j \left(S_{ij} + \sum_s k_{js}^* M_{is} \right) \bar{e}_j^* + \bar{m}^* Q_i. \tag{12.31}
$$

$$
\overline{M}_1' = \bar{e}_2^* M_{11} - \bar{e}_1^* M_{12}, \quad \overline{M}_2' = \bar{e}_2^* M_{21} - \bar{e}_1^* M_{22} \tag{12.32}
$$

and equating the coefficients of the unit vectors $\bar{e}_i^*$ and $\overline{m}^*$ we find:

$$
S_{11} + \sum_i^2 M_{1i} k_{1i}^* = \Phi_{21}, \quad S_{12} + \sum_i^2 M_{1i} k_{2i}^* = \Phi_{22},
$$

$$
S_{21} + \sum_i^2 M_{2i} k_{1i}^* = -\Phi_{11}, \quad S_{22} + \sum_i^2 M_{2i} k_{2i}^* = -\Phi_{12},
$$

$$
Q_1 = \Phi_2, \quad Q_2 = -\Phi_1, \quad M_{11} = \Psi_{22}, \quad M_{22} = \Psi_{11},
$$

$$
M_{12} = -\Psi_{21} - \varphi, \quad M_{21} = -\Psi_{12} + \varphi.
$$

From the condition $M_{12} = M_{21}$ it results that

$$
2\varphi = \Psi_{12} - \Psi_{21}. \tag{12.33}
$$

Therefore,

$$
M_{11} = \Psi_{22}, \quad M_{22} = \Psi_{11}, \quad M_{12} = M_{21} = -\frac{1}{2}(\Psi_{12} + \Psi_{21}); \tag{12.34}
$$

$$
\left.
\begin{aligned}
S_{11} &= \frac{1}{A_2^*} \cdot \frac{\partial \Psi_2}{\partial \alpha_2} + \frac{\Psi_1}{A_1^* A_2^*} \cdot \frac{\partial A_2^*}{\partial \alpha_1} + k_{12}^* \Psi_{12} - k_{11}^* \Psi_{22}, \\
S_{12} &= -\frac{1}{A_2^*} \cdot \frac{\partial \Psi_1}{\partial \alpha_2} + \frac{\Psi_2}{A_1^* A_2^*} \cdot \frac{\partial A_2^*}{\partial \alpha_1} + k_{22}^* \Psi_{12} - k_{12}^* \Psi_{22}, \\
Q_1 &= \frac{1}{2A_2^*} \cdot \frac{\partial}{\partial \alpha_2} (\Psi_{12} - \Psi_{21}) + k_{22}^* \Psi_1 - k_{12}^* \Psi_2.
\end{aligned}
\right\}
\tag{12.35}
$$

From the latter we obtain S_{22}, S_{21} and Q_2 by permutation of the indexes 1, 2. The condition $S_{12} = S_{21}$ will be satisfied after substituting for ψ_{ik} and ψ_i according to (12.28) and (12.29) and then using the Codazzi conditions (2.27) for orthogonal coordinates.

Thus, the vectors of forces and moments are expressed in terms of one vector $\bar{\psi}(\psi_1, \psi_2, \psi)$. ★

§ 13. The Variational Principle for the State of Stress of the Shell

As shown in § 10, several geometrically possible states may be allowed from the point of view of the energy functional. Let us construct a functional for which several statically possible states may be allowed, i. e., states that do not disturb the condition of equilibrium inside the shell and at the boundary. In the theory of elasticity, the corresponding variational principle is called Castigliano's principle. This principle has been worked out for the theory of shells by N. A. Alumyae /III. 2/ and K. Z. Galimov /III. 3/. Another derivation of Castigliano's variational formula for homogeneous and laminar cynlindrical shells has been given in the paper by Wang Ehi-The /III. 8/.

Castigliano's variational principle for the three-dimensional problem has been expounded in its general form in the author's paper /III. 9/. A mixed variational method for the three-dimensional problem is given in the paper by Reissner /III. 10/; the application of this principle to the theory of finite deformation of shells is given in Galimov's paper /III. 11/. Another variant of the mixed variational method for the theory of shells has been given by N. A. Alumyae in /III. 2/.

In this section we shall describe the variational principle for the state of stress of the shell (Castigliano's principle of variation in generalized form) for small deformations and arbitrary bending.

★ Castigliano's variation formula may be obtained from the energy functional by using Friedrich's transformation (known from the calculus of variations) as has been done in /III. 8/. We shall, however, follow the methods developed in /III. 3/. We shall first derive some new relations which will be necessary for the following development*.

One of these relations will be obtained by assuming in (11.12) that $\bar{f} = \bar{v}$ (where $\bar{v}$ is the displacement vector), i. e., assuming that $f_i = u^*_i$ and $f_3 = w^*$. Here u^*_i and w^* are the projections of the displacement vector on the coordinate axes of the deformed shell. For the quantities Ω_i, Ω_{ij}, and $\hat{x}_{ij}$, given by (11.9) and (11.11), we find:

$$\Omega_i = \omega_i^*, \quad \Omega_{ij} = \vec{e}^{\,*}_{ij}, \quad A_i^* \hat{x}_{ij} = -\vec{e}_j^{\,*} \sum_{s=1}^{2} (\vec{e}_s^{\,*} \omega_s^*)_{,i}, \qquad (13.1)$$

where ω^*_i and e^*_{ij} may be expressed by (13.37)*.

The integral relation (11.12) is also valid for the symmetric components of forces and moments, introduced in § 12, provided that T^*_{ij} and M^*_{ij} are replaced by $\hat{T}_{ij}$ and M_{ij}. Taking into account (13.1) we then obtain from (11.12) the integral relation for the symmetrical forces and moments:

----

* Some relations from the theory of deformations of a surface, which are used here, are given at the end of the section.

★

$$\iint_\sigma (\bar{X}\bar{v} - M_1\omega_1^* - M_2\omega_2^*)\, d\sigma + \int_{\check{C}} (\bar{\Phi}_s \bar{v} - \tilde{G}^*\omega_n^*)\, ds + \tilde{H}^* w^* \Big|_C =$$

$$= \iint_\sigma \sum_{i,j} (\hat{T}_{ij} e_{ij}^* + M_{ij} \tilde{\varkappa}_{ij})\, d\sigma =$$

$$= \iint_\sigma \sum_{i,j} \left[S_{ij}^* e_{ij}^* + M_{ij} \left(\tilde{\varkappa}_{ij} + \sum_{s=1}^{2} k_{js}^* e_{is}^* \right) \right] d\sigma. \tag{13.2}$$

where $\hat{T}_{ij}$ were taken from (12.6).

Here $\varkappa_{ij}$ may be expressed by the last of formulas (13.1), and

$$\omega_n^* = \omega_1^* n_1 + \omega_2^* n_2.$$

In this section we shall use the symmetrical components of forces and moments because in this case Love's formulas of the type (9.5) are exact in the first approximation.

Therefore, it is not necessary to choose any variant of the elasticity relations with additional terms depending on parameters of the curvature.

It should be noted that according to (13.42)

$$\tilde{\varkappa}_{ij} + \sum_{s=1}^{2} k_{js}^* e_{is}^* = \varkappa_{ij} - (\bar{m}\,\bar{m}^* - 1)\, k_{ij}.$$

and furthermore, by using (13.40) we obtain

$$\sum_{i,j} S_{ij} e_{ij}^* = \frac{1}{2}\sum_{i,j} S_{ij} (e_{ij}^* + e_{ji}^*) = \frac{1}{2}\sum_{i,j}\left[e_{ij}^* + e_{ji}^* - \frac{1}{A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} \right] S_{ij} + \frac{1}{2}\sum_{i,j} \frac{S_{ij}}{A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} =$$

$$= \sum_{i,j}\left(S_{ij}\varepsilon_{ij} + \frac{1}{2} \frac{S_{ij}}{A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} \right). \quad (i, j = 1, 2)$$

Taking these into account, we may write the integral equation (13.2) as follows:

$$\iint_\sigma (\bar{X}\bar{v} - M_1\omega_1^* - M_2\omega_2^*)\, d\sigma + \int_{\check{C}} (\bar{\Phi}_s \bar{v} - \tilde{G}^*\omega_n^*)\, ds + \tilde{H}^* w^* \Big|_C =$$

$$= \iint_\sigma \sum_{i,j} \left[S_{ij}\varepsilon_{ij} + M_{ij}\varkappa_{ij} + \frac{S_{ij}}{2A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} + M_{ij}(1 - \bar{m}\,\bar{m}^*)\, k_{ij} \right] d\sigma. \tag{13.3}$$

This relation holds, therefore, if the conditions of equilibrium (12.17), (12.18), and (12.19) are fulfilled. In (13.3) $\bar{\Phi}_s$ is the vector of the external load at the contour, $\tilde{G}^*$ is the external bending moment at the contour, and $\tilde{H}^*$ is the external twisting moment at the contour. For small displacements (13.3) gives Clapeyron's theorem for the linear theory of shells.

In addition to the relation (13.3) we also need Lagrange's equation of variation (12.4) for symmetrical components of the forces and moments. Taking (10.13) into account, we may write this equation as follows:

$$\iint_\sigma (\bar{X}\,\delta\bar{v} + [\bar{M}, \bar{m}^*]\,\delta\bar{m}^*)\, d\sigma + \int_C (\bar{\Phi}_s \delta\bar{v} + \tilde{G}^*\, \bar{n}^* \delta\bar{m}^*)\, ds +$$

$$+ \tilde{H}^*\, \bar{m}^* \delta\bar{v} \Big|_C = \iint_\sigma \sum_{i,j} (S_{ij}\delta\varepsilon_{ij} + M_{ij}\delta\varkappa_{ij})\, d\sigma. \tag{13.4}$$

Here

$$\bar{M} = M_1\bar{e}_2^* - M_2\bar{e}_1^*, \quad [\bar{M}, \bar{m}^*]\,\delta\bar{m}^* = \sum_i M_i \bar{e}_i^* \delta\bar{m}^*,$$

$$\bar{n}^* \delta\bar{m}^* = \sum_i n_i \bar{e}_i^* \delta\bar{m}^* \tag{13.5}$$

★ One more integral relation may be obtained by means of (13.3) and (13.4). Let us vary (13.3) with respect to the forces, moments, and displacements, and subtract (13.4) from the result.

Then, taking (13.5) into account, we obtain:

$$
\begin{aligned}
U = & \iint_{\sigma} \Big\{ \bar{v}\,\delta\bar{X} - \sum_{i} \Big(\omega_i^* \delta M_i + M_i A_i^{*-1} \frac{\partial \bar{v}}{\partial \alpha_i}\, \delta \bar{m}^* \Big) \Big\} d\sigma + \\
& + \int_{C} \Big(\bar{v}\,\delta\bar{\Phi}_s - \omega_n^* \delta \tilde{G}^* - \sum_{i} \tilde{G}^* n_i A_i^{*-1} \frac{\partial \bar{v}}{\partial \alpha_i}\, \delta \bar{m}^* \Big) ds + \\
& + (w^* \delta \tilde{H}^* + \hat{H} \cdot \bar{v}\, \delta \bar{m}^*) \Big|_{C} = \iint_{\sigma} \sum_{i,j} \Big\{ \varepsilon_{ij} \delta S_{ij} + \varkappa_{ij} \delta M_{ij} + \\
& + \delta \Big[\frac{S_{ij}}{2A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} + M_{ij} (1 - \bar{m}\,\bar{m}^*) k_{ij} \Big] \Big\} d\sigma,
\end{aligned}
\tag{13.6}
$$

where U represents the left-hand side of this equation.

Furthermore, we used the equation

$$
\omega_i^* = -\bar{e}_i \bar{m}^*, \quad \bar{e}_i^* = \bar{e}_i + A_i^{*-1} \frac{\partial \bar{V}}{\partial \alpha_i},
\qquad
\tilde{H}^* w^* = \hat{H}^* \bar{m}^* \bar{V}.
\tag{13.6'}
$$

But, since by (12.21) the variation of the work of deformation is

$$
\delta W = \sum_{i,j} \Big(\frac{\partial W}{\partial S_{ij}} \delta S_{ij} + \frac{\partial W}{\partial M_{ij}} \delta M_{ij} \Big) = \sum_{i,j} (\varepsilon_{ij} \delta S_{ij} + \varkappa_{ij} \delta M_{ij}).
$$

the above equation may be written as follows:

$$
U = \delta \iint_{\sigma} \Big\{ W + \sum_{i,j} \Big[\frac{S_{ij}}{2A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} + M_{ij} (1 - \bar{m}\,\bar{m}^*) k_{ij} \Big] \Big\} d\sigma.
\tag{13.7}
$$

where W is the work of deformation expressed in terms of forces and moments.

Let us look for the conditions for which the equation of variation (13.7) holds. For that purpose we have to do the inverse calculation. The variation of the right-hand side of (13.7) will be :

$$
\begin{aligned}
& \sum_{i,j} \Big\{ \varepsilon_{ij} \delta S_{ij} + \varkappa_{ij} \delta M_{ij} + \frac{\delta S_{ij}}{2A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} + (1 - \bar{m}\,\bar{m}^*) k_{ij} \delta M_{ij} \Big\} + \\
& \quad + \sum_{i,j} \Big\{ S_{ij}\, \delta \Big(\frac{1}{2A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} \Big) - k_{ij} M_{ij}\, \delta (\bar{m}\,\bar{m}^*) \Big\}.
\end{aligned}
\tag{*}
$$

The form of the expression in the first braces is analogous to the integrand in the right-hand side of (13.3). Therefore, the expression in braces is equal to $\sum_{i,j} (e_{ij}^* \delta \hat{T}_{ij} + \tilde{\varkappa}_{ij} \delta M_{ij})$, where the forces $\hat{T}_{ij}$ have been introduced according to (12.6). As a result, the right-hand side of (13.7) becomes:

$$
\begin{aligned}
& \iint_{\sigma} \sum_{i,j} (e_{ij}^* \delta \hat{T}_{ij} + \tilde{\varkappa}_{ij} \delta M_{ij})\, d\sigma + \\
& + \iint_{\sigma} \sum_{i,j} \Big\{ S_{ij}\, \delta \Big(\frac{1}{2A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_j} - M_{ij} k_{ij} \bar{m}\, \delta \bar{m}^* \Big\} d\sigma \qquad (i, j = 1, 2).
\end{aligned}
\tag{13.8}
$$

Here the first integral may be transformed by (11.8) provided I is not zero, because the displacements vary. We introduce the differential operators

★

$$
\begin{aligned}
L_1 = \left(A_2^* \delta \hat{T}_{11}\right)_{,1} + \left(A_1^* \hat{T}_{21}\right)_{,2} + A_{1,2}^* \delta \hat{T}_{12} - A_{2,1}^* \delta \hat{T}_{22} + \\
+ A_1^* A_2^* (k_{11}^* \delta Q_1 + k_{12}^* \delta Q_2 + \delta X_1^*) \qquad \overrightarrow{\underleftarrow{1,2}}
\end{aligned}
\tag{13.9}
$$

$$
L_3 = (A_2^* \delta Q_1)_{,1} + (A_1^* \delta Q_2)_{,2} + \sum_{i,j} k_{ij} \delta \hat{T}_{ij} + \delta X_3^*.
$$

After substituting for T^*_{ij} and M^*_{ij} in (11.3) in terms of the variations $\delta\hat{T}_{ij}$ and δM_{ij} and assuming that $f = \bar{v}$, $f_i = u^*_i$ and $f_3 = w^*$, we obtain

$$
\begin{aligned}
I = \iint_\sigma (L_1 u_1^* + L_2 u_2^* + L_3 w^*)\, da_1 da_2 = -\iint_\sigma \sum_{i,j} (e_{ij}^* \delta \hat{T}_{ij} + \tilde{x}_{ij} \delta M_{ij})\, d\sigma + \\
+ \iint_\sigma \left\{ \sum_i u_i^* \delta X_i^* - \omega_i^* \delta M_i) + w^* \delta X_3^* \right\} d\sigma + \\
+ \int_C \left(\sum_i u_i^* \delta \Phi_i^* + w^* \delta X_3^* - \omega_i^* \delta G \right) ds + w^* \delta H^* \Big|_C
\end{aligned}
\tag{13.10}
$$

Here $\delta\Phi_i^*$ and $\delta\Phi_3^*$ are the variations of the internal forces at the contour, and δG and δH are the variations of the internal bending and twisting moments at the contour. We now introduce in (13.8) the value of the double integral $\iint_\sigma \sum_{i,j} (e_{ij}^* \delta \hat{T}_{ij} + \tilde{x}_{ij} \delta M_{ij})\, d\sigma$ and equate the result to the left-hand side of (13.7).

Omitting the details of calculation, (13.7) becomes:

$$
\begin{aligned}
\iint_\sigma \left\{ \sum_{i=1}^{2} \left(\bar{v} X_i^* \delta \bar{e}_i^* - M_i A_i^{*-1} \frac{\partial \bar{v}}{\partial a_i} \delta \bar{m}^* \right) + \bar{v} X_3^* \delta \bar{m}^* \right\} d\sigma + \left\{ w^* (\delta \tilde{H}^* - \delta H^*) + \right. \\
\left. + \tilde{H} \bar{v} \delta \bar{m}^* \right\} \Big|_C + \int_C \left\{ \bar{v} \delta \bar{\Phi}_s - \sum_i u_i^* \delta \Phi_i^* - w^* \delta \Phi_3^* - \omega_n^* (\delta \tilde{G}^* - \delta G^*) - \right. \\
\left. - \sum_i \tilde{G}^* n_i A_i^{*-1} \frac{\partial \bar{v}}{\partial a_i} \delta \bar{m}^* \right\} ds = - \iint_\sigma \left(\sum_{i=1}^{2} L_i u_i^* + L_3 w^* \right) da_1 da_2 + \\
+ \iint_\sigma \sum_{i,j} \left\{ S_{ij} \delta \left(\frac{1}{2A_i^* A_j^*} \cdot \frac{\partial \bar{v}}{\partial a_i} \cdot \frac{\partial \bar{v}}{\partial a_j} \right) - M_{ij} k_{ij} \bar{m} \delta \bar{m}^* \right\} d\sigma;
\end{aligned}
\tag{13.11}
$$

Here we took into account that $\bar{v} \delta \bar{X} = \bar{v} \sum_i (\bar{e}_i^* \delta X_i^* + X_i^* \delta \bar{e}_i^*) + \bar{v} (\bar{m}^* \delta X_3^* +$
Since the forces and moments $\hat{T}_{ij}$ and M_{ij} satisfy the same equations of equilibrium as T^*_{ij} and M^*_{ij}, we have for $\hat{T}_{ij}$ and M_{ij} the integral formula (11.2). Assuming here

$$
f_i = \bar{v} \delta \bar{e}_i^* = \bar{v} \delta \left(A_i^{*-1} \frac{\partial \bar{v}}{\partial a_i} \right); \; f_3 = \bar{v} \delta \bar{m}^*,
\tag{13.12}
$$

we obtain:

$$
\begin{aligned}
U_1 = \iint_\sigma \left\{ \sum_i (X_i^* \bar{v} \delta \bar{e}_i^* - M_i \Omega_i) + X_3^* \bar{v} \delta \bar{m}^* \right\} d\sigma + \\
+ \int_C \left\{ \sum_i (\Phi_i^* \bar{v} \delta \bar{e}_i^* - G^* n_i \Omega_i) + \Phi_3^* \bar{v} \delta \bar{m}^* \right\} ds + H^* \bar{v} \delta \bar{m}^* \Big|_C = \\
= \iint_\sigma \sum_{i,j} \left\{ S_{ij} \Omega_{ij} + M_{ij} \left(\tilde{x}_{ij} + \sum_s k_{js}^* \Omega_{is} \right) \right\} d\sigma.
\end{aligned}
\tag{13.13}
$$

For brevity, we denote here the left-hand side by J. We obtain the quantities U_1, and Ω_i; Ω_{ij}, and $\tilde{x}_{ij}$ from (13.12) according to (11.9) and (11.11):

$$\Omega_i = \frac{\delta\,\overline{m}^*}{A_i^*}\cdot\frac{\partial\bar v}{\partial\alpha_i} + \bar v\,\bar a_i,\quad \Omega_{ij} = \frac{\delta\,\bar e_j^*}{A_i^*}\cdot\frac{\partial\bar v}{\partial\alpha_i} + \bar v\,\bar a_{ij},$$

$$(i, j = 1, 2),$$

$$\tilde\varkappa_{ij} = -\frac{1}{A_i^*}\cdot\frac{\partial\Omega_j}{\partial\alpha_i} - \frac{(-1)^{i-j}}{A_i^* A_{3-j}^*}\cdot\frac{\partial A_i^*}{\partial\alpha_{3-j}}\,\Omega_{3-j} \tag{13.14}$$

Here we introduced the vectors

$$\bar a_i = \frac{1}{A_i^*}\cdot\frac{\partial\delta\,\overline{m}^*}{\partial\alpha_i} - \sum_{s=1}^{2} k_{is}^*\,\delta\,\bar e_s^*,$$

$$\bar a_{ij} = \frac{1}{A_i^*}\cdot\frac{\partial\delta\,\bar e_j^*}{\partial\alpha_i} + \frac{(-1)^{i-j}}{A_i^* A_{3-i}^*}\cdot\frac{\partial A_i^*}{\partial\alpha_{3-i}}\,\delta\,\vec e_{3-j} + k_{ij}^*\,\delta\,\overline{m}^*. \tag{13.15}$$

By using (13.14) we find:

$$\sum_{i,j} M_{ij}\,(\tilde\varkappa_{ij} + \sum_s k_{is}^*\,\Omega_{js}) = \sum_{i,j} M_{ij}\,\Big(\varkappa_{ij}^\circ + \sum_s k_{is}^*\,\bar a_{js}\,\bar v\Big) -$$

$$-\sum_{i,j} M_{ij}\left\{\frac{\partial}{\partial\alpha_i}(\bar e_i^* - \bar e_j) + \frac{(-1)^{i-j}(\bar e_{3-i} - \bar e_{3-i})}{A_{3-i}^*}\,\frac{\partial A_i^*}{\partial\alpha_{3-i}}\right\}\frac{\delta\,\overline{m}^*}{A_i^*}.$$

where we set

$$\varkappa_{ij}^\circ = -\frac{1}{A_i^*}\cdot\frac{\partial(\bar v\,\bar a_j)}{\partial\alpha_i} - \frac{(-1)^{i-j}}{A_i^* A_{3-i}^*}\cdot\frac{\partial A_i^*}{\partial\alpha_{3-i}}\,(\bar v\,\bar a_j). \tag{13.16}$$

Using the differentiation formulas (2.12) and (7.3) for the unit vectors, the previous equation simplifies to

$$\sum_{i,j} M_{ij}\,(\tilde\varkappa_{ij} + \sum_s k_{is}^*\,\Omega_{js}) = \tag{*}$$

$$= \sum_{i,j} M_{ij}\left\{\varkappa_{ij}^\circ + \sum_s k_{is}^*\,\bar a_{js}\,\bar v - k_{ij}^*\,\overline{m}\,\delta\,\overline{m}^*\right\}.$$

By virtue of the symmetry $S_{ij} = S_{ji}$ we obtain:

$$\sum_{i,j} S_{ij}\,\delta\left(\frac{1}{2A_i^* A_j^*}\cdot\frac{\partial\bar v}{\partial\alpha_i}\cdot\frac{\partial\bar v}{\partial\alpha_j}\right) = \sum_{i,j}\frac{S_{ij}}{A_i^*}\cdot\frac{\partial\bar v}{\partial\alpha_i}\,\delta\bar e_j^*.$$

Therefore,

$$\sum_{i,j} S_{ij}\,\Omega_{ij} = \sum_{i,j} S_{ij}\left\{\delta\left(\frac{1}{2A_i^* A_j^*}\cdot\frac{\partial\bar v}{\partial\alpha_i}\cdot\frac{\partial\bar v}{\partial\alpha_j}\right) + \bar v\,\bar a_{ij}\right\}. \tag{**}$$

Taking into account (*) and (**), (13.13) becomes

$$u_1 = \iint_\sigma \sum_{i,j}\left\{S_{ij}\,\delta\left(\frac{1}{2A_i^* A_j^*}\cdot\frac{\partial\bar v}{\partial\alpha_i}\cdot\frac{\partial\bar v}{\partial\alpha_j}\right) - M_{ij}\,k_{ij}\,\overline{m}\,\delta\,\overline{m}^*\right\}d\sigma +$$

$$+\iint_\sigma \sum_{i,j}(\hat T_{ij}\,\bar v\,\bar a_{ij} + M_{ij}\,\varkappa_{ij}^\circ)\,d\sigma. \tag{13.17}$$

After subtracting (13.17) from (13.11) we finally obtain:

$$I_C + \iint_\sigma\left(\sum_{i=1}^{2} L_i\,u_i^* + L_3\,w^*\right)d\alpha_1\,d\alpha_2 + I_1 = 0. \tag{13.18}$$

Where we set

$$I_C = \int_C\left\{\sum u_i^*\,\delta(\Phi_{is}^\circ - \Phi_i^*) +\right. \tag{13.19}$$

$$\left.+ w^*\,\delta(\Phi_{3s}^\circ - \Phi_3^*) - w_n^*\,\delta(\tilde G^* - G^*)\right\}ds + w^*\,\delta(\tilde H^* - H^*)\Big|_C;$$

$$I_1 = \int_C\left\{\sum_{i=1}^{2}(\Phi_{is}^\circ - \Phi_i^*)\,\bar v\,\delta\,\bar e_i^* + (\Phi_{3s}^\circ - \Phi_3^*)\,\bar v\,\delta\,\overline{m}^* - (\tilde G^* - G^*)\,\frac{d\bar v}{dn}\,\delta\,\overline{m}^* +\right. \tag{13.20}$$

$$\left.+ G\,\bar v\,\bar a_n\right\}ds + (\tilde H^* - H^*)\,\bar v\,\delta\,\overline{m}^*\Big|_C + \iint_\sigma\sum_{i,j}(\hat T_{ij}\,\bar v\,\bar a_{ij} + M_{ij}\,\varkappa_{ij}^\circ)\,d\sigma + \iint_\sigma\sum_{i=1}^{2} M_i\,\bar v\,\bar a_i\,d\sigma.$$

★ where

$$\frac{d\bar{v}}{dn}=\sum_{i=1}^{2}\frac{n_i}{A_i}\cdot\frac{\partial\bar{v}}{\partial\alpha_i};\ \bar{a}_n=\sum_{i=1}^{2}\bar{a}_i n_i;$$

Φ^*_{ir}, Φ^*_{3s} are the projections of the external load along $\bar{e}^*_1$ and $\bar{m}^*$; Φ_i and Φ_3 are the projections of the internal forces along the same directions; G* and H* are the internal moments at the contour which may be expressed by (8.12) and (8.15).

Let us first consider the case when the displacements are not varied: $\delta\bar{v}=0$, i.e., when the integral I_1 vanishes. Equation (13.18) will be satisfied if the varied state is statically possible and hence the conditions of equilibrium

$$L_1=0,\quad L_2=0,\quad L_3=0. \tag{13.21}$$

are fulfilled inside the shell and the static boundary conditions*

$$\delta\Phi^*_{is}=\delta\Phi_i^*,\quad \delta\Phi^*_{3s}=\delta\Phi_3^*,\quad \delta\bar{G}^*=\delta G^*,\quad (\delta\bar{H}^*-\delta H^*)\big|_C=0. \tag{13.22}$$

are fulfilled at the contour.

It should be noted that in deriving the equation of equilibrium (13.21) we used the Gauss-Weingaerten formulas for the deformed surface, thus assuming the continuity of the deformations of the actual state. Therefore, in the functional (13.7) only those statically possible states are allowed which do not disturb the conditions of continuity of the deformations ε_{ik} and $\varkappa_{ik}$ of the actual state.

The contour integral I_C in (13.18) vanishes for other boundary conditions, for example, if the edge of the shell is hinged or clamped, or for some mixed boundary conditions.

Let us now consider the case when the displacements are varied and the allowable states are statically impossible, i.e., where the variation of the forces and moments disturbs the conditions of equilibrium inside the shell and at the boundary. If in (7.3) we vary not only the unit vectors $\bar{e}^*_i$ and $\bar{m}^*$ but also the coefficients of the first and second quadratic forms of the undeformed surface, we shall find, according to (13.14)

$$\bar{a}_i=\sum_{s=1}^{2}\bar{e}_s^*\,\delta(A_i^*k^*_{is}),\quad A_i^*\bar{a}_{ij}=(-1)^{i-j+1}\bar{e}^*_{3-j}\delta\left(\frac{1}{A^*_{3-j}}\frac{\partial A_i^*}{\partial\alpha_{3-i}}\right)-\bar{m}^*\delta(A_i^*k^*_{ij}). \tag{13.23}$$

With these we obtain the second surface integral of (13.17):

$$\iint_\sigma\sum_{i,j}(\hat{T}_{ij}\,\bar{v}\,\bar{a}_{ij}+M_{ij}\varkappa^*_{ij})\,d\sigma=-\iint_\sigma\sum_{i,j}\Big\{T_{ij}\Big[\frac{(-1)^{i-j}}{A_i^*}u^*_{3-j}\delta\Big(\frac{1}{A^*_{3-i}}\cdot\frac{\partial A_i^*}{\partial\alpha_{3-j}}\Big)-$$

$$\frac{w^*}{A_i}\delta(A_i^*k^*_{ij})\Big]+M_{ij}\Big[\frac{1}{A_i^*}\cdot\frac{\partial}{\partial\alpha_i}(\bar{v}\,\bar{a}^*_{ij})+\frac{(-1)^{i-j}}{A_i^*A^*_{3-i}}\cdot\frac{\partial A_i^*}{\partial\alpha_{3-i}}\,\bar{v}\,\bar{a}_j\Big]\Big\}\,d\sigma. \tag{*}$$

Let us indicate the subsequent course of calculations. The terms $M_{ij}A_i^{*-1}\partial(\bar{v}\,\bar{a}_{ij})/\partial\alpha_i$ have to be integrated by parts by (10.19), and then the equation of moments (12.18) should be used. After introducing the result in (13.17) and subtracting from (13.11) we obtain

$$I_C+\iint_\sigma\Big(\sum_{i=1}^{2}L_i^*u_i^*+L_3^*w^*\Big)\,d\alpha_1\,d\alpha_2+I'_C=0. \tag{13.24}$$

----

* The latter condition relates to the localized moment.

★ where I_C is given by (13.19) and I'_C is the contour integral

$$I_C'=\int_C\left\{\sum_{i=1}^{2}(\Phi_{is}^{*}-\Phi_i^{*})\bar{v}\,\delta\bar{e}_i^{*}+(\Phi_{3s}^{*}-\dot{\Phi}_3^{*})\bar{v}\,\delta\bar{m}^{*}-(\tilde{G}^{*}-G^{*})\frac{d\bar{v}}{dn}\delta\bar{m}^{*}\right\}ds+(\tilde{H}^{*}-H^{*})\bar{v}\,\delta\bar{m}^{*} \qquad (13.25)$$

In addition, we introduced the operators

$$L_1^{*}=\delta(A_2^{*}\hat{T}_{11})_{,1}+\delta(A_1^{*}\hat{T}_{21})_{,2}+\delta(A_{1,2}^{*}\hat{T}_{12})-\delta(A_{2,1}^{*}\hat{T}_{22})+\delta[A_1^{*}A_2^{*}(k_{11}^{*}Q_1+k_{12}^{*}Q_2+X_1^{*})] \quad \overrightarrow{\underleftarrow{1,2}} \qquad (13.26)$$

$$L_3^{*}=\delta(A_2^{*}Q_1)_{,1}+\delta(A_1^{*}Q_2)_{,2}+\delta\sum_{i,j}(k_{ij}\hat{T}_{ij})+\delta X_3^{*}.$$

Thus, if statically impossible displacements and states of stress are allowed in the functional (13.7), the expression (13.7) will be valid, provided that

$$L_1^{*}=0;\quad L_2^{*}=0,\quad L_3^{*}=0 \text{ (inside the region)} \qquad (13.27)$$

and that the static boundary conditions or the geometrical boundary conditions (if the edges of the shell are hinged or clamped) are satisfied.

The variational equations (13.18) or (13.27) are equations of the Bubnov-Galerkin method. The first one is formally identical with the equations of the linear Bubnov-Galerkin theory of shells.

We shall derive one more equation which is similar in content to the equation of the Bubnov-Galerkin method for the elasticity relations and which we shall apply to the theory of flat shells. We subtract (13. 11) from (13. 6) and obtain

$$\iint_\sigma\left\{\sum_{i=1}^{2}(u_i^{*}\delta X_i^{*}-\omega_i^{*}\delta M_i^{*})\,d\sigma+\int_C\left(\sum_{i=1}^{2}u_i^{*}\delta\Phi_i^{*}-\omega_n^{*}\delta G^{*}\right)ds+w^{*}\delta H^{*}\right|_C= \qquad (13.28)$$

$$=\iint_\sigma\sum_{i,j}\left\{\varepsilon_{ij}\delta S_{ij}+\varkappa_{ij}\delta M_{ij}+\frac{\delta S_{ij}}{2A_i^{*}A_j^{*}}\cdot\frac{\partial\bar{v}}{\partial\alpha_i}\cdot\frac{\partial\bar{v}}{\partial\alpha_j}+(1-\bar{m}\bar{m}^{*})k_{ij}\delta M_{ij}\right\}d\sigma;$$

Here we assume that the equations of equilibrium (13.21) and the static boundary conditions are satisfied. Therefore, S_{ij} and M_{ij} may be replaced in (13.2) by δS_{ij} and δM_{ij}. Then the left-hand side in (13.28) is equal to the double integral

$$\iint_\sigma\sum_{i,j}\left[e_{ij}^{*}\delta S_{ij}+\left(\tilde{\varkappa}_{ij}+\sum_{s=1}^{2}k_{is}e_{is}^{*}\right)\delta M_{ij}\right]d\sigma,$$

and (13.28) will therefore be equivalent to

$$\iint_\sigma\sum_{i,j}\left\{\left(\varepsilon_{ij}-e_{ij}^{*}+\frac{1}{2A_i^{*}A_j^{*}}\cdot\frac{\partial\bar{v}}{\partial\alpha_i}\cdot\frac{\partial\bar{v}}{\partial\alpha_j}\right)\delta S_{ij}+\left[\hat{\varkappa}_{ij}-\tilde{\varkappa}_{ij}-\sum_{s=1}^{2}k_{sj}^{*}e_{is}^{*}+(1-\bar{m}\bar{m}^{*})k_{ij}\right]\delta M_{ij}\right\}d\sigma=0.$$

From this, after simple transformations by formulas listed at the end of this section, we obtain

$$\iint_\sigma\sum_{i,j}\left\{\left(\hat{\varepsilon}_{ij}-\varepsilon_{ij}\right)\delta S_{ij}+\left(\hat{\varkappa}_{ij}-\varkappa_{ij}\right)\delta M_{ij}\right\}d\sigma=0. \qquad (13.29)$$

It is evident from the deduction just outlined that $\hat{\varepsilon}_{ij}$ and $\hat{\varkappa}_{ij}$ are here the components of deformation of the surface expressed in terms of forces and moments, and ε_{ij} and $\varkappa_{ij}$ are the same quantities expressed in terms of displacement.

Since the variations δS_{ij} and δM_{ij} are arbitrary, the elasticity relations follow from (13.29).

★ The variational equation (13.29) also holds if the allowed states are statically impossible /III. 9/.

The functional (13.7) may be obtained from the energy functional by means of Friedrichs' transformation if the coefficients of the equations of equilibrium are considered to be expressed in terms of the displacements. Therefore, for geometrically non-linear problems, one may also take as allowed states of displacement in addition to the states of stress of the shell.

In concluding this section, let us consider the functional (13.7) for a particular problem, namely, for the case $\delta\overline{X} = 0$, $\overline{M} = 0$, and the contour integral

$$\int_C \left(\bar{v}\,\delta\overline{\Phi}_s - \omega_n^*\,\delta\,\overline{G}^* - \tilde{G}^*\,\frac{d\bar{v}}{dn}\,\delta\overline{m}^* \right) ds = 0 \tag{13.30}$$

vanishes. Then the following theorem holds: the actual state of equilibrium of the shell differs from the statically possible states by the fact that for the former the functional

$$R = \iint_{\sigma} \left\{ W + \sum_{i,j} \left[\frac{S_{ij}}{2A_i^* A_j^*} \cdot \frac{\partial\bar{v}}{\partial\alpha_i} \cdot \frac{\partial\bar{v}}{\partial\alpha_j} + M_{ij}\left(1 - \overline{m}\,\overline{m}^*\right) k_{ij} \right] \right\} d\sigma \tag{13.31}$$

has a stationary value $\delta R = 0$.

The condition at the contour (13.30) will, for instance, be satisfied if:

1. the contour is free

$$\overline{\Phi}_s = \ddot{G}^* = 0; \tag{13.32}$$

2. the contour is clamped

$$\bar{v} = 0, \quad \omega_n^* = 0 \quad (\delta\overline{m}^* = 0); \tag{13.33}$$

3. the contour is immovably hinged

$$\bar{v} = 0, \quad \tilde{G}^* = 0; \tag{13.34}$$

4. the contour is freely supported

$$w = 0, \quad \tilde{G}^* = 0, \quad \delta\Phi_{1s}^* = \delta\Phi_{2s}^* = 0; \tag{13.35}$$

5. the boundary conditions are mixed, consisting of several of the preceding ones.

If these conditions are fulfilled, the functional (13.31) enables one to solve the problem by the Ritz method. The theorem holds also if the variations of the forces and moments are statically impossible /III. 3/.

Let us derive those relations of the theory of surface deformation to which we referred at the beginning of this section. In order to simplify the derivation we shall neglect the shear in comparison with unity, introducing the same error as in the equations of equilibrium. The derivation of these relations for finite deformations in general coordinates has been given in /0.7/ and /III. 3/.

We resolve the displacement vector $\bar{v}$ along the unit vectors $\bar{e}_1^*$, $\bar{e}_2^*$, and $\overline{m}^*$ of the deformed shell: $\bar{v} = \bar{e}_1^* u_1^* + \bar{e}_2^* u_2^* + \overline{m}^* w^*$. Then the derivatives of this vector will be:

★

$$\frac{\partial \bar{v}}{\partial \alpha_i} = A_i^* \left(\sum_{k=1}^{2} e_{ik}^* \bar{e}_k^* + \bar{m}^* \omega_i^* \right) \qquad (i = 1,\, 2); \tag{13.36}$$

Here we have used the formulas (2.18) and (2.22) for the deformed surface. The quantities $\bar{e}^*_{ik}$ and ω^*_i may be expressed by formulas of the type (3.5) for this surface:

$$\begin{aligned} e_{11}^* &= \frac{1}{A_1^*} \cdot \frac{\partial u_1^*}{\partial \alpha_1} + \frac{u_2^*}{A_1^* A_2^*} \cdot \frac{\partial A_1^*}{\partial \alpha_2} + w^* k_{11}^*, \\ \omega_1^* &= \frac{1}{A_1^*} \cdot \frac{\partial \omega^*}{\partial \alpha_1} - u_1^* k_{11}^* - u_2^* k_{12}^*, \qquad \underset{\longleftarrow}{\overset{\longrightarrow}{1,\,2}} \\ e_{12}^* &= \frac{1}{A_1^*} \cdot \frac{\partial u_2^*}{\partial \alpha_1} - \frac{u_1^*}{A_1^* A_2^*} \cdot \frac{\partial A_1^*}{\partial \alpha_2} + k_{12}^* w^* \end{aligned} \tag{13.37}$$

By (13.36) we find for the unit vectors $\bar{e}_i$ and $\bar{m}$ the expressions

$$\bar{e}_i = \sum_{i=1}^{2} \left\{ \left(\delta_{ik} - e_{ik}^* \right) \bar{e}_k^* - \bar{m}^* \omega_i^* \right\}, \quad \bar{m} = E_3^* \bar{m}^* + \bar{e}_1^* E_1^* + \bar{e}_2^* E_2^*. \tag{13.38}$$

where $\bar{E}^*_3$ and $\bar{E}^*_i$ may be obtained from formulas like (3.20) by substituting $-e^*_{ik}$ and $-\omega^*_i$ instead of e_{ik} and ω_i ($\delta_{ik} = 1$ for $i = k$; $\delta_{ik} = 0$ for $i \neq k$). The expressions (3.13) may be written as follows:

$$2\varepsilon_{ik} = \bar{e}_i \frac{\partial \bar{v}}{\partial \alpha_k} + \bar{e}_k \frac{\partial \bar{v}}{\partial \alpha_i} + \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_k}. \tag{13.39}$$

Introducing $\frac{\partial \bar{v}}{\partial \alpha_i}$ and $\bar{e}_i$ from (13.36) and (13.38) we find

$$2\varepsilon_{ik} = e_{ik}^* + e_{ki}^* - \frac{1}{A_i^* A_k^*} \frac{\partial \bar{v}}{\partial \alpha_i} \cdot \frac{\partial \bar{v}}{\partial \alpha_k} = e_{ik}^* + e_{ki}^* - \sum_{j=1}^{2} e_{ij}^* e_{kj}^* - \omega_i^* \omega_k^*. \tag{13.40}$$

By adding two different expressions for ε_{ik} we obtain

$$2\varepsilon_{ik} = e_{ik} + e_{ki}^* = e_{ik}^* + e_{ki}. \tag{13.41}$$

Formally, this expression recalls the formulas of the linear theory. In order to find other expressions for the components of the bending deformation, we have to calculate the derivatives $\frac{\partial \omega_i^*}{\partial \alpha_j}$.

Since $\omega^*_i = \bar{m}^* \bar{e}_i$, we obtain

$$\frac{\partial \omega_i^*}{\partial \alpha_j} = -\bar{m}^* \frac{\partial \bar{e}_i}{\partial \alpha_j} - \bar{e}_i \frac{\partial \bar{m}^*}{\partial \alpha_j} \qquad (i,\, j = 1,\, 2).$$

Substituting for $\bar{e}_{ij}$ from (2.18) and for $\bar{m}^*_j$ from formulas like (2.22) for the deformed surface, we obtain:

$$\begin{aligned} \frac{A}{A_1} \frac{\partial \omega_1^*}{\partial \alpha_1} &= E_3 k_{11} + k_{11}^* \left(e_{11}^* - 1 \right) + k_{12}^* e_{12}^* - \frac{\omega_2^*}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} \qquad \underset{\longleftarrow}{\overset{\longrightarrow}{1,\,2}} \\ \frac{1}{A_1} \cdot \frac{\partial \omega_2^*}{\partial \alpha_1} &= E_3 k_{12} + k_{12}^* \left(e_{22}^* - 1 \right) + k_{11}^* e_{21}^* + \frac{\omega_1^*}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2}; \end{aligned} \tag{13.42}$$

where the quantities $\bar{e}_i$ were taken from (13.38).

Using (3.31) we obtain $E_3 k_{ij} - k_{ij}^* = k_{ij} (E_3 - 1) - \varkappa_{ij}$ Introducing this expression in the previous equations for the components of the bending deformation we obtain

$$\begin{aligned} \varkappa_{11} &= k_{11} (E_3 - 1) - \frac{1}{A_1} \cdot \frac{\partial \omega_1^*}{\partial \alpha_1} - \frac{\omega_2^*}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} + k_{11}^* e_{11}^* + k_{12}^* e_{12}^* \qquad \underset{\longleftarrow}{\overset{\longrightarrow}{1,\,2}} \\ \varkappa_{21} &= k_{12} (E_3 - 1) - \frac{1}{A_1} \cdot \frac{\partial \omega_2^*}{\partial \alpha_1} + \frac{\omega_1^*}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} + k_{11}^* e_{21}^* + k_{12}^* e_{22}^* \end{aligned} \tag{13.43}$$

★ The quantity E_3 may be expressed in terms of the rotation angles ω_1 and ω_2; squaring (3.16) we obtain

$$E_3 = \sqrt{1 - \omega_1^2 - \omega_2^2},$$

where we neglected elongations and shear in comparison with unity. ★

Chapter IV

CLASSIFICATION OF PROBLEMS OF THE THEORY OF SHELLS AND SIMPLIFICATION OF ITS FUNDAMENTAL RELATIONS

§ 14. Small Bending. Linear Theory of Shells

The general relations of the preceding sections have been obtained by assuming that the deformations and the relative thickness of the shell t/R may be neglected in comparison with unity, i. e., assuming that

$$1/2t\varkappa \lesssim \varepsilon_p \ll 1, \quad \varepsilon \lesssim \varepsilon_p, \quad t/R \ll 1, \tag{14.1}$$

where ε_p is the limit of proportionality of the material of the shell. The symbol ~ shows that the compared quantities are of the same order of magnitude. R is the smallest radius of curvature of the middle surface of the shell, $\varkappa$ is the largest value of the quantities $\varkappa_{ij}$; and ε is the largest value of the ε_{ij}. For particular cases, these relations may be considerably simplified.

For α_i we shall take such dimensionless coordinates as to obtain coefficients for the first quadratic form of the surface σ of the order of magnitude of L

$$A_i \sim L \qquad i = 1 \text{ or } 2 \tag{14.2}$$

where L is the characteristic dimension, for instance, the width of the part of the shell under consideration or, if we consider the entire shell, the minimal radius of curvature.

We shall also assume that the geometrical parameters of the shell vary smoothly i. e., that

$$A_{i,j} \lesssim A_i; \quad k_{ij,m} \lesssim k_{ij}, \tag{14.3}$$

where, as in the preceding, the comma before the index j or m denotes partial differentiation with respect to α_j or α_m, respectively.

We say that the bending of the shell is "small" if the rotations of its linear elements in bending are everywhere negligibly small in comparison with unity:

$$\omega_j << 1. \tag{14.4}$$

In this case, the maxima of the tangential displacements and the bending may be of the same order of magnitude as the thickness of the shell, if they are slowly varying functions of α_i, whose derivatives with respect to σ_i are smaller than or of the same order of magnitude as the functions themselves. The maximum deflection of the points of a cylindrical tube compressed by an external and internal pressure uniformly distributed over its surface, may, for instance, be of the same order of magnitude as the thickness of the shell. In the contrary case, we obtain according to (3.5) and (3.13):

$$e_{22} \simeq w k_{22}; \ \varepsilon_{22} \simeq e_{22} \simeq w/R >> t/R > \varepsilon_p, \quad \text{if} \quad w > t.$$

From (3.5), in the case under consideration the quantities e_{ik} will also be small (of the same order of magnitude as ε_p or smaller).

If the moduli of the projections of the displacement increases $\varepsilon_p^{-1/2}$ times on differentiation with respect to α_i, we get $w_{,i} \sim w\varepsilon_p^{-1/2}$ and according to (3.5) and (14.2)

$$\omega_i \sim w_{,i}/L \sim \frac{w}{L}\varepsilon_p^{-1/2}.$$

In that case, we may assume that, in our theory, based on neglecting (in comparison with unity) quantities of the same order as ε_p or t/R, the condition (14.4) will be satisfied only for deflections which are small in comparison with the thickness of the shell and for which $w \leqslant t\varepsilon_p^{1/2}$. The tangential displacements u_i are such quantities and the e_{ik} are negligible in comparison with unity.

Therefore, if the condition (14.4) is satisfied with the degree of approximation assumed, one can use the formulas of the linear theory of shells:

$$\varepsilon_{11} = e_{11}, \quad 2\varepsilon_{12} = e_{12} + e_{21}, \qquad \overrightarrow{\underleftarrow{1,2}} \tag{14.5}$$

$$A_1A_2\varkappa_{11} = -A_2\omega_{1,1} - A_{1,2}\omega_2, \quad A_1A_2\varkappa_{12} = -A_2\omega_{2,1} + A_{1,2}\omega_1,$$

where e_{ij} and ω_i may be determined by (3.5). One also has to put $k^*_{ij} = k_{ij} + \varkappa_{ij} \approx k_{ij}$ in the equations of equilibrium (7.4), and in the boundary conditions the components X^*_i of the external stress in the directions $\bar{e}^*_i$ and $\bar{m}^*$ of the deformed shell have to be equated to the components X_i of that force in the directions $\bar{e}_i$ and $\bar{m}$.

Let us consider the possible particular states of stress of an entire shell or of a considerable part of it (when $L \sim R$).

A. If the bending elongations are negligibly small in comparison with the elongations of the middle surface, we shall call it a "membrane stress state". In that case, "the membrane forces" may be determined from the approximate equations

$$(A_2T^*_{11})_{,1} + (A_1T^*_{21})_{,2} + T^*_{12}A_{1,2} - T^*_{22}A_{2,1} + A_1A_2X_1 = 0, \quad \overrightarrow{\underleftarrow{1,2}} \tag{14.6}$$

$$T^*_{11}k_{11} + 2T^*_{21}k_{21} + T^*_{22}k_{22} - X_3 = 0. \tag{14.7}$$

Their components in the directions $\bar{e}^*_1$, $\bar{e}^*_2$, and $\bar{m}^*$ after deformation may be equated to the corresponding components along the directions $\bar{e}_1$, $\bar{e}_2$, and $\bar{m}$ before deformation.

B. The bending and the membrane elongations may be of the same order of magnitude. We shall call such a state a "composite state of stress". In particular, deformations of this kind may occur in the neighborhood of the fixed edges of the shell ("edge effect"). In this case, there may be considerable local variations of curvature even with small deflections, because the deflection function w increases considerably upon differentiation with respect to α_i.

★ From (7.5) and (9.5), we have

$$\overline{T}^*_{ij} \sim Et\varepsilon, \quad M^*_{ij} \sim Et^3\varkappa, \quad A_iN_i^* \sim M^*_{ij,k} \sim Et^3\varkappa_{,k}$$
$$i, j, k = 1, 2.$$

Evidently, the maximum elongation by bending occurs in the extreme fibers $z = \pm t/2$; the order of magnitude of this elongation is $t\varkappa$. So, in this case

★

$$t\varkappa \sim \varepsilon, \quad tk^*_{ij} \sim \varepsilon, \quad (A_i T^*_{jk})_{,m} \sim LEt\varepsilon_{,m} \cdot$$
$$A_1 A_2 N_i^* K^*_{jj} \sim LEt^2 k^*_{jj} \varkappa_{,k} \sim LEt\varepsilon \cdot \varepsilon_{,k} .$$

Therefore, in the first two equations (7.4) we can neglect the terms containing N_i^* and use the approximate equations (14.6)*. In the third equation (7.4) the shearing forces have greater importance because, for instance,

$$(A_2 N_1^*)_{,1} \sim M_{11,11} \sim Et^3\varkappa_{,11} \sim Et^2\varepsilon_{,11}$$

may be of the same order of magnitude as the terms

$$A_1 A_2 T^*_{ij} k_{ij} \sim L^2 Et\varepsilon/R .$$

In fact, the ratio between these quantities is $\sim t\varepsilon_{,11}/L\varepsilon$. According to our assumption, $\varepsilon_{,11} >> \varepsilon$, but $t/L << 1$. ★

Therefore, in the case considered we have to use, instead of (14.7), the equation

$$(A_2 N_1^*)_{,1} + (A_1 N_2^*)_{,2} - A_1 A_2 (T^*_{11} k_{11} + 2T^*_{21} k_{21} + T^*_{22} k_{22} - X_3) = 0 \qquad (14.8)$$

where N_i^* is given by (7.5).

C. If the bending of the shell occurs almost without elongations,

$$\varepsilon << t\varkappa , \qquad (14.9)$$

and the simplifications of the cases A and B will no longer be valid. This case corresponds first of all to pure bending of the middle surface of the shell. It will occur when the surface σ is not closed and is not fixed to some contour which does not lie partly on the asymptotic line of the surface**.

As we know from the theory of surfaces, a line along which the curvature of the normal section of the surface is zero is called an asymptotic line of the surface. In particular, this property belongs to the straight-line generators of developable surfaces. Cylindrical and conical surfaces are the simplest surfaces of this kind. A thin shell which has a middle surface of this kind, with a rigid contour only along an asymptotic line, may be easily bent and is not very stable. Therefore, with thin-walled structures of the shell type at least one rigid fixture is usually introduced which does not coincide with the asymptotic line of the middle surface of the shell. In view of that, in the following we shall not consider details of such cases of equilibrium of a shell and we recommend to the reader the monographs /0.5/, /0.8/, and /0.15/***, which deal with the linear theory of shells. But let us note here that, in our case by retaining the terms depending on N_i^* in (14.6), it is not yet possible to ensure the required degree of accuracy of the theory based on Kirchhoff's hypothesis, because when using this hypothesis an error ****

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* A particular case, when the above estimates become less exact owing to the mutual cancellation of the principal terms, will be considered in § 18.

** See S. P. Finikov, Teoriya poverkhnostei (Theory of Surfaces), Chap. IV.

*** The most comprehensive study of this problem may be found in the monograph /0.8/.

**** See /0.14/, /0.15/, and /0.19/.

$$M^*_{ij}k_{ij} \sim N_i^*.$$

is, in fact, introduced in the expression for T^*_{ij}, ... Besides, for small bending the bending stress in thin shells is very small; therefore, the case C for which the membrane stresses are small is not of interest for the determination of the state of stress. When the displacements are small, the case C is also not of interest even for studying the form of the deformed surface. Therefore, in all cases, unless otherwise specified, we shall determine the state of stress of the shell for small bending before the loss of stability by superposing the solutions for the cases A and B. This is admissible because of the linearity of the equations of equilibrium for small displacements.

Long thin shells occupy a special place in the theory of shells because there is a small bending even when the end sections are fixed. The problem of stability of such shells will be considered briefly in the following. The state of stress of these shells far from the edges and before the loss of stability may be determined to a sufficient degree of accuracy according to the membrane theory, provided that the following condition* is fulfilled:

$$t^2n^4(n^2-1)^2l^4 << 3.4 R^6, \tag{14.10}$$

where l is the length of the shell and n is the frequency of the sinusoidal load applied to the end contours of the shell. Evidently this condition will always be satisfied in the case of symmetrical loads (n = 0) and of bending of the shell as a beam (n = 1). If the load applied to each edge is in equilibrium with itself (n ≥ 2), the fulfilling of condition (14.10) will depend on the quantity l/R. We shall call this the "thinness" of the shell.

In order to illustrate the theory of small bending, let us consider an example of determining the influence of the edge effect on the state of stress of the shell. Let a circular, cylindrical shell be in equilibrium under the action of a uniformly distributed internal pressure of density p, and an axial compressive force T_0, uniformly distributed over the circular end sections.

We shall assume that at these sections, which are rigidly fixed against bending in their plane by rings, the conditions for hinging are fulfilled. We shall take as coordinate lines the lines of curvature of the middle surface so that

$$ds^2 = R^2(d\alpha_1^2 + d\alpha_2^2),$$

where $R\alpha_1$ is the coordinate measured along the generating line from the middle of the shell, and α_2 the polar angle. Here,

$$A_1 = A_2 = R, \quad k_{11} = k_{12} = 0, \quad k_{22} = 1/R.$$

At the ends of the shell (for $\alpha_1 = \pm l/2R$) the following conditions must be satisfied:

$$\begin{gathered} T^*_{11} = -T_0, \quad w = 0, \quad M^*_{11} = D(\varkappa_{11} + \nu\varkappa_{22}) = 0, \\ u_2 = 0, \quad T^*_{12} = 0. \end{gathered} \tag{14.11}$$

The latter two are automatically satisfied owing to the axial symmetry of the load and the boundary conditions; besides, all quantities which characterize the deformation do not depend on α_2. Hence, according to (3.5) and (14.5):

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* See formula (7.5) in Chapter II of /0.8/.

$$2\varepsilon_{12} = \frac{1}{R}(u_{2,1} + u_{1,2}), = 0, \ T^*_{12} = K(1-\nu)\,\varepsilon_{12} = 0, \ \varepsilon_{22} = w/R,$$
$$\omega_1 = w_{,1}/R, \ \omega_2 = 0, \ \varkappa_{11} = -w_{,11}/R^2, \ \varkappa_{22} = \varkappa_{12} = 0.$$

Besides,

$$X_1 = X_2 = 0, \ X_3 = p.$$

From (14.6) we obtain

$$\begin{gathered} T^*_{11} = \text{const} = -T_0 = Et(\varepsilon_{11} + \nu\varepsilon_{22}) : (1-\nu^2), \\ \varepsilon_{11} = -\nu\varepsilon_{22} - T_0(1-\nu^2)/Et, \ T^*_{22} = -\nu T_0 + Et\varepsilon_{22}. \end{gathered} \tag{14.12}$$

Let

$$w = w^{\text{b}} + w^{\varkappa},$$

where w^{b} is the deflection due to the membrane deformation and w^{K} the deflection due to the edge effect. Then

$$T^*_{22} = T^{\text{b}}_{22} + T^{\varkappa}_{22},$$

and according to (14.7)

$$T^{\text{b}}_{22} = pR.$$

On the other hand, according to (14.12)

$$T^{\text{b}}_{22} = -\nu T_0 + E\varepsilon^{\text{b}}_{22} = -\nu T_0 + Etw^{\text{b}}/R.$$

Hence,

$$w^{\text{b}} = (pR + \nu T_0)R/Et. \tag{14.13}$$

[illegible] rt [illegible] we have

$$T^{\varkappa}_{22} = Et\varepsilon^{\varkappa}_{22} = Etw^{\varkappa}/R.$$

[illegible] to (7.5)

$$A_2 N^*_1 = M^*_{11,1} = D\varkappa_{11,1}.$$

Introdu[illegible] expressions in (14.8) we obtain the equation for w^{K}:

$$w^{\varkappa}_{,1111} + 4\lambda^4 w^{\varkappa} = 0, \ 4\lambda^4 = 12(1-\nu^2)R^2/t^2. \tag{14.14}$$

In view of the symmetry of the boundary conditions

$$w^{\varkappa} = c_1 \operatorname{ch}\lambda\alpha_1 \cos\lambda\alpha_1 + c_2 \operatorname{sh}\lambda\alpha_1 \sin\lambda\alpha_1.$$

Therefore, from

$$w^{\text{b}} + w^{\varkappa} = 0, \ w^{\varkappa}_{,11} = 0 \text{ when } \alpha_1 = l/2R,$$

we obtain

$$c_1 = -\frac{w^{\text{b}} \operatorname{ch}\mu \cos\mu}{\cos^2\mu + \operatorname{sh}^2\mu}, \ c_2 = -\frac{w^{\text{b}} \operatorname{sh}\mu \sin\mu}{\cos^2\mu + \operatorname{sh}^2\mu}, \ \mu = \lambda l/2R.$$

If the shell is not short, then $l \gtrsim 2R$, $\mu \gtrsim \sqrt{R/t}$, and admitting a negligible error we may assume $\operatorname{ch}\mu \approx \operatorname{sh}\mu \gg 1$. Besides, the moduli of the functions $\operatorname{ch}\lambda\alpha_1$ and $\operatorname{sh}\lambda\alpha_1$ decrease rapidly with increase of the distance from the end. Therefore, w^K is significant only near the ends, where $|\operatorname{ch}\lambda\alpha_1| \approx |\operatorname{sh}\lambda\alpha_1|$. Consequently, in the boundary zone $\alpha_1 = l/2R$ we have

$$w^\kappa \approx -w^b \operatorname{ch}\lambda\alpha_1 \cos(\mu - \lambda\alpha_1)/\operatorname{ch}\mu.$$

Whence, applying the usual rule for determination of the extremum of a function, we find:

$$\begin{aligned} &\max w^\kappa \approx w^b \operatorname{ch}\left(\mu - \tfrac{3}{4}\pi\right)\Big/\sqrt{2}\operatorname{ch}\mu \text{ where } \lambda\alpha_1 = \mu - \tfrac{3}{4}\pi; \\ &\max T_2^\kappa \approx pR + (pR + \nu T_0)\operatorname{ch}\left(\mu - \tfrac{3}{4}\pi\right)\Big/\sqrt{2}\operatorname{ch}\mu. \end{aligned} \tag{14.15}$$

After reaching this maximum, the edge effect rapidly decreases and at $\lambda\alpha_1 = \mu - \pi/2$ it becomes small. Therefore, the width of its effective zone at each edge is

$$R\pi/2\lambda \approx 1.7\sqrt{Rt}.$$

§ 15. Medium Bending of a Shell. Theory of Shallow Shells

We shall denote by medium bending those cases when the maximum deflection is of the same order of magnitude as the thickness or larger, but is small in comparison with other linear dimensions of the shell. For the sake of definiteness, we shall assume that in this case the squares of the rotations of an element due to bending may be neglected in comparison with unity

$$\omega_i^2 \ll 1. \qquad (15.1)$$

Evidently in this case the rotations of the elements e_{12} and e_{21} in the plane tangent to σ, and also e_{ii}, are small. Therefore, it results from (3.13) and (15.1) that

$$e_{11} = \varepsilon_{11} - \frac{1}{2}(e_{11}^2 + e_{12}^2 + \omega_1^2).$$

Neglecting the squares of elongations and the fourth powers of the quantities e_{ik} and ω_i, we obtain

$$\begin{gathered} \varepsilon_{11} \approx e_{11} + \frac{1}{2}e_{12}^2 + \frac{1}{2}\omega_1^2, \\ 2\varepsilon_{12} = (1 + e_{11})e_{21} + (1 + e_{22})e_{12} + \omega_1\omega_2. \end{gathered} \qquad (15.2)$$

These expressions may be further simplified for the case of shells which satisfy the condition

$$\theta_0^2 \approx (L/R)^2 \ll 1, \qquad (15.3)$$

where θ_0 is the rise angle of the shell before deformation. They may also be simplified for the case of non-shallow shells which divide themselves during deformation into many shallow portions, provided that the condition of small bending elongations is satisfied as hitherto:

$$\frac{t\varkappa}{2} \leqslant \varepsilon_p. \qquad (15.4)$$

In order to illustrate the following simplifications of the theory, let us consider at first the bending of a thin straight bar AB = L with small elongation; we shall assume that the middle line of the bar turns into the arc AB' (Figure 10).

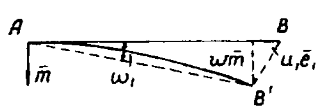

Figure 10

In view of the small elongations and of the relative smallness of w, we may write the approximate equalities:

$$\begin{gathered} AB' \approx \smile AB' \approx AB; \quad |\omega_1| \approx \frac{|w|}{L}; \\ |u_1| \approx |w\omega_1|, \quad |u_{1,1}| \sim |w_{,1}\omega_1|. \end{gathered} \qquad (a)$$

In this case, the change in curvature of an element is small and is $\varkappa_{11} = \omega_1/L$. As distinct from this, during the bending of a bar with a large initial deflection, if condition (15.1) is satisfied, the change in the curvature may be of the same order of magnitude as the initial curvature. Let us consider, for instance, the bending of the arc $L = \smile AB$ of a circle of radius R from the position AC_1B to the symmetric position AC_2B. As shown in Figure 11, the maximum deflection in this case is

$$w = c_1c_2 = 2R\left(1-\cos\frac{\omega_1}{2}\right) \approx \frac{R\omega_1^2}{4},$$

where ω_1 is the maximum rotation of the linear element from the position $\bar{e}_1$ to position $\bar{e}_1^*$. On the other hand, $L \approx R\omega_1$. Therefore $w = \frac{L\omega_1}{4}$ (b), whereby, as earlier,

$$u_1 \sim w\omega_1.$$

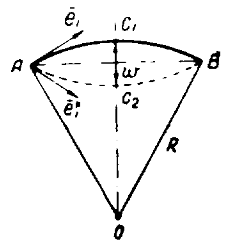

Figure 11

The same calculations for the projections of the displacements are also valid for the general case of medium bending of a shallow shell. From (3.5)(a) and (b) for shallow shells:

$$e_{11} \sim e_{12} \sim \omega_1^2, \qquad \text{(c)}$$
$$u_1 \sim A_1 w k_{11} \sim Lw/R,$$
$$A_1 k_{11} u_1 \sim wL^2/R^2,$$

That is, when neglecting ω_1^2 and ω_2^2 in comparison with unity, we may replace the formulas (15.2) by the approximate formulas

$$\varepsilon_{11} = e_{11} + \frac{1}{2}\omega_1^2, \quad \varepsilon_{22} = e_{22} + \frac{1}{2}\omega_2^2, \quad 2\varepsilon_{12} = e_{12} + e_{21} + \omega_1\omega_2, \qquad (15.5)$$

where

$$\omega_1 \approx w_{,1}/A_1, \quad \omega_2 \approx w_{,2}/A_2. \qquad (15.6)$$

As in the preceding, we shall calculate the quantities $e_{11}, \ldots, e_{21}$ from (3.5). In this case, according to (3.17) the quantities which determine the direction of the unit normal $\bar{m}^*$ to the deformed middle surface will be:

$$E_1 = \bar{m}^*\bar{e}_1 \approx -\omega_1, \quad E_2 \approx -\omega_2; \quad E_3 = \bar{m}^*\bar{m} \approx 1.$$

Therefore, taking into account (c) and (3.29), the expressions for the change in curvature of the coordinate lines and for their torsion may be replaced within assumed degree of approximation by the linear formulas

$$A_1A_2\varkappa_{11} \approx -A_2\omega_{1,1} - A_{1,2}\omega_2, \quad A_1A_2\varkappa_{12} \approx A_{1,2}\omega_1 - A_2\omega_{2,1}, \quad \overrightarrow{\underleftarrow{1,2}}. \qquad (15.7)$$

★ In order to evaluate the errors of these formulas, let us analyze them.

Let

$$k_{11} = \max|k_{ij}|. \qquad \text{(d)}$$

and L be the width of the shell $tk_{11} \sim \varepsilon_p$, and ω_1 the maximum rotation of an element of the middle surface.

Furthermore, let

$$\omega_1 \sim \varepsilon_p^r, \quad Lk_{11} \sim L/R \sim \varepsilon_p^s,$$

where r and s are quantities still to be defined; on differentiating with respect to the dimensionless coordinate, the quantities which characterize the deformation acquire the factor $\sim \varepsilon_p^{-\lambda}$. According to (15.6) and (14.2),

★ $$L\omega_1 \sim w_{,1} \sim w\epsilon_p^{-\lambda}, \quad w/L \sim \epsilon_p^{r+\lambda}$$

By (15.7)

$$L\varkappa_{11} \sim \omega_{1,1} \sim \epsilon_p^{r-\lambda}.$$

In the most unfavorable case $\varkappa_{11} \sim k_{11}$, because if we assume $tk_{11} \sim \varepsilon_p$, condition (15.4) will not be satisfied for larger changes in curvature.

Therefore,

$$L\varkappa_{11} \sim Lk_{11} \sim \epsilon_p^{s} \sim \epsilon_p^{r-\lambda}, \quad s = r - \lambda \tag{e}$$

Let us consider further the quantities in (3.5) for e_{11}:

$$\frac{u_{1,1}}{A_1} \sim \frac{u_1}{L}\,\epsilon_p^{-\lambda} \sim \frac{w\omega_1}{L}\,\epsilon_p^{-\lambda} \sim \epsilon_p^{r+\lambda+r-\lambda} \sim \epsilon_p^{2r},$$

$$wk_{11} = \frac{w}{L}\cdot Lk_{11} \sim \epsilon_p^{r+\lambda+s} \sim \epsilon_p^{2r} \sim \omega_1^2.$$

Thus, all quantities in the expression (15.5) for the elongation are of the same order. If the majority of them cancel each other, our calculation of the order of magnitude will not be sufficiently exact.

Let us consider several variants:

A. Let $\epsilon_{11} \sim \epsilon_p^{2r} \sim \epsilon_p$, i.e., there is no loss of accuracy. Furthermore,

$$r = \frac{1}{2}, \quad t\varkappa_{11} \sim tk_{11} \sim \epsilon_p, \quad w \sim t,$$

and both the bending elongations and the membrane elongations are of the same order of magnitude. Then the error in formulas (13.5) and (15.7) will be ε_p in comparison with unity and according to (e)

$$s = \frac{1}{2} - \lambda, \quad Lk_{11} \sim L/R \sim \epsilon_p^{1/2-\lambda}. \tag{f}$$

Further, according to (7.5), (9.5), (e), and (d):

$$A_1N_1^* \sim M_{11,1}^* \sim Et^3\varkappa_{11,1}, \quad A_1N_1^* k_{11} \sim Et^3\epsilon_p^{-\lambda} k_{11}^2 \sim Et\epsilon_p^{2-\lambda},$$
$$T_{11,1}^* \sim Et\epsilon_{11,1} \sim Et\epsilon_p^{1-\lambda}.$$

Thus, on assuming in this case an error ε_p in comparison with unity, we may neglect the terms which contain shearing forces in the first two equations (7.4). It should also be noted that according to (f) in the sufficiently shallow part of the shell (where $L/R \sim \sqrt{\epsilon_p}$ and $\lambda = 0$) the change in curvature and hence the elongation due to bending may reach the admissible value even for a smooth change of the shell.

B. Let $r = 1/3$. The main terms in the expression for ε_{11} cancel each other in such a way that $\epsilon_{11} \sim \epsilon_p^{2/3}$. In this case, when determining the elongations from (15.5), the error will be $\omega_1^2 \sim \epsilon_p^{2/3}$ in comparison to unity; but since the membrane elongations have a smaller influence on the deformation than in case A, the error indicated is admissible in determining these deformations as well as for the equations of equilibrium of the forces in the plane tangent to σ. If $\varkappa_{11} < k_{11}$ or for a thin plate with small curvature but with $\varkappa_{11} \sim k_{11}$, the above simplification produces a still smaller error. ★

Thus, for shallow shells or for non-shallow shells which may be divided into a large number of shallow portions, equations (7.4) may be replaced by the following approximate equations:

$$(A_2 T_{11}^*)_{,1} + (A_1 T_{21}^*)_{,2} + T_{12}^* A_{1,2} - T_{22}^* A_{2,1} + A_1 A_2 X_1^* = 0 \quad \overrightarrow{\underrightarrow{1, 2}}; \tag{15.8}$$

$$(A_2 N_1^*)_{,1} + (A_1 N_2^*)_{,2} - A_1 A_2 [T_{11}^* k_{11}^* + 2T_{12}^* k_{12}^* + T_{22}^* k_{22}^* - X_3^*] = 0; \tag{15.9}$$

$$k_{ij}^* = k_{ij} + \varkappa_{ij} \quad (i, j = 1, 2).$$

Introducing (3.5), (15.5)-(15.7), and (7.5) in these equations, we obtain a system of three equations for u_1, u_2, w. These equations are called equations of equilibrium in the components of the displacement.

The elongations and the changes in curvature must satisfy the conditions of compatibility of the deformation, i. e., Codazzi's equations (3.35) and Gauss' equation (3.32). For flat shells the former may be simplified by neglecting the quantities of the order of magnitude of tk_{11} in comparison with unity. Thus, the equations become

$$(A_1 \varkappa_{11})_{,2} - A_{1,2} \varkappa_{22} - (A_2 \varkappa_{12})_{,1} - A_{2,1} \varkappa_{12} = 0 \qquad \overrightarrow{\underleftarrow{1, 2}}. \tag{15.10}$$

as in the linear theory of thin shells. Gauss' equation remains non-linear, and for shallow shells it has the form (3.32) as before.

On replacing in (15.10) and (3.32) the quantities $\varepsilon_1, \ldots, \varkappa_{12}$ by $T_{11}^*, \ldots, M_{12}^*$ according to (9.5), we obtain three equations which together with (15.8) and (15.9) form after elimination of N_1^* by (7.5) a system of six equations for T_{11}^*, T_{12}^*, T_{22}^*, M_{11}^*, M_{12}^* and M_{22}^*. That enables one to solve the problem of equilibrium for the components of the elastic force and the moment, without introducing the formulas connecting the latter with the displacement, provided that the boundary conditions are given independent of the displacement components.

If $X_1^* = X_2^* = 0$, the composite form of equations is very convenient. In order to derive these equations we have to satisfy approximately the static equations (15.8) by introducing a force function ψ according to

$$\begin{aligned} A_2 T_{11}^* &= (\psi_{,2}/A_2)_{,2} + A_{2,1}\psi_{,1}/A_1^2, \qquad \overrightarrow{\underleftarrow{1,2}}, \\ A_1 A_2 T_{12}^* &= -\psi_{,12} + A_{2,1}\psi_{,2}/A_2 + A_{1,2}\psi_{,1}/A_1 \end{aligned} \tag{15.11}$$

and taking into account the relations (2.27) between the parameters of the shell A_i and k_{ij} and the definitions (2.19) and (2.24).

In fact, after introducing (15.11), the left-hand sides of the equations (15.8) become

$$-\frac{A_1 A_2}{R_1 R_2} \cdot \frac{\psi_{,1}}{A_1} \qquad \overrightarrow{\underleftarrow{1, 2}},$$

while the terms which cancel each other contain the derivatives of ψ up to the third order included, without containing the factor $A_1 A_2 / R_1 R_2$. For the equilibrium of the shallow part of the shell, while choosing the dimensionless coordinates α_i by (14.2), this factor will be small in comparison with unity owing to (15.3). Therefore, we may consider the equations (15.8) as satisfied, although

$$\psi_{,i} \sim \psi.$$

It should be noted that this approximate theory may also be applied to a non-shallow shell, when considering the kind of deformation for which the shell devides into a large number of shallow portions (for example, at buckling when a large number of half-waves are formed on the surface of the shell). In the latter case, $A_i \sim R \sim L$, but the force function sharply increases with differentiation so that the second order derivative with respect to α_i is large in comparison to the function itself.

Thus, considering that the equations (15.8) are approximately satisfied, we introduce (15.11) in (15.9). We shall assume that the moment of the external surface forces may be neglected:

$$L_i^* = 0. \qquad (15.12)$$

Taking into account (15.10) we obtain from (7.5) and (9.5)

$$A_i N_i^* = D(\varkappa_{11} + \varkappa_{22})_{,i} \qquad i = 1,2. \qquad (15.13)$$

But according to (15.7)

$$\varkappa_{11} + \varkappa_{22} = -\Delta w,$$

where Δ is the Laplace operator in orthogonal curvilinear coordinates:

$$\Delta w = \frac{1}{A_1 A_2}\left[\left(\frac{A_2}{A_1} w_{,1}\right)_{,1} + \left(\frac{A_1}{A_2} w_{,2}\right)_{,2}\right]. \qquad (15.14)$$

Therefore, equation (15.9) becomes:

$$D\Delta\Delta w + T_{11}^*(k_{11} + \varkappa_{11}) + 2T_{12}^*(k_{12} + \varkappa_{12}) + T_{22}^*(k_{22} + \varkappa_{22}) - X_3^* = 0. \qquad (15.15)$$

Here T^*_{ij} and $\varkappa_{ij}$ may be expressed in terms of the force function ψ and the deflection function w according to (15.7) and (15.11). Thus, (15.15) represents a non-linear differential equation in w and ψ. Another relation between these quantities is given by Gauss' equation (3.32) for the surface σ^*, which may also be written in compact form.

We shall not deal in detail with the transormfation of this equation. We shall only point out that in order that this equation should be more readily satisfied, one has to use the equations (15.8) and (2.27). Besides, according to (15.11) and (15.14),

$$T_{11}^* + T_{22}^* = \Delta\psi, \quad T_{11}^* = Et(\varepsilon_{11} + \nu\varepsilon_{22})/(1 - \nu^2), \ldots .$$

Thus, (3.32) becomes

$$\Delta\Delta\psi - Et(\varkappa_{12}^2 - \varkappa_{11}\varkappa_{22} - \varkappa_{11}k_{22} - \varkappa_{22}k_{11} + 2\varkappa_{12}k_{12}) = 0. \qquad (15.16)$$

On using the expressions (15.6), (15.7), and (15.11), the equations (15.15) and (15.16) form the required system of non-linear equations in ψ and w.

Let us consider briefly the boundary conditions.

A. Let the end contour be free. Then, only the static boundary conditions need be satisfied at the contour. Usually it is more convenient to reduce the forces and moments to the principal directions of the deformed shell. We shall assume, as in Section 8, that the normal $\bar{n}^*$ to the end section lies in the plane tangent to the middle surface σ^* of the deformed shell. We shall also assume that the external forces acting on this section are reduced to the normal force $\tilde{\Phi}_n^*$, the tangential force Φ_t^*, the shearing force Φ_3^*, and the bending moment $\tilde{G}^*$.

Taking into account that for small deformations

$$n_i^* \approx n_i, \quad \tau_i^* \approx \tau_i, \quad ds^* \approx ds,$$

we may write, according to (8.5)

$$\frac{\partial H^*}{\partial s} = \frac{\partial H^*}{\partial \alpha_1} \cdot \frac{d\alpha_1}{ds} + \frac{\partial H^*}{\partial \alpha_2} \cdot \frac{d\alpha_2}{ds} = -\frac{\partial H^*}{\partial \alpha_1} \cdot \frac{n_2}{A_1} + \frac{\partial H^*}{\partial \alpha_2} \cdot \frac{n_1}{A_2}.$$

Therefore, using (8.11) and (8.12) and neglecting tk_{ii} in comparison with unity, the boundary conditions (8.21) may be expressed in the simplified form

$$\Phi_n^* = T_{11}^* n_1^2 + 2T_{12}^* n_1 n_2 + T_{22}^* n_2^2, \qquad (15.17)$$
$$\Phi_\tau^* = (T_{22}^* - T_{11}^*) n_1 n_2 + T_{12}^* (n_1^2 - n_2^2);$$

$$\varphi_3^* = N_1^* n_1 + N_2^* n_2 + n_2 [(M_{11}^* - M_{22}^*) n_1 n_2 - M_{12}^* (n_1^2 - n_2^2)]_{,1}/A_1 -$$
$$- n_1 [(M_{11}^* - M_{22}^*) n_1 n_2 - M_{12}^* (n_1^2 - n_2^2)]_{,2}/A_2; \qquad (15.18)$$

$$\widetilde{G}^* = M_{11}^* n_1^2 + 2M_{12}^* n_1 n_2 + M_{22}^* n_2^2 = G^*;$$
$$n_1 = \sin\varphi, \quad n_2 = -\cos\varphi, \qquad (15.19)$$

where φ is the angle between the positive direction of the α_1 axis and the positive direction $\bar{\tau}$ of the tangent to the contour before deformation, and the trihedron of the axes $\{\bar{n}, \bar{\tau}, \bar{m}\}$ before deformation is right-handed.

If the end contour coincides with one of the coordinate lines, the conditions (15.17)-(15.19) will be considerably simplified. For instance, if it is the line α_1= const then $\varphi = \pi/2$.

B. If any geometrical constraints are put on the end contour, the external load on the contour will be usually composed of the forces and moments $\Phi_n^{\text{н}}$, $\Phi_\tau^{\text{н}}$, $\Phi_3^{\text{н}}$ and $\widetilde{G}^{\text{н}}$ along the principal directions of the surface σ. In this case the conditions (8.35) must be satisfied at the contour. But for medium bending

$$e_{ij} \ll 1, \quad E_1 \approx -\omega_1, \quad E_2 \approx -\omega_2; \quad E_3 \approx 1, \quad N_i^* \omega_i \ll T_{ij}^*;$$

therefore it follows from (8.30) that

$$T_{ij}^{\text{н}} \approx T_{ij}^*, \quad M_{ij}^{\text{н}} \approx M_{ij}^*,$$
$$N_1^{\text{н}} \approx N_1^* + T_{11}^* \omega_1 + T_{12}^* \omega_2, \quad N_2^{\text{н}} \approx N_2^* + T_{12}^* \omega_1 + T_{22}^* \omega_2. \qquad (15.20)$$

Consequently, in this case, the boundary conditions are given by the equations (15.17)-(15.19) after replacing the quantities $\Phi_n^*, \ldots, T_{ij}^*, M_{ij}^*, N_i^*$ by $\Phi_n^{\text{н}}, \ldots, T_{ij}^{\text{н}}, M_{ij}^{\text{н}}$ and $N_i^{\text{н}}$. It is not difficult to see from (15.20) that only condition (15.18) is considerably different.

Before closing this section, let us add a few notes on the stages of development of the theory of shallow shells. As far as we know L. Donnell the first to show that it is possible to simplify the equations of equilibrium for the membranes forces by neglecting the shearing forces and bringing these equations to the form (15.8). He made this remark in his paper /IV. 6/ in connection with the problem of stability of a cylindrical shell. He also proved that it is possible, in that case, to use the simplified expressions (15.7) for the parameters of change of the curvature and the torsion. We have developed this theory in /0.13/ for any shells which were divided during buckling into a large number of shallow portions; we have solved the relevant problems, mainly in the components of displacements, and only for cylindrical and conical shells. We have introduced the stress function according to equations like (15.11) where the equations (15.8) for $X_1^* = X_2^* = 0$ have been exactly satisfied. In the same work we have determined the degree of accuracy of the approximate theory. Already in the works of I. G. Bubnov /0.2/ and /0.3/, a slightly curved bar or plate were considered as plates having an initial deviation from a plane surface. The non-linear theory of shallow shells has been expounded in this formulation by K. Marguerre in his work /VI. 11/. In the monograph by Wei-Tsang Chien /0.19/ a shell is called shallow when the ratio between the width of the portion of the shell under consideration and the minimal radius of curvature is small, i.e., when $L/R \ll 1$. By this assumption, he introduced the stress function and other simplifications in the theory of shallow shells. At the same time V. Z. Vlasov created the general framework of the theory of shallow shells in terms of lines of curvature /IV. 2/ as above, which is equivalent to neglecting L^2/R^2 in comparison with unity. Later, in his work /0.4/, the same author gave more valid reasons for neglecting this, for the shallow part of a shell for which the intrinsic geometry is approximately Euclidean. The papers by Yu. N. Rabotnov /IV. 3/, N. A. Alumyae /IV. 1/, etc, have also undoubtedly contributed to the theory of shallow shells.

It may be seen from the above that the theory of shallow shells has been developed by the efforts of a number of scientists (as V. V. Novozhilov has rightly pointed out in his monograph /0.15/).

§ 16. Large Bending of a Shell*

We shall say that the bending of a shell is large when the deflection of the points of the shell is large in comparison with its thickness and comparable with its characteristic linear dimension. In this case the rotation of the linear elements will also be large. Thus

$$t/w \sim \varepsilon_p, \quad \omega_i \sim 1. \tag{16.1}$$

Besides, the condition

$$\frac{1}{2} t\varkappa \leqslant \varepsilon_p, \tag{16.2}$$

must be fulfilled in order to avoid plastic deformations. Hence, it results that the functions that characterize the deformations should vary smoothly;

$$w_{,i} \sim w, \quad u_{1,i} \sim u_1, \quad u_{2,i} \sim u_2 \quad (i = 1,2). \tag{16.3}$$

Therefore,

$$M_{ij}^* \sim D\varkappa \sim Et^3\varkappa \sim A_i N_i^* \lesssim Et^2\varepsilon_p. \tag{16.4}$$

A. Composite deformation of the shell. Let:

$$L \sim R, \quad \varepsilon \sim \varepsilon_p, \tag{16.5}$$

this means that bending occurs with considerable membrane elongations.

Then,

$$T_{ij}^* \sim Et\varepsilon \sim Et\varepsilon_p,$$

and we may neglect in (7.4) the terms which contain N_i^*. We thus obtain the equations

$$(A_2 T_{11}^*)_{,1} + (A_1 T_{21}^*)_{,2} + T_{12}^* A_{1,2} - T_{22}^* A_{2,1} + A_1 A_2 X_1^* = 0 \quad \overleftarrow{\overrightarrow{1, 2}} \tag{16.6}$$

$$T_{11}^* k_{11}^* + T_{22}^* k_{22}^* + 2T_{12}^* k_{12}^* - X_3^* = 0, \tag{16.7}$$

which may be called equations of equilibrium for a membrane. They differ from the equations of the usual membrane theory (zero moments) by the fact that all quantities refer to the deformed state. In particular, in this case $k_{ij}^* = k_{ij} + \varkappa_{ij}$. The forces are also non-linear in the displacements. Thus, (16.6 and (16.7) form a system of three non-linear equations in u_i and w. When the deformations are not expressed in terms of the displacements it will be necessary to add to the equations (16.6) and (16.7) the condition of compatibility of deformations. This condition may be expressed in the case considered by equations (15.10) and (3.32). Neglecting ε in comparison with unity, and taking into account that $\varepsilon_{,i} \sim \varepsilon$, the latter equation becomes

$$\varkappa_{12}^2 - \varkappa_{11}\varkappa_{22} - \varkappa_{11}k_{22} - \varkappa_{22}k_{11} + 2\varkappa_{12}k_{12} = 0. \tag{16.8}$$

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* The material of this paragraph has been treated, in general coordinates, in the paper by Mushtari /IV.4/.

Let us furthermore consider the deformation near the fixed edge $\alpha_1 = \alpha_1^0$ = const which has no part in common with the asymptotic line of the surface σ. Here the deflections cannot be considered as large in comparison with the thickness. If the damping is rapid enough, then in the zone of the edge effect

$$w \lesssim t.$$

Therefore, assuming

$$w = w^{M} + w^{K}, \tag{16.9}$$

where w^M is the membrane deflection and w^K is the deflection due to the edge effect, near to the edge we have

$$w^{M}_{,11} \sim w^{M}, \quad w^{K}_{,11} \gg w^{K}, \quad \varkappa_{11} = \varkappa^{M}_{11} + \varkappa^{K}_{11} \approx \varkappa^{K}_{11},$$
$$A_1 A_2 N_1^* \approx (A_2 M^{*K}_{11})_{,1} \approx D(A_2 \varkappa^{K}_{11})_{,1}; \quad T^*_{ij} \approx T^{*M} + T^{*K}_{ij}.$$

By subtracting from (16.6) the corresponding equations for a membrane we obtain equations of the same kind for T^{*k}_{ij} (without the terms that contain X^*_i). Similarly, by subtracting from (7.5) the equation (16.7) for a membrane, we obtain the equation

$$\left[\frac{D}{A_1}(A_2 \varkappa^{K}_{11})_{,1}\right]_{,1} + A_1 A_2 \left[T^{*M}_{11} \varkappa^{K}_{11} + T^{*M}_{22} \varkappa^{K}_{22} + 2 T^{*M}_{12} \varkappa^{K}_{12} + \right.$$
$$\left. + T^{*K}_{11}(k_{11} + \varkappa^{K}_{11}) + T^{*K}_{22}(k_{22} + \varkappa^{K}_{22}) + 2 T^{*K}_{12}(k_{12} + \varkappa^{K}_{12})\right] = 0. \tag{16.10}$$

Gauss' condition of compatibility is applied in its complete form (3.32).

B. Deformations with prevailing bending. Let

$$t\varkappa \sim \varepsilon_p, \quad \varepsilon \sim \varepsilon_{ij} \lesssim \varepsilon_p^2, \quad L \sim R, \quad \varkappa_{,i} \sim \varkappa. \tag{16.11}$$

In this case the external forces must be small:

$$X_1^* \sim X_2^* \sim E\varepsilon_p^3, \quad X_3^* \sim E\varepsilon_p^4. \tag{16.12}$$

The bending must satisfy the equations of compatibility (15.10) and (16.8). If, in addition to bending moments the membrane forces are also given at the edge section, the problem reduces to the determination of the integrals in the system of equations (15.10), (16.8), and (7.4). If the membrane forces at the edge are not known, it will be necessary to solve first the system of equations (15.10) and (16.8) for the given moments at the contour, and afterwards to determine the forces, by integrating the linear system of equations (7.4).

In the case of developable surfaces the Gaussian curvature $1/R_1R_2$ is zero and the equations (15.10) will be satisfied by substituting (15.7) and (15.6). If we take the straight-line generators and their orthogonal trajectories instead of the coordinate lines α_1 and α_2 it will be necessary to set $k_{11} = k_{12} = 0$ and $A_1 = 1$ in (16.8). We thus obtain the equation for determining w:

$$w_{,11} w_{,22} - w_{,12}^2 - A_2^2 k_{22} w_{,11} + A_2 A_{2,1} w_{,1} w_{,11} +$$
$$+ 2 w_{,12} w_{,2} A_{2,1}/A_2 - (A_{2,1} w_{,2}/A_2)^2 - A_{2,2} w_{,11} w_{,2}/A_2 = 0. \tag{16.13}$$

As is known from the theory of partial differential equations of second order, both families of the characteristics coincide and may be expressed by the following equations*:

$$dw - w_{,1} d\alpha_1 - w_{,2} d\alpha_2 = 0, \quad dw_{,1} - A_{2,1} \frac{w_{,2}}{A_2} d\alpha_2 = 0,$$
$$dw_{,2} - \frac{A_{2,1}}{A_2} w_{,2} d\alpha_1 + (A_{2,1} A_2 w_{,1} - A_{2,2} A_2 w_{,2} - A_2^2 k_{22}) d\alpha_2 = 0. \tag{16.14}$$

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* See /IV.5/, Vol IV.

In the particular case of a cylindrical shell, on setting $\alpha_1 = x$ and $\alpha_2 = y$, where x is the coordinate measured along the generating line and y is the length of the orthogonal trajectory, we have $A_2 = 1$, $k_{22} = k_{22}(y)$, and the equations of the characteristics allow three integrable combinations

$$d\theta_1 = 0, \quad d\theta_2 = 0, \quad d\theta_3 = 0, \tag{16.15}$$

where

$$\theta_1 = w_{,x}, \quad \theta_2 = w_{,y} - \int k_{22} dy, \quad \theta_3 = w - x\theta_1 - y\theta_2 - \int\int k_{22} dy dy.$$

Therefore, the complete integral of the system is determined by the equations

$$\theta_3 = C, \quad \theta_1 = \varphi_1(C), \quad \theta_2 = \varphi_2(C),$$

where φ_1 and φ_2 are arbitrary functions.

We may obtain in this case the general integral of the equation (16.13) by eliminating C from the equations

$$\begin{gathered} \theta = \theta_3 - C = w - C - x\varphi_1(C) - y\varphi_2(C) + f(y) = 0; \\ \partial\theta/\partial C = -1 - x\varphi_1'(C) - y\varphi_2'(C) = 0, \end{gathered} \tag{16.16}$$

$$f(y) = -\int\int k_{22} dy dy. \tag{16.17}$$

We shall further assume that the edges of the thin cylindrical plate are free and subjected to the action of those distributed bending and twisting moments which cause a change in curvature:

$$\begin{gathered} \varkappa_{11} = -w_{,xx} = -\alpha \neq 0, \quad \varkappa_{22} = -w_{,yy} = -\beta, \\ \varkappa_{12} = -w_{,xy} = -\gamma. \end{gathered} \tag{16.18}$$

From (16.16) and (16.13) we obtain, using (16.17):

$$\begin{gathered} w_{,x} = \varphi_1, \quad w_{,y} = \varphi_2 - f_{,y}, \quad w_{,xx} = \varphi_1' C_{,x} = \alpha, \\ w_{,yy} = \varphi_2' C_{,y} + k_{22} = \beta, \quad w_{,xy} = \varphi_2' C_{,x} + \varphi_1' C_{,y} = \gamma, \\ \gamma^2 = \alpha(\beta - k_{22}) = \alpha\varphi_2' C_{,y}, \quad \frac{\gamma^2 y}{\alpha C_{,y}} + \frac{\alpha x}{C_{,y}} = -1. \end{gathered}$$

Therefore,

$$\begin{gathered} -dC = (\alpha x + \gamma y)dx + \left(\frac{\gamma^2}{\alpha} y + \gamma x\right)dy, \\ d\varphi_1 = \alpha dx + \gamma dy, \quad d\varphi_2 = \gamma dx + (\beta - k_{22})dy. \end{gathered} \tag{16.19}$$

From this, we can find the expressions for C, φ_1, and φ_2 in terms of α, β, γ. At the same time, the integrability condition for (16.19) must be satisfied

$$\frac{\partial\alpha}{\partial y} = \frac{\partial\gamma}{\partial x}, \quad \frac{\partial\gamma}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\gamma^2}{\alpha}\right) = \frac{\partial\beta}{\partial x}. \tag{16.20}$$

For instance, if $\alpha = \text{const}$, $\gamma = \text{const}$, $\alpha(\beta - k_{22}) = \gamma^2$, we readily find from (16.19):

$$\begin{gathered} C = -\frac{\gamma^2 y^2}{2\alpha} - \gamma xy - \frac{\alpha x^2}{2}, \quad \varphi_1 = \alpha x + \gamma y, \quad \varphi_2 = \frac{\gamma^2}{\alpha} y + \gamma x, \\ w = \frac{\gamma^2 y^2}{2\alpha} + \frac{\alpha x^2}{2} + \gamma xy - f(y). \end{gathered} \tag{16.21}$$

If $\alpha = 0$, this solution will be useless.

C. The edge effect at free edges for large displacements. In order to illustrate such a problem, let us consider the determination of forces and moments which have to be applied to the straight edges of a thin, circular, cylindrical plate to maintain its cylindrical form (or any other form very near to that) after a large bending.

Using the same notations as in B, we may in this case write $A_1 = A_2 = 1$, $k_{22} = 1/R$, $k^*_{22} = 1/R^*$. The equations (15.10) and (16.8) will be satisfied if

$$\varkappa_{12} = 0, \quad \varkappa_{11} = \varkappa_{11}(x), \\ \varkappa_{22} = 1/R^* - 1/R = \varkappa^0_{22} = \text{const.} \tag{16.22}$$

The equation (16.6) will be satifeid if

$$T^*_{12} = 0, \quad T^*_{11,x} = 0, \quad T^*_{22,y} = 0, \tag{16.23}$$

because $X^*_i = 0$.

The boundary conditions

$$T^*_{11} = 0, \quad M^*_{11} = D(\varkappa_{11} + \nu\varkappa_{22}), \quad N^*_1 = D\varkappa_{11,1} = 0. \tag{16.24}$$

will be satisfied for $x = \pm l/2$.

It follows from (16.23) and from the first of (16.24) that

$$T^*_{11} = 0, \quad \varepsilon_{11} = -\nu\varepsilon_{22}, \quad T^*_{22} = Et\varepsilon_{22}. \tag{16.25}$$

The last of equations (7.5) and Gauss' equation (3.32) become:

$$D\varkappa_{11,xx} + Etk^*_{22}\varepsilon_{22} = 0, \quad \varepsilon_{22,xx} - k^*_{22}\varkappa_{11} = 0, \tag{16.26}$$

or

$$\varkappa_{11,xxxx} + 4\lambda^4\varkappa_{11} = 0, \quad \lambda = \left(\sqrt{3(1-\nu^2)}/tR^*\right)^{1/2}. \tag{16.27}$$

The solution must be an even function of x. Therefore,

$$\varkappa_{11} = C_1 \operatorname{ch}\lambda x \cos\lambda x + C_2 \operatorname{sh}\lambda x \sin\lambda x.$$

Assuming

$$\operatorname{ch}\mu \approx \operatorname{sh}\mu, \quad \mu = \lambda l/2,$$

we obtain from the second and third conditions of (16.24)

$$C_1 \approx -\nu\varkappa^0_{22}\frac{\cos\mu + \sin\mu}{\operatorname{ch}\mu}, \quad C_2 \approx \nu\varkappa^0_{22}\frac{\cos\mu - \sin\mu}{\operatorname{ch}\mu}.$$

With these, the force may be determined from (16.26) and (16.25), and the bending moment from

$$M_{22} = D(\varkappa^0_{22} + \nu\varkappa_{11}).$$

A particular case of this problem for $R = \infty$, has been examined by another method, in the monograph by Love /0.11/.

The classification of problems of the non-linear theory of shells suggested in this chapter has been given in a more general form in the article by Kh. M. Mushtari /IV.4/. A somewhat modified classification has been given in the monograph by Wei-Tsang Chien /0.19/.

Chapter V

GENERAL THEORY OF STABILITY OF THIN SHELLS

§ 17. Fundamental Equations of the Theory of Stability of Shells

Let the middle surface of the shell before deformation be referred to the orthogonal coordinates α_1, α_2, z with the unit vectors $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$. We shall consider two successive deformations of the shell. After the first deformation defined by the displacements σ

$$\bar{v}^{\mathrm{I}} = u_1^{\mathrm{I}}\bar{e}_1 + u_2^{\mathrm{I}}\bar{e}_2 + w^{\mathrm{I}}\bar{m}, \tag{17.1}$$

the surface σ^{I} becomes the middle surface of the shell. We shall fix the points of the surface σ^{I} by the former coordinates α_1, α_2 and z. In spite of the smallness of the elongations, we shall not neglect them initially in comparison with unity, since that may sometimes lead to inexact results. We shall neglect only the shear in comparison with unity. As shown below, his leads only to an unimportant error, even when reducing the main terms in the equations of equilibrium. For the present we shall not put any limitations on the magnitudes of the displacements. From equations (3.4), (3.5), (3.16), (3.17), (3.13)-(3.15), (3.29), (9.5), and (6.13), we shall find the quantities that characterize the first state of deformation and stress of the shell; we shall denote them by superscripts I; the quantities which are not specified here may be obtained by permutation of the indices 1 and 2.

The following are the most important quantities:

1. the fundamental vectors of the deformed coordinates

$$\begin{gathered} \bar{r}_{,1}^{\mathrm{I}} = A_1\{(1+e_{11}^{\mathrm{I}})\bar{e}_1 + e_{12}^{\mathrm{I}}\bar{e}_2 + \omega_1^{\mathrm{I}}\bar{m}\}, \\ \bar{m}^{\mathrm{I}} = E_1^{\mathrm{I}}\bar{e}_1 + E_2^{\mathrm{I}}\bar{e}_2 + E_3^{\mathrm{I}}\bar{m}. \end{gathered} \tag{17.2}$$

where

$$\begin{gathered} A_1A_2e_{11}^{\mathrm{I}} = A_2u_{1,1}^{\mathrm{I}} + u_2^{\mathrm{I}}A_{1,2} + w^{\mathrm{I}}k_{11}A_1A_2, \\ A_1\omega_1^{\mathrm{I}} = w_{,1}^{\mathrm{I}} - A_1(u_1^{\mathrm{I}}k_{11} + u_2^{\mathrm{I}}k_{12}), \\ A_1A_2e_{12}^{\mathrm{I}} = A_2u_{2,1}^{\mathrm{I}} - u_1A_{1,2} + w^{\mathrm{I}}k_{12}A_1A_2,\ E_1^{\mathrm{I}} = e_{12}^{\mathrm{I}}\omega_2^{\mathrm{I}} - (1+e_{22}^{\mathrm{I}})\omega_1^{\mathrm{I}}, \\ E_3^{\mathrm{I}} = (1+e_{11}^{\mathrm{I}})(1+e_{22}^{\mathrm{I}}) - e_{12}^{\mathrm{I}}e_{21}^{\mathrm{I}}; \end{gathered} \tag{17.3}$$

2. the elongations and shears

$$\begin{gathered} \varepsilon_{11}^{\mathrm{I}} = e_{11}^{\mathrm{I}} + \frac{1}{2}(e_{11}^{\mathrm{I}\,2} + e_{12}^{\mathrm{I}\,2} + \omega_1^{\mathrm{I}\,2}), \\ 2\varepsilon_{12}^{\mathrm{I}} = e_{12}^{\mathrm{I}} + e_{21}^{\mathrm{I}} + e_{11}^{\mathrm{I}}e_{21}^{\mathrm{I}} + e_{22}^{\mathrm{I}}e_{12}^{\mathrm{I}} + \omega_1^{\mathrm{I}}\omega_2^{\mathrm{I}}; \end{gathered} \tag{17.4}$$

3. the changes in curvature

$$\begin{gathered} \varkappa_{11}^{\mathrm{I}} = -e_{21}^{\mathrm{I}}k_{12} + e_{22}^{\mathrm{I}}k_{11} - (E_1^{\mathrm{I}}e_{11,1}^{\mathrm{I}} + E_2^{\mathrm{I}}e_{12,1}^{\mathrm{I}} + \\ + E_3^{\mathrm{I}}\omega_{1,1}^{\mathrm{I}} + \omega_2^{\mathrm{I}}A_{1,2}/A\ ) : A_1, \\ \varkappa_{12}^{\mathrm{I}} = -e_{12}^{\mathrm{I}}k_{11} + e_{11}^{\mathrm{I}}k_{12} - (E_1^{\mathrm{I}}e_{21,1}^{\mathrm{I}} + E_2^{\mathrm{I}}e_{22,1}^{\mathrm{I}} + \\ + E_3^{\mathrm{I}}\omega_{2,1}^{\mathrm{I}} - \omega_1^{\mathrm{I}}A_{1,2}/A\ ) : A_1; \end{gathered} \tag{17.5}$$

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4. the internal forces and moments

$$T_{11}^{I} = K(\varepsilon_{11}^{I} + \nu\varepsilon_{22}^{I}), \quad T_{12}^{I} = K(1-\nu)\varepsilon_{12}^{I},$$
$$M_{11}^{I} = D(\varkappa_{11}^{I} + \nu\varkappa_{22}^{I}). \quad M_{12}^{I} = D(1-\nu)\varkappa_{12}^{I}; \tag{17.6}$$

5. the external forces and the external moment per unit area of the middle surface

$$\bar{X}^{I} = X_1^{I}\bar{e}_1^{I} + X_2^{I}\bar{e}_2^{I} + X_3^{I}\bar{m}^{I}, \quad \bar{L}^{I} = L_1^{I}\bar{e}_2^{I} - L_2^{I}\bar{e}_1^{I}. \tag{17.7}$$

These have to satisfy equations of equilibrium similar to (7.4) and (7.5):

$$(A_2^{I}T_{11}^{I})_{,1} + (A_1^{I}T_{21}^{I})_{,2} + T_{12}^{I}A_{1,2} - T_{22}^{I}A_{2,1}^{I} + \\ + A_1A_2(k_{11}^{I}N_1^{I} + k_{12}^{I}N_2^{I} + X_1^{I}) = 0; \tag{17.8}$$

$$(A_2N_1^{I})_{,1} + (A_1N_2^{I})_{,2} - A_1A_2(k_{11}^{I}T_{11}^{I} + 2k_{12}^{I}T_{12}^{I} + \\ + k_{22}^{I}T_{22}^{I} - X_3^{I}) = 0; \tag{17.9}$$

$$(A_2M_{11}^{I})_{,1} + (A_1M_{21}^{I})_{,2} + M_{12}^{I}A_{1,2} - M_{22}^{I}A_{2,1} + \\ + A_1A_2(L_1^{I} - N_1^{I}) = 0 \\ (k_{ij}^{I} = k_{ij} + \varkappa_{ij}^{I}). \tag{17.10}$$

If, for some values of the external contour and surface forces, a second equilibrium form of the middle surface denoted σ^* becomes possible in addition to the form σ^I, one says that the shell is in a state of "neutral equilibrium". The corresponding load is called a critical load, because for an increase, however little it may be exceeded, the form of equilibrium σ^I will lose its stability.

We shall assume that the additional displacement

$$\bar{v} = u_1\bar{e}_1 + u_2\bar{e}_2 + w\bar{m},$$

which turns σ^I into σ^* is infinitesimal. The components of the displacement in the disturbed state σ^* are

$$u_1^* = u_1^{I} + u_1, \quad u_2^* = u_2^{I} + u_2, \quad w^* = w^{I} + w.$$

After substituting these in formulas like (3.5) we find

$$e_{11}^* = e_{11}^{I} + e_{11}, \quad e_{12}^* = e_{12}^{I} + e_{12}, \quad \omega_1^* = \omega_1^{I} + \omega_1, \quad \overleftrightarrow{1,2} \tag{17.11}$$

Where e_{ij} and ω_i are infinitesimals which may be calculated from (3.5), and e_{ij}^{I} and ω^{I} are quantities of the maximal order of magnitude unity, which may be determined from (17.3). We denote by ε_{ii}^* and $2\ \varepsilon_{12}^*$ the elongations and the shear of the middle surface σ^*. They may be calculated from formulas similar to (3.13)-(3.15), using (3.5), (17.11) and (17.14). In the expressions for the quantities which are of interest to us we shall retain, for the moment, the infinitesimals of second order, neglecting only infinitesimals of the third and higher orders.

Thus

$$\varepsilon_{11}^* = e_{11}^* + \frac{1}{2}(e_{11}^{*2} + e_{12}^{*2} + \omega_1^{*2}), \quad \varepsilon_{11}^* = \varepsilon_{11}^{I} + \varepsilon_{11}' + \varepsilon_{11}'', \quad \overleftrightarrow{1,2}$$
$$2\varepsilon_{21}^* = e_{12}^* + e_{21}^* + e_{11}^*e_{21}^* + e_{12}^*e_{22}^* + \omega_1^*\omega_2^* = \\ = 2(\varepsilon_{12}^{I} + \varepsilon_{12}' + \varepsilon_{12}''), \tag{17.12}$$

where we denoted by ε_{ij}' and ε_{ij}'' the infinitesimals of the first and second orders respectively. These characterize the additional elongations and shear due to the displacement $\bar{v}$:

$$\varepsilon_{11}' = (1 + e_{11}^{\mathrm{I}}) e_{11} + e_{12}^{\mathrm{I}} e_{12} + \omega_1^{\mathrm{I}} \omega_1, \quad \varepsilon_{11}'' = \frac{1}{2} (e_{11}^2 + e_{12}^2 + \omega_1^2),$$
$$2\varepsilon_{12}' = e_{11} e_{21}^{\mathrm{I}} + e_{22} e_{12}^{\mathrm{I}} + (1 + e_{11}^{\mathrm{I}}) e_{21} + (1 + e_{22}^{\mathrm{I}}) e_{12} + \omega_1 \omega_2^{\mathrm{I}} + \omega_2 \omega_1^{\mathrm{I}}, \tag{17.13}$$
$$2\varepsilon_{12}'' = e_{11} e_{21} + e_{22} e_{12} + \omega_1 \omega_2 \quad \overrightarrow{\underleftarrow{1, 2}}.$$

The components of the elastic force along the principal directions of the surface σ^* are:

$$T_{11}^* = K(\varepsilon_{11}^* + \nu\varepsilon_{22}^*) = T_{11}^{\mathrm{I}} + T_{11}' + T_{22}'',$$
$$T_{12}^* = K(1 - \nu)\varepsilon_{12}^* = T_{12}^{\mathrm{I}} + T_{12}' + T_{12}'',$$
$$T_{11}' = K(\varepsilon_{11}' + \nu\varepsilon_{22}'), \quad T_{12}' = K(1 - \nu)\varepsilon_{12}', \tag{17.14}$$
$$T_{11}'' = K(\varepsilon_{11}'' + \nu\varepsilon_{22}''), \quad T_{12}'' = K(1 - \nu)\varepsilon_{12}'' \quad \overrightarrow{\underleftarrow{1, 2}}.$$

In the same manner, the total change in curvature on going from the undeformed middle surface σ to the surface σ^* may be determined from (3.29)

$$\varkappa_{ij}^* = \varkappa_{ij}^{\mathrm{I}} + \varkappa_{ij}' + \varkappa_{ij}'', \quad i, j = 1 \text{ and } 2, \tag{17.15}$$

where the $\varkappa_{ij}^{\mathrm{I}}$, given by (17.5), are the changes in curvature due to the passage from σ to σ^{I} and $\varkappa_{ij}'$ are the changes due to the additional displacement.

Linearizing (3.29), we find:

$$\varkappa_{11} = -e_{21}k_{12} + e_{22}k_{11} - (\omega_2 A_{1,2} + A_2\omega_{1,1})/A_1A_2,$$

since in this case

$$te_{22}k_{11} = t\varepsilon_{22}k_{11} \sim \varepsilon_{22}t/R,$$

and $e_{22}k_{11}$ may be neglected in the expression for $\varkappa_{11}$ as $(1/2)t\varkappa_{11} \sim \varepsilon_{22}$. If $(1/2)t\varkappa_{ij} \ll \varepsilon_{kl}$ the influence of the bending on the equilibrium of the shell is in general insignificant. Therefore, in all cases:

$$\varkappa_{11} \approx -e_{21}k_{12} - (\omega_2 A_{1,2} + A_2\omega_{1,1})/A_1A_2,$$
$$\varkappa_{12} \approx -e_{12}k_{11} - (A_2\omega_{2,1} - \omega_1 A_{1,2})/A_1A_2, \quad \overrightarrow{\underleftarrow{1, 2}} \tag{17.16}$$

It should be noted that since $e_{12} = 2\varepsilon_{12} - e_{21}$, we can replace e_{12} by e_{21} in the second formulas after neglecting the shear as before. We thus obtain the expression for the torsion that is given, for instance, in the monograph by V. V. Novozhilov /0.15/.

Introducing (17.11) in (3.29) applied to $\varkappa_{ij}^*$ and using (3.20), (17.5), (17.11) and (17.15), we obtain*:

$$A_1\varkappa_{11}' = -E_1^{\mathrm{I}}e_{11,1} - E_2^{\mathrm{I}}e_{12,1} - E_3^{\mathrm{I}}\omega_{1,1} - E_1'e_{11,1}^{\mathrm{I}} - E_2'e_{12,1}^{\mathrm{I}} -$$
$$- E_3'\omega_{1,1}^{\mathrm{I}} - \omega_2 A_{1,2}/A_2 + k_{11}e_{12}A_1 - e_{21}k_{12}A_1,$$
$$A_1\varkappa_{12}' = -E_1^{\mathrm{I}}e_{21,1} - E_2^{\mathrm{I}}e_{22,1} - E_3^{\mathrm{I}}\omega_{2,1} - E_1'e_{21,1}^{\mathrm{I}} - E_2'e_{22,1}^{\mathrm{I}} -$$
$$- E_3'\omega_{2,1}^{\mathrm{I}} + \omega_1 A_{1,2}/A_2 + k_{12}e_{11}A_1 - e_{12}k_{11}A_1, \tag{17.17}$$
$$A_1\varkappa_{11}'' = -E_1''e_{11,1}^{\mathrm{I}} - E_2''e_{12,1}^{\mathrm{I}} - E_3''\omega_{1,1}^{\mathrm{I}} - E_1'e_{11,1} -$$
$$- E_2'e_{12,1} - E_3'\omega_{1,1},$$
$$A_1\varkappa_{12}'' = -E_1''e_{21,1}^{\mathrm{I}} - E_2''e_{22,1}^{\mathrm{I}} - E_3''\omega_{2,1}^{\mathrm{I}} - E_1'e_{21,1} -$$
$$- E_2'e_{22,1} - E_3'\omega_{2,1},$$

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* Formulas (17.13) and (17.17) for general coordinates have been given in a somewhat different form in the works /0.7/ and /V. 4/.

where the E_i^I are given by

$$E_1' = e_{12}^I \omega_2 + e_{12}\omega_2^I - \omega_1 - e_{22}^I \omega_1 - e_{22}\omega_1^I,$$
$$E_3' = - e_{12}^I e_{21} - e_{12} e_{21}^I + e_{11}(1 + e_{22}^I) + e_{22}(1 + e_{11}^I), \tag{17.18}$$
$$E_1'' = e_{12}\omega_2 - e_{22}\omega_1, \quad E_3'' = e_{11}e_{22} - e_{12}e_{21} \quad \overrightarrow{\underleftarrow{1, 2.}}$$

We may find the bending and twisting moments of the internal forces from the well-known formulas

$$M_{11}^* = D(\varkappa_{11}^* + \nu\varkappa_{22}^*) = M_{11}^I + M_{11}' + M_{11}'', \ldots \tag{17.19}$$

where

$$M_{11}^I = D(\varkappa_{11}^I + \nu\varkappa_{22}^I), \quad M_{11}' = D(\varkappa_{11}' + \nu\varkappa_{22}'),$$
$$M_{11}'' = D(\varkappa_{11}'' + \nu\varkappa_{22}''), \ldots$$

As seen from (7.5) and (17.19), the shearing forces are also composed of three parts

$$N_i^* = N_i^I + N_i' + N_i'', \tag{17.20}$$

which may be calculated from the corresponding equations similar to (7.5). If the external surface density of load does not vary during the deformation (as, for instance, the shell's own weight),

$$\bar{X}^* = X_1^*\bar{e}_1^* + X_2^*\bar{e}_2^* + X_3^*\bar{m}^* = X_1\bar{e}_1 + X_2\bar{e}_2 + X_3\bar{m}, \tag{17.21}$$

where X_1, X_2, X_3 are the projections of the load density on the principal directions of the shell before deformation. Multiplying this by $\bar{e}_i^*$ we find

$$X_1^* = X_1\bar{e}_1\bar{e}_1^* + X_2\bar{e}_2\bar{e}_1^* + X_3\bar{m}\bar{e}_1^*, \ldots.$$

Whence, using (3.19) and (3.20), we obtain

$$X_1^* = X_1(1 + e_{11}^I + e_{11}) + X_2(e_{12}^I + e_{12}) + X_3(\omega_1^I + \omega_1), \tag{17.22}$$
$$X_3^* = X_1E_1^* + X_2E_2^* + X_3E_3^*.$$

Similarly,

$$X_1^I = X_1(1 + e_{11}^I) + X_2e_{12}^I + X_3\omega_1^I, \tag{17.23}$$
$$X_3^I = X_1E_1^I + X_2E_2^I + X_3E_3^I.$$

From these we can determine the components of the surface load along the principal-directions of σ^* and σ^I if the components along the principal directions before deformation are given. If the vector $\bar{X}$ changes during the deformation in such a way that its components along the principal directions of the deformed shell remain invariant, then

$$X_j^* = X_j^I = X_j, \quad j = 1, 2, 3.$$

The hydrostatic pressure, for which

$$X_i^* = X_i^I = X_i = 0 \text{ when } i = 1, 2; \quad X_3^* = X_3^I = X_3 = -p, \tag{17.24}$$

is an example for such a "self-adjusting" load. Here we assumed $p > 0$ for the external pressure.

★ We thus obtain expressions for all the quantities occurring in the equations of equilibrium (7.4) for the state σ^*.

★ Subtracting from the latter the equations of equilibrium (17.8) and (17.9) for the state σ^I, and taking into account (17.14), (17.19), and (17.20) and the equations

$$A_i^* = A_i(1+\varepsilon_{ii}^I)(1+e_{ii}), \quad A_i^I = A_i(1+\varepsilon_{ii}^I), \tag{17.25}$$

we obtain the equations of neutral equilibrium, which contain the components of the additional elastic forces in the principal directions of σ^*. Here we have neglected the terms containing infinitesimals of the second order. We shall also neglect the small quantities of the order of $T_{ij}^I \varepsilon_{ij}^I e_{km}$. Besides, we have neglected before the shears in comparison with unity; therefore, we may also neglect, in the equations of equilibrium, the small terms like $T_{21}^I e_{11}$ or $T_{21}' \varepsilon_{11}^I$, retaining, however, the terms like $T_{11}^I e_{11}$ or $T_{11}' \varepsilon_{11}^I$.

For example

$$A_2^* T_{11}^* - A_2^I T_{11}^I \approx A_2 e_{22} T_{11}^I + A_2(1+\varepsilon_{22}^I) T_{11}',$$
$$A_1^* T_{21}^* - A_1^I T_{21}^I \approx A_1 e_{11} T_{21}^I + A_1(1+\varepsilon_{11}^I) T_{21}' \approx A_1 T_{21}'. \text{ ★}$$

Thus, we bring the equations of neutral equilibrium to the following simpler form

$$\begin{gathered}[A_2 e_{22} T_{11}^I + A_2(1+\varepsilon_{22}^I) T_{11}']_{,1} + (A_1 T_{21}')_{,2} + T_{12}' A_{1,2} - T_{22}^I (A_2 e_{22})_{,1} - \\ - T_{22}' [A_2(1+\varepsilon_{22}^I)]_{,1} + A_1 A_2 [N_1^I \varkappa_{11}' + N_1'(k_{11} + \varkappa_{11}^I) + \\ + N_2^I \varkappa_{12}' + N_2'(k_{12} + \varkappa_{12}^I) + X_1^* - X_1^I] = 0 \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}}\end{gathered} \tag{17.26}$$

$$\begin{gathered}(A_2 N_1')_{,1} + (A_1 N_2')_{,2} - A_1 A_2 [T_{11}^I \varkappa_{11}' + T_{11}'(k_{11} + \varkappa_{11}^I) + \\ + 2T_{12}^I \varkappa_{12}' + 2T_{12}'(k_{12} + \varkappa_{12}^I) + T_{22}^I \varkappa_{22}' + \\ + T_{22}'(k_{22} + \varkappa_{22}^I) - X_3^* + X_3^I] = 0,\end{gathered} \tag{17.27}$$

where N_1' and N_2' may be determined from the equations (7.5) after replacing the $\varkappa_{ij}^*$ by $\varkappa_{ij}'$.

Let us, further, derive the energy criterion for stability of the shell.

★ Let Φ_1^H, Φ_2^H, Φ_3^H be the projections of the external contour force on the axes $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$, and $\tilde{G}$ the bending moment of the external contour forces. From the formulas (8.30) for the first state of equilibrium we find the projections of the internal elastic force on the undeformed axes which we shall mark by the superscripts H:

$$\begin{gathered}T_{11}^{IH} = T_{11}^I(1+e_{11}^I) + T_{12}^I e_{21}^I + N_1^I E_1^I; \quad T_{12}^{IH} = T_{11}^I e_{12}^I + T_{12}^I(1+e_{22}^I) + N_1^I E_2^I, \\ N_1^{IH} = T_{11}^I \omega_1^I + T_{12}^I \omega_2^I + E_3^I N_1^I \qquad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}}.\end{gathered} \tag{17.28}$$

According to (8.32), for this state we have the following static boundary conditions:

$$\begin{gathered}\Phi_1^H = T_{11}^{IH} n_1 + T_{21}^{IH} n_2 - E_1^I \frac{\partial H^I}{\partial s} \qquad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}}, \\ \Phi_3^H = N_1^{IH} n_1 + N_2^{IH} n_2 - E_3^I \frac{\partial H^I}{\partial s}, \\ \tilde{G} = M_{11}^I n_1^2 + 2M_{12}^I n_1 n_2 + M_{22}^I n_2^2 = G^I,\end{gathered} \tag{17.29}$$

where, from (8.12) for $n_i^I \approx n_i$

$$H^I = (M_{11}^I - M_{22}^I) n_1 n_2 - M_{12}^I (n_1^2 - n_2^2), \tag{17.30}$$

and from (8.7)

$$\frac{\partial H^I}{\partial s} = \frac{\partial H^I}{\partial \alpha_1} \cdot \frac{d\alpha_1}{ds} + \frac{\partial H^I}{\partial \alpha_2} \cdot \frac{d\alpha_2}{ds} = -\frac{n_2}{A_1} \cdot \frac{\partial H^I}{\partial \alpha_1} + \frac{n_1}{A_2} \cdot \frac{\partial H^I}{\partial \alpha_2}. \tag{17.31}$$

★ Analogously, neglecting the squares of the infinitesimal quantities, we find for the state σ^*

$$\begin{gathered} T_{11}^{*\text{н}} = T_{11}^{*}(1+e_{11}^{*}) + T_{12}^{*}e_{21}^{*} + N_1^{*}E_1^{*} = T_{11}^{\mathrm{I}}(1+e_{11}^{\mathrm{I}}+e_{11}) + T_{11}'(1+e_{11}^{\mathrm{I}}) + \\ + T_{12}^{\mathrm{I}}(e_{21}^{\mathrm{I}}+e_{21}) + T_{12}'e_{21}^{\mathrm{I}} + N_1^{\mathrm{I}}(E_1^{\mathrm{I}}+E_1) + N_1'E_1^{\mathrm{I}}, \\ T_{12}^{*\text{н}} = T_{11}^{\mathrm{I}}(e_{12}^{\mathrm{I}}+e_{12}) + T_{11}'e_{12}^{\mathrm{I}} + T_{12}^{\mathrm{I}}(1+e_{22}^{\mathrm{I}}+e_{22}) + T_{12}'(1+e_{22}^{\mathrm{I}}) + \\ + N_1^{\mathrm{I}}(E_2^{\mathrm{I}}+E_2') + N_1'E_2^{\mathrm{I}}, \quad N_1^{*\text{н}} = T_{11}^{\mathrm{I}}(\omega_1^{\mathrm{I}}+\omega_1) + T_{11}'\omega_1^{\mathrm{I}} + \\ + T_{12}^{\mathrm{I}}(\omega_2^{\mathrm{I}}+\omega_2) + T_{12}'\omega_2^{\mathrm{I}} + E_3^{\mathrm{I}}(N_1^{\mathrm{I}}+N_1') + E_3'N_1^{\mathrm{I}}. \end{gathered} \tag{17.32}$$

Since the external contour load remains invariant with loss of stability, we have the following static boundary conditions which are similar to (17.29)

$$\begin{gathered} \Phi_1^{\text{н}} = T_{11}^{*\text{н}}n_1 + T_{21}^{*\text{н}}n_2 - (E_1^{\mathrm{I}}+E_1')\frac{\partial H^{\mathrm{I}}}{\partial s} - E_1^{\mathrm{I}}\frac{\partial H'}{\partial s} \quad \overrightarrow{\underleftarrow{1,2}}, \\ \Phi_3^{\text{н}} = N_1^{*\text{н}}n_1 + N_2^{*\text{н}}n_2 - (E_3^{\mathrm{I}}+E_3')\frac{\partial H^{\mathrm{I}}}{\partial s} - E_3^{\mathrm{I}}\frac{\partial H'}{\partial s}, \\ \tilde{G} = (M_{11}^{\mathrm{I}}+M_{11}')n_1^2 + 2(M_{12}^{\mathrm{I}}+M_{12}')n_1n_2 + (M_{22}^{\mathrm{I}}+M_{22}')n_2^2; \\ H' = (M_{11}'-M_{22}')n_1n_2 - M_{12}'(n_1^2-n_2^2), \\ \frac{\partial H'}{\partial s} = -n_2H'_{,1}/A_1 + n_1H'_{,2}/A_2. \bigstar \end{gathered} \tag{17.33}$$

Subtracting (17.29) from (17.33) we find the static boundary conditions for the additional forces and moments

$$\begin{gathered} [T_{11}^{\mathrm{I}}e_{11} + T_{11}'(1+e_{11}^{\mathrm{I}}) + T_{12}^{\mathrm{I}}e_{21} + T_{12}'e_{21}^{\mathrm{I}} + N_1^{\mathrm{I}}E_1' + N_1'E_1^{\mathrm{I}}]n_1 + \\ + [T_{22}^{\mathrm{I}}e_{21} + T_{22}'e_{21}^{\mathrm{I}} + T_{21}^{\mathrm{I}}e_{11} + T_{21}'(e_{11}^{\mathrm{I}}+1) + \\ + N_2^{\mathrm{I}}E_1' + N_2'E_1^{\mathrm{I}}]n_2 - E_1'\,\partial H^{\mathrm{I}}/\partial s - E_1^{\mathrm{I}}\partial H'/\partial s = 0 \quad \overrightarrow{\underrightarrow{1,2}}; \end{gathered} \tag{17.34}$$

$$\begin{gathered} (T_{11}^{\mathrm{I}}\omega_1 + T_{11}'\omega_1^{\mathrm{I}} + T_{12}^{\mathrm{I}}\omega_2 + T_{12}'\omega_2^{\mathrm{I}} + E_3^{\mathrm{I}}N_1' + E_3'N_1^{\mathrm{I}})n_1 + \\ + (T_{22}^{\mathrm{I}}\omega_2 + T_{22}'\omega_2^{\mathrm{I}} + T_{21}^{\mathrm{I}}\omega_1 + T_{21}'\omega_1^{\mathrm{I}} + E_3^{\mathrm{I}}N_2' + E_3'N_2^{\mathrm{I}})n_2 - \\ - E_3'\,\partial H^{\mathrm{I}}/\partial s - E_3^{\mathrm{I}}\partial H'/\partial s = 0; \end{gathered} \tag{17.35}$$

$$M_{11}'n_1^2 + 2M_{12}'n_1n_2 + M_{22}'n_2^2 = 0. \tag{17.36}$$

Besides these conditions, on those parts of the edge contour where the displacements are given, the corresponding geometrical boundary conditions must be satisfied.

Thus, after replacing the forces and deformations by their expressions in terms of u_1, u_2, w, from (17.26) and (17.27), we obtain a system of three homogeneous linear equations with respect to these quantities for the homogeneous boundary conditions (17.34), (17.35), and (17.36). Evidently, the solution of this boundary problem will be different from zero only for certain combinations of T_{ij}^{I} and M_{ij}^{I} which characterize the critical load.

Since in both the equations of neutral equilibrium and the boundary conditions most of the coefficients are mainly variable, an exact solution for this boundary problem may be obtained only in the simplest cases. Therefore, it is often necessary to resort to approximate methods; among these, the energy criterion for stability, based on the principle of virtual displacements, is the most widely used.

In § 10 we deduced the equilibrium condition (10.14) where δA is the elementary work of the external load in the possible displacements from the state of equilibrium determined by (10.13), and δW is the variation of the specific work of deformation, given by (10.11).

Let us at first apply (10.14) to the first state of equilibrium. The specific work of deformation corresponding to this state is:

$$W^{\mathrm{I}} = \frac{1}{2}\left\{K\left[(\varepsilon_{11}^{\mathrm{I}} + \varepsilon_{22}^{\mathrm{I}})^2 - 2(1-\nu)(\varepsilon_{11}^{\mathrm{I}}\varepsilon_{22}^{\mathrm{I}} - \varepsilon_{12}^{\mathrm{I}2})\right] + \right.$$
$$\left. + D\left[(\varkappa_{11}^{\mathrm{I}} + \varkappa_{22}^{\mathrm{I}})^2 - 2(1-\nu)(\varkappa_{11}^{\mathrm{I}}\varkappa_{22}^{\mathrm{I}} - \varkappa_{12}^{\mathrm{I}2})\right]\right\}. \tag{17.37}$$

Therefore,

$$\delta W^{\mathrm{I}} = K\left[(\varepsilon_{11}^{\mathrm{I}} + \varepsilon_{22}^{\mathrm{I}})(\delta\varepsilon_{11}^{\mathrm{I}} + \delta\varepsilon_{22}^{\mathrm{I}}) - (1-\nu)(\varepsilon_{11}^{\mathrm{I}}\delta\varepsilon_{22}^{\mathrm{I}} + \right.$$
$$\left. + \varepsilon_{22}^{\mathrm{I}}\delta\varepsilon_{11}^{\mathrm{I}} - 2\varepsilon_{12}^{\mathrm{I}}\delta\varepsilon_{12}^{\mathrm{I}})\right] + D\left[(\varkappa_{11}^{\mathrm{I}} + \varkappa_{22}^{\mathrm{I}})(\delta\varkappa_{11}^{\mathrm{I}} + \delta\varkappa_{22}^{\mathrm{I}}) - \right.$$
$$\left. - (1-\nu)(\varkappa_{11}^{\mathrm{I}}\delta\varkappa_{22}^{\mathrm{I}} + \varkappa_{22}^{\mathrm{I}}\delta\varkappa_{11}^{\mathrm{I}} - 2\varkappa_{12}^{\mathrm{I}}\delta\varkappa_{12}^{\mathrm{I}})\right], \tag{17.38}$$

where, according to (17.4)

$$\delta\varepsilon_{11}^{\mathrm{I}} = (1 + e_{11}^{\mathrm{I}})\,\delta e_{11}^{\mathrm{I}} + e_{12}^{\mathrm{I}}\delta e_{12}^{\mathrm{I}} + \omega_1^{\mathrm{I}}\delta\omega_1^{\mathrm{I}},$$
$$2\delta\varepsilon_{12}^{\mathrm{I}} = (1 + e_{22}^{\mathrm{I}})\,\delta e_{12}^{\mathrm{I}} + (1 + e_{11}^{\mathrm{I}})\,\delta e_{21}^{\mathrm{I}} + e_{12}^{\mathrm{I}}\delta e_{22}^{\mathrm{I}} +$$
$$+ e_{21}^{\mathrm{I}}\delta e_{11}^{\mathrm{I}} + \omega_1^{\mathrm{I}}\delta\omega_2^{\mathrm{I}} + \omega_2^{\mathrm{I}}\delta\omega_1^{\mathrm{I}}.$$

As has been pointed out in § 10, the last term in the right-hand side of (19.13) is zero.

An elementary virtual displacement from the state σ^{I} is given by

$$\delta\bar{v}^{\mathrm{I}} = \bar{e}_1\delta u_1^{\mathrm{I}} + \bar{e}_2\delta u_2^{\mathrm{I}} + \bar{m}\delta w^{\mathrm{I}}. \tag{17.39}$$

We assume that the external surface force $\overline{X}$ density may be resolved into the components $\overline{X}_1$ and $\overline{X}_2$. The first component is independent of the deformation (for instance, the shell's own weight) and the second varies in such a manner that its projections on the principal directions of the deformed shell remain constant (for instance, a hydrostatic pressure). Consequently, the external surface force for the state σ^{I} equals

$$\overline{X}^{\mathrm{I}} = \overline{X}_1 + \overline{X}_2^{\mathrm{I}} = X_{11}\bar{e}_1 + X_{12}\bar{e}_2 + X_{13}\bar{m} +$$
$$X_{21}\bar{e}_1^{\mathrm{I}} + X_{22}\bar{e}_2^{\mathrm{I}} + X_{23}\bar{n}^{\mathrm{I}}, \tag{17.40}$$

where $X_{11}, \ldots, X_{23}$ are quantities which do not depend on the deformation. Using (3.19) and (3.20) for $\bar{e}_i^{\mathrm{I}}$ and m^{I}, and multiplying (17.39) by (17.40) we find the elementary work of this force in the virtual displacement

$$\overline{X}^{\mathrm{I}}\delta\bar{v}^{\mathrm{I}} = \left[X_{11} + X_{21}(1 + e_{11}^{\mathrm{I}}) + X_{22}e_{21}^{\mathrm{I}} + X_{23}E_1^{\mathrm{I}}\right]\delta u_1^{\mathrm{I}} +$$
$$+ \left[X_{12} + X_{21}e_{12}^{\mathrm{I}} + X_{22}(1 + e_{22}^{\mathrm{I}}) + X_{23}E_2^{\mathrm{I}}\right]\delta u_2^{\mathrm{I}} +$$
$$+ (X_{13} + X_{21}\omega_1^{\mathrm{I}} + X_{22}\omega_2^{\mathrm{I}} + X_{23}E_3^{\mathrm{I}})\,\delta w^{\mathrm{I}}. \tag{17.41}$$

The equivalent moment of the external surface forces with respect to a point of the middle surface is usually small, and we shall therefore neglect it. Let the external load for the state σ^{I}, equal $\overline{\Phi}^{\mathrm{I}}$, be expressed according to (17.40) in terms of the quantities Φ_{ij} which do not depend on the deformation. We shall consider that the external bending moment $\widetilde{G}$ is independent of the variations of the deformation. Then $\overline{\Phi}^{\mathrm{I}}\delta\bar{v}^{\mathrm{I}}$ may be determined from formulas like (17.41)

$$\bar{m}^{\mathrm{I}} = \bar{e}_1E_1^{\mathrm{I}} + \bar{e}_2E_2^{\mathrm{I}} + \bar{m}E_3^{\mathrm{I}}, \quad \delta m^{\mathrm{I}} = \bar{e}_1\delta E_1^{\mathrm{I}} + \bar{e}_2\delta E_2^{\mathrm{I}} + \bar{m}\delta E_3^{\mathrm{I}},$$
$$\bar{n}^{\mathrm{I}} = n_1^{\mathrm{I}}\bar{e}_1^{\mathrm{I}} + n_2^{\mathrm{I}}\bar{e}_2^{\mathrm{I}} \Rightarrow n_1\bar{e}_1^{\mathrm{I}} + n_2\bar{e}_2^{\mathrm{I}}. \tag{17.42}$$

Using (3.20) once more, we find:

$$\bar{n}^{\mathrm{I}}\delta\bar{m}^{\mathrm{I}} = \left[n_1(1 + e_{11}^{\mathrm{I}}) + n_2e_{21}^{\mathrm{I}}\right]\delta E_1^{\mathrm{I}} +$$
$$+ \left[n_1e_{12}^{\mathrm{I}} + n_2(1 + e_{22}^{\mathrm{I}})\right]\delta E_2^{\mathrm{I}} + (n_1\omega_1 + n_2\omega_2^{\mathrm{I}})\,\delta E_3^{\mathrm{I}}; \tag{17.43}$$

$$
\begin{aligned}
&\delta\omega_2^{I} + \omega_2^{I}\delta e_{12}^{I} - (1 + e_{22}^{I})\,\delta\omega_1^{I} - \omega_1^{I}\delta e_{22}^{I} \quad \overrightarrow{\underleftarrow{1,2}}, \\
&\delta E_3^{I} = (1 + e_{11}^{I})\,\delta e_{22}^{I} + (1 + e_{22}^{I})\,\delta e_{11}^{I} - e_{12}^{I}\delta e_{21}^{I} - e_{21}^{I}\delta e_{12}^{I}.
\end{aligned}
\tag{17.44}
$$

We have thus obtained the necessary expressions for all the quantities occurring in the variational equation of equilibrium (10.14) for the state σ^{I}:

$$
\begin{aligned}
&\iint_{(\sigma)} \overline{X}^{I}\,\delta\overline{v}^{I} d\sigma + \int_{C} (\overline{\Phi}^{I}\,\delta\overline{v}^{I} + \overline{G}\,\overline{n}^{I}\delta\,\overline{m}^{I})\,ds = \\
&\qquad = \iint_{(\sigma)} \delta W^{I} A_1 A_2 d\alpha_1 d\alpha_2 .
\end{aligned}
\tag{17.45}
$$

By successively transforming this equation (as shown in § 10) and equating to zero the coefficients of the virtual displacements, we obtain the equations of equilibrium and the static boundary conditions. We shall now write down the same equation for the equilibrium state σ^*. For this purpose we replace everywhere, in the previous formulas, u_1^I, u_2^I and w^I by $u_1^I + u_1$, $u_2^I + u_2$ and $w^I + w$, retaining the squares of the additional displacements. Since e_{ij}^* and w_i^* are linear functions of $u_i^I + u_i$ and $w^I + w$, we put:

$$
e_{ij}^* = e_{ij}^{I} + e_{ij}; \quad \omega_i^* = \omega_i^{I} + \omega_i .
$$

Introducing these and (17.12)-(17.18) into (10.11), we find

$$
W^* = W^{I} + W'' + W' ,
\tag{17.46}
$$

where

$$
\begin{aligned}
W' = {} & K\{(\varepsilon_{11}^{I} + \varepsilon_{22}^{I})(\varepsilon_{11}^{\prime} + \varepsilon_{22}^{\prime}) - (1-\nu)(\varepsilon_{11}^{I}\varepsilon_{22}^{\prime} + \varepsilon_{22}^{I}\varepsilon_{11}^{\prime} - \\
& - 2\varepsilon_{12}^{I}\varepsilon_{12}^{\prime})] + D[(\varkappa_{11}^{I} + \varkappa_{22}^{I})(\varkappa_{11}^{\prime} + \varkappa_{22}^{\prime}) - \\
& - (1-\nu)(\varkappa_{11}^{I}\varkappa_{22}^{\prime} + \varkappa_{22}^{I}\varkappa_{11}^{\prime} - 2\varkappa_{12}^{I}\varkappa_{12}^{\prime})], \\
W'' = {} & K\Big[\frac{1}{2}(\varepsilon_{11}^{\prime} + \varepsilon_{22}^{\prime})^2 + (\varepsilon_{11}^{I} + \varepsilon_{22}^{I})(\varepsilon_{11}^{\prime\prime} + \varepsilon_{22}^{\prime\prime}) - \\
& - (1-\nu)(\varepsilon_{11}^{I}\varepsilon_{22}^{\prime\prime} + \varepsilon_{22}^{I}\varepsilon_{11}^{\prime\prime} + \varepsilon_{11}^{\prime}\varepsilon_{22}^{\prime} - \varepsilon_{12}^{\prime 2} - 2\varepsilon_{12}^{I}\varepsilon_{12}^{\prime\prime})\Big] + \\
& + D\Big[\frac{1}{2}(\varkappa_{11}^{\prime} + \varkappa_{22}^{\prime})^2 + (\varkappa_{11}^{I} + \varkappa_{22}^{I})(\varkappa_{11}^{\prime\prime} + \varkappa_{22}^{\prime\prime}) - \\
& - (1-\nu)(\varkappa_{11}^{I}\varkappa_{22}^{\prime\prime} + \varkappa_{22}^{I}\varkappa_{11}^{\prime\prime} + \varkappa_{11}^{\prime}\varkappa_{22}^{\prime} - \varkappa_{12}^{\prime 2} - 2\varkappa_{12}^{I}\varkappa_{12}^{\prime\prime})\Big].
\end{aligned}
\tag{17.47}
$$

When performing an infinitesimal virtual displacement of the state σ^*

$$
\delta\overline{v} = \overline{e}_1\delta u_1 + \overline{e}_2\delta u_2 + \overline{m}\delta w,
$$

we have to vary only the additional displacement. Here $\delta W^I = 0$, and δW may be determined from (17.37) and (17.38) after replacing δe_{ij}^I and $\delta\omega_i^I$ by δe_{ij} and $\delta\omega_i$ respectively.

The surface force for the state σ^* is

$$
\begin{aligned}
\overline{X}^* = \overline{X}_1 + \overline{X}_2^* = {} & X_{11}\overline{e}_1 + X_{12}\overline{e}_2 + X_{13}\overline{m} + \\
& + X_{21}\overline{e}_1^* + X_{22}\overline{e}_2^* + X_{23}\overline{m}^*.
\end{aligned}
\tag{17.48}
$$

Consequently, using (3.19) and (3.20) we obtain the expression for the elementary work of the surface load in the virtual displacement $\delta\bar{v}$:

$$
\begin{aligned}
\bar{X}^*\delta\bar{v} = [X_{11} + X_{21}(1+e_{11}^{\mathrm{I}}+e_{11}) + X_{22}(e_{21}^{\mathrm{I}}+e_{21}) + \\
+ X_{23}(E_1^{\mathrm{I}}+E_1')]\delta u_1 + [X_{12} + X_{21}(e_{12}^{\mathrm{I}}+e_{12}) + \\
+ X_{22}(1+e_{22}^{\mathrm{I}}+e_{22}) + X_{23}(E_2^{\mathrm{I}} - E_2')]\delta u_2 + \\
+ [X_{13} + X_{21}(\omega_1^{\mathrm{I}}+\omega_1) + X_{22}(\omega_2^{\mathrm{I}}+\omega_2) + X_{23}(E_3^{\mathrm{I}}+E_3')]\delta w.
\end{aligned} \tag{17.49}
$$

The external contour force for the state σ^* is

$$
\bar{\Phi}^* = \Phi_{11}\bar{e}_1 + \Phi_{12}\bar{e}_2 + \Phi_{13}\bar{m} + \Phi_{21}\bar{e}_1^* + \Phi_{22}\bar{e}_2^* + \Phi_{23}\bar{m}^*, \tag{17.50}
$$

and the elementary work of this force may be determined from formulas similar to (17.49).

We obtain further

$$
\begin{aligned}
\bar{n}^*\delta\bar{m}^* = [n_1(1+e_{11}^{\mathrm{I}}+e_{11}) + n_2(e_{21}^{\mathrm{I}}+e_{21})]\delta(E_1'+E_1'') + \\
+ [n_1(e_{12}^{\mathrm{I}}+e_{12}) + n_2(1+e_{22}^{\mathrm{I}}+e_{22})]\delta(E_2'+E_2'') + \\
+ [n_1(\omega_1^{\mathrm{I}}+\omega_1) + n_2(\omega_2^{\mathrm{I}}+\omega_2)]\delta(E_3'+E_3''),
\end{aligned} \tag{17.51}
$$

where

$$
\begin{aligned}
\delta E_1' = e_{12}^{\mathrm{I}}\delta\omega_2 + \omega_2^{\mathrm{I}}\delta e_{12} - \delta\omega_1 - e_{22}^{\mathrm{I}}\delta\omega_1 - \omega_1^{\mathrm{I}}\delta e_{22}; \\
\delta E_1'' = e_{12}\delta\omega_2 + \omega_2\delta e_{12} - e_{22}\delta\omega_1 - \omega_1\delta e_{22}, \quad \overrightarrow{\underleftarrow{1,2}} \\
\delta E_3' = -e_{12}^{\mathrm{I}}\delta e_{21} - e_{21}^{\mathrm{I}}\delta e_{12} + (1+e_{22}^{\mathrm{I}})\delta e_{11} + (1+e_{11}^{\mathrm{I}})\delta e_{22}; \\
\delta E_3'' = e_{11}\delta e_{22} + e_{22}\delta e_{11} - e_{12}\delta e_{21} - e_{21}\delta e_{12}.
\end{aligned} \tag{17.52}
$$

If the shell is also in equilibrium in the state σ^*, then in addition to (17.45) it is necessary to satisfy the variational equation

$$
\begin{aligned}
\iint_{(\sigma)} \bar{X}^*\delta\bar{v}\,d\sigma + \int_C (\bar{\Phi}^*\delta\bar{v} + \bar{G}\,\bar{n}^*\,\bar{m}^*)\,ds - \\
- \delta\iint_{(\sigma)} A_1A_2(W'+W'')\,d\alpha_1 d\alpha_2 = 0.
\end{aligned} \tag{17.53}
$$

As may be seen from formulas (17.47)-(17.52), both sides of this equation contain quantities of the first and second order of smallness with respect to the infinitesimal displacements u_i and w and their variations. It may easily be seen that the quantities of the first order of smallness contain infinitesimal factors only in the form of the variations δu_i and δw, the coeffflents of these variations being equal to the corresponding coefficients of δu_i^{I} and δw^{I} in equation (17.45). Whence, remembering that we are considering arbitrary variations of additional displacement which occur without disturbing the geometrical relations, and that the state σ^{I} is a state of equilibrium, we conclude that the sum of the first order terms in the left-hand side of (17.53) is zero. Consequently, if we neglected the quantities of the second order of smallness we should obtain, according to (17.45) and (17.53), the same equations of equilibrium and static boundary conditions which characterize the state of equilibrium σ^{I} but do not enable us to estimate the stability of this state. Equating to zero the remaining terms in the left-hand side of equation (17.53), i. e., the sum of terms of the second order of smallness, we obtain the equation:

$$
\begin{aligned}
&\iint_{(\sigma)} [(X_{21}e_{11} + X_{22}e_{21} + X_{23}E_1')\,\delta u_1 + (X_{21}e_{12} + X_{22}e_{22} + X_{23}E_2')\,\delta u_2 + \\
&\quad + (X_{21}\omega_1 + X_{22}\omega_2 + X_{23}E_3')\,\delta w]\,d\sigma + \int_C [(\Phi_{21}e_{11} + \Phi_{22}e_{22} + \Phi_{23}E_1')\,\delta u_1 + \\
&+ (\Phi_{21}e_{12} + \Phi_{22}e_{22} + \Phi_{23}E_2')\,\delta u_2 + (\Phi_{21}\omega_1 + \Phi_{22}\omega_2 + \Phi_{23}E_3')\,\delta w]\,ds + \\
&+ \int_C \widetilde{G}\,\{(n_1e_{11} + n_2e_{21})\,\delta E_1' + (n_1e_{12} + n_2e_{22})\,\delta E_2' + (n_1\omega_1 + n_2\omega_2)\,\delta E_3' + \qquad (17.54)\\
&\quad + [n_1(1 + e_{11}^{\mathrm{I}}) + n_2e_{21}^{\mathrm{I}}]\,\delta E_1'' + [n_1e_{12}^{\mathrm{I}} + n_2(1 + e_{22}^{\mathrm{I}})]\,\delta E_2'' + \\
&\qquad + [n_1\omega_1^{\mathrm{I}} + n_2\omega_2^{\mathrm{I}}]\,\delta E_3''\}\,ds - \delta \iint_{(\sigma)} W''A_1A_2\,d\alpha_1 d\alpha_2 = 0.
\end{aligned}
$$

This equation gives the energy criterion for the determination of the limit of stability of the equilibrium of the shell*.

Thus, when the critical load is reached, the variational equations (17.45) and (17.54) must be satisfied simultaneously. For solving them approximately by the Ritz-Timoshenko method, we shall take at first the analytic expressions of the projections of displacements in the state σ^{I}

$$u_1^{\mathrm{I}} = C_{11}^{\mathrm{I}}u_{11}^{\mathrm{I}} + C_{12}^{\mathrm{I}}u_{12}^{\mathrm{I}} + \ldots,\quad w^{\mathrm{I}} = C_{31}w_1^{\mathrm{I}} + C_{32}w_2^{\mathrm{I}} + \ldots, \qquad (17.55)$$

where each of the functions u_{ij}^{I} and w_j^{I} satisfies the geometrical boundary conditions, and C_{ij}^{I} are the amplitudes of displacement which we wish to determine. Besides, we shall assume that these functions are dependent on some parameters, m, n, p, ... which have also to be determined. Substituting (17.55) in (17.45), using the expressions for virtual displacements

$$\delta u_1^{\mathrm{I}} = \sum_{k=1}^{\infty} \left(u_{1k}^{\mathrm{I}}\,\delta C_{1k}^{\mathrm{I}} + C_{1k}^{\mathrm{I}}\,\frac{\partial u_{1k}^{\mathrm{I}}}{\partial m}\,\delta m + \ldots \right), \ldots \qquad (17.56)$$

and equating to zero the coefficients of the variations $\delta C_{ik}^{\mathrm{I}}$, δm, ..., we obtain an infinite system of equations for C_{ik}^{I}, m, ..., corresponding to the state of equilibrium of the shell.

Proceeding analogously for the determination of the additional displacements, we obtain from the variational equation (17.54) a system of equations for the parameters C_{ik}, m, n, ..., which characterize the displacement at the beginning of the loss of stability of the state of equilibrium σ^{I}. The condition of compatibility of these equations gives the required relations between the critical magnitudes of the load parameters. In concluding this section, it should be noted that the fundamental equations of the theory of stability of shells derived here are highly complicated. This is because we tried not to restrict, for the present, the range of application by retaining the generality of the deductions. In particular, we did not make any assumption on the nature of the deformed state and the state of stress of the shell before the loss of stability, assuming that the state σ^{I} is necessarily a membrane state which results from the initial state σ by large bending.

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* The energy criterion for the stability of equilibrium of an elastic body has been formulated by E. Trefftz in his works /V.16/ and /V.13/, assuming that the external forces are independent of the deformation. This criterion was developed for shells in the monograph /0.13/.

The generalization of this criterion for the case where the external forces depend on the deformation has been given in /0.14/ (see also /V.14/).

§ 18. The Stability of a Non-Shallow Shell under Small Bending

Let us assume that the contour forces and the surface forces acting on a shell in the state σ^I do not bend it, or bend it very slightly. Then

$$\frac{1}{2} t\varkappa_{ij}^I << \varepsilon_{ij}^I, \quad \omega_i^I << 1, \quad e_{ij}^I << 1, \tag{18.1}$$

and the first state of the shell is a membrane state. This state may be determined from the linear theory of shells in membrane state

$$\varepsilon_{11}^I \approx e_{11}^I; \quad 2\varepsilon_{12}^I \approx e_{12}^I + e_{21}^I \qquad \overrightarrow{\underleftarrow{1, 2}}; \tag{18.2}$$

$$(A_2^I T_{11}^I)_{,1} + (A_1^I T_{21}^I)_{,2} + T_{12}^I A_{1,2}^I - T_{22}^I A_{2,1}^I + A_1^I A_2^I X_1^I = 0 \quad \overrightarrow{\underleftarrow{1, 2}}; \tag{18.3}$$

$$k_{11} T_{11}^I + 2k_{12} T_{12}^I + k_{22} T_{22}^I - X_3^I = 0, \tag{18.4}$$

where e_{ij}^I and T_{ij}^I are given by (17.3) and (17.6). The additional deformation of the shell with buckling is characterized by additional elongations and shears $\varepsilon_{ij}^{\prime}$ and by additional changes in curvature $\varkappa_{ij}^{\prime}$, whose expressions (17.13) and (17.17) become less simple.

Since, in this case, according to (17.3) and (17.18)

$$E_i^I \approx -\omega_i^I, \; |E_i^I| \ll 1, \; E_3^I \approx 1 + \varepsilon_{11}^I + \varepsilon_{22}^I \approx 1,$$
$$E_i^{\prime} \approx -\omega_i, \; E_3^{\prime}\, \omega_{1,1}^I \approx 0,$$

neglecting (as in calculating the change in curvature), the elongations and shears in comparison with unity, and using (18. 1) and (18. 2), we find

$$\varkappa_i^{\prime} \approx \varkappa_i, \; \varkappa_{12}^{\prime} \approx \varkappa_{12}, \qquad \varepsilon_{11}^{\prime} \approx e_{11} + e_{12}^I e_{12} + \omega_1^I \omega_1, \tag{18.5}$$

where $\varkappa_i$ and $\varkappa_{12}$ are given from (17.16). The approximate value of ε_{12} may be calculated, as in the general case, from (17.13).

The equations of neutral equilibrium (17.26) and (17.27) can also be simplified because there it is possible to neglect the terms containing $\varkappa_{ij}^I$ and N_i^I and to put $N_i^{\prime} \approx N_i$, where N_i are shearing forces expressed in terms of $\varkappa_{ij}$ by the linear formulas (7.5). Besides, in the following we shall assume that the hydrostatic pressure (17.24) is the only surface load. Therefore, with small bending the equations of neutral equilibrium become

$$\begin{aligned}&[A_2 e_{22} T_{11}^I + A_2(1 + \varepsilon_{22}^I) T_{11}^{\prime}]_{,1} + (A_1 T_{21}^{\prime})_{,2} + T_{12}^{\prime} A_{1,2} - \underline{T_{22}^I (A_2 e_{22})_{,1}} - \\ &\quad - T_{22}^{\prime} [A_2(1 + \underline{\varepsilon_{22}^I})]_{,1} + A_1 A_2 (N_1 k_{11} + N_2 k_{12}) = 0;\end{aligned} \tag{18.6}$$

$$\begin{aligned}&\underline{[A_1 e_{11} T_{22}^I} + A_1(1 + \underline{\varepsilon_{11}^I}) T_{22}^{\prime}]_{,2} + (A_2 T_{12}^{\prime})_{,1} + T_{21}^{\prime} A_{2,1} - T_{11}^I (A_1 e_{11}) \\ &\quad - T_{11}^{\prime} [A_1(1 + \varepsilon_{11}^I)]_{,2} + A_1 A_2 (N_1 k_{12} + N_2 k_{22}) = 0;\end{aligned} \tag{18.7}$$

$$\begin{aligned}&(A_2 N_1)_{,1} + (A_1 N_2)_{,2} - A_1 A_2 (T_{11}^I \varkappa_{11} + 2T_{12}^I \varkappa_{12} + T_{22}^I \varkappa_{22} + \\ &\quad + T_{11}^{\prime} k_{11} + 2T_{12}^{\prime} k_{12} + T_{22}^{\prime} k_{22}) = 0\end{aligned} \tag{18.8}$$

With the loss of stability of the membrane state of equilibrium, a transition occurs to a state of equilibrium for which the bending stresses begin to play a considerable role. Therefore, the additional bending elongations cannot be small in comparison with the additional membrane elongations. On this basis we assume that

$$t\varkappa \sim \varepsilon, \tag{18.9}$$

where $\varkappa$ and ε are the maxima of $\varkappa_{ij}$ and ε_{ij} respectively.

Let
$$A_1 \sim A_2 \sim L, \quad t/R \sim \varepsilon_p << 1. \tag{18.10}$$

(where L is the characteristic linear dimension of the middle surface of the shell). In this section we shall assume that $L \sim R$ (R being the smallest of the R_i), (18.11), i. e., we consider the stability of the entire shell or of a major part of it.

Evidently, owing to the smallness of the bending in the state σ^1 the quantities characterizing this state vary smoothly or in particular are constant:

$$\varepsilon^1_{ij,m} \lesssim \varepsilon^1_{ij}, \quad T^1_{ij,m} \lesssim T^1_{ij} \qquad i, j, m = 1, 2. \tag{18.11}$$

Let the quantities characterizing the additional deformation be varying upon differentiation with respect to the dimensionless coordinate α_i, so that

$$\varepsilon'_{ij,1} \sim \varepsilon'_{ij} \cdot \varepsilon_p^{-\lambda_1}, \quad \varepsilon'_{ij,2} \sim \varepsilon'_{ij} \cdot \varepsilon_p^{-\lambda_2}, \ldots \varepsilon_p << 1. \tag{18.12}$$

where λ_1 and λ_2 are real numbers.

Let us consider the various possibilities.

A. Let

$$\lambda_1 > 0, \quad \lambda_2 \geqslant 0. \tag{18.13}$$

In other words, we assume that the parameters of the additional deformation increase considerably upon differentiation with respect to at least one of the coordinates. For a non-shallow shell such a deformation is possible when the surface σ^* divides into a large number of "waves" each of which may be considered as a shallow shell. Therefore, the additional deflection w is large in comparison with the components of the tangential displacement, and the rotations ω_i are large in comparison with the elongations and the rotations e_{12} and e_{21}.

This important case of loss of stability will be examined in detail in the following chapters. It should only be noted here that in this case not only the shear may be neglected in comparison with unity—as we did in setting up the general theory—but also the elongations. Thus, after making all the simplifications which can be used for shallow shells, as we did in § 15, we obtain the equations of neutral equilibrium*:

$$(A_2 T_{11})_{,1} + (A_1 T_{21})_{,2} + T_{12} A_{1,2} - T_{22} A_{2,1} = 0 \quad \overleftrightarrow{1,2}; \tag{18.14}$$

$$D\Delta\Delta w + T^1_{11}\varkappa_{11} + 2T^1_{12}\varkappa_{12} + T^1_{22}\varkappa_{22} + T_{11}k_{11} + 2T_{12}k_{12} + T_{22}k_{22} = 0, \tag{18.15}$$

where

$$T_{11} = K(\varepsilon_{11} + \nu\varepsilon_{22}), \quad T_{12} = K(1-\nu)\varepsilon_{12}, \quad \varepsilon_{11} = e_{11},$$
$$2\varepsilon_{12} = e_{12} + e_{21}, \tag{18.16}$$

and $\varkappa_{ij}$ are given by (15.7).

---

* See monograph /0.13/ and the articles /V.4/ and /V.1/.

B. Condition (18.9) may also be satisfied for the case when

$$\lambda_1 < 0, \ \lambda_2 = 0, \quad (\text{or } \lambda_1 = 0, \ \lambda_2 < 0), \tag{18.17}$$

i. e., when the parameters of the additional deformation decrease considerably upon differentiation with respect to α_1.

★ As a concrete example, let us consider the stability of a shell of revolution referred to the lines of curvature and let the lines α_2 = const be the meridians. Here $A_{1,2} = 0$.

The equations (18.6) and (18.7) may be replaced, in the first approximation, by the equations (18.14), from which we obtain an estimate of the order of magnitude of T_{ij}:

$$T_{11,1} \sim T_{21,2} \sim T_{21}, \quad T_{21,1} \sim T_{22,2} \sim T_{22}$$

or

$$T_{22} \sim T_{21}\varepsilon_\rho^{-\lambda_1}, \ T_{12} = T_{21} \sim T_{11}\varepsilon_\rho^{-\lambda_1}, \ T_{22} \sim T_{11}\varepsilon_\rho^{-2\lambda_1}. \tag{18.18}$$

Let us assume, for instance, $\lambda_1 = -\frac{1}{2}$. Then

$$T_{22} \sim T_{11}\varepsilon_\rho, \quad T_{22} \ll T_{11}$$

Therefore, from (18.16), in the first approximation

$$\varepsilon_{22} = -\nu\varepsilon_{11}, \ T_{11} \sim Et\,\varepsilon_{11}, \quad T_{12} \sim Et\,\varepsilon_{11}\varepsilon_\rho^{-\lambda_1} \sim Et\,\varepsilon_{12}.$$

Consequently,

$$\varepsilon_{12} \sim \varepsilon_{11}\varepsilon_\rho^{-\lambda_1}, \ \varepsilon_{12} \sim \varepsilon_{11}\sqrt{\varepsilon_\rho} \ \left(\text{where } \lambda_1 = -\frac{1}{2}\right). \tag{18.19}$$

Here $\varkappa_{22}$ is the largest of the $\varkappa_{ij}$. In order that conditions (18.9) should be satisfied,

$$t\varkappa_{22} \sim \varepsilon_{22} \sim \varepsilon_{11}. \tag{18.20}$$

must hold, where

$$(A_2N_1)_{,1} \ll (A_1N_2)_{,2}, \ (A_1N_2)_{,2} \sim D\varkappa_{22,2} \sim Et^3\varkappa_{22} \sim Et^2\varepsilon_{22}. \tag{18.21}$$

Let us consider two further possibilities:

1. Let $k_{12} = 0$, $k_{11} = k_{12} \sim 1/R$. Then (18.19), (18.20), and the condition of compatibility (3.32) will not be simultaneously satisfied. In fact, the left-hand side of this equation is the largest quantity (of an order of magnitude ε_{22}) while its right-hand side equals

$$\sim A_1A_2\varkappa_{22}k_{11} \sim R\varkappa_{22} \sim t\varkappa_{22}R\,t.$$

In other words, it contradicts (18.20).

2. Let $k_{12} = k_{11} = 0$, $k_{22} \sim 1/R$, i. e., σ is a developable surface. Then condition (3.32) is satisfied, but the first of Codazzi's equations (3.35) can hold only for A_2 = const.

★ Thus, the case considered may occur only for the loss of stability of a cylindrical shell, for which

$$A_1A_2T'_{22}k_{22} \sim REt\,\varepsilon_{22}\,\varepsilon_p^{-2\lambda_1}.$$

This is of the same order of magnitude as (18.21), if

$$t/R \sim \varepsilon_p^{-2\lambda_1},\ \lambda_1 = -1/2;\quad t/R \sim \varepsilon_p.$$

Besides, according to (18.8), (18.21), and (18.20)

$$A_1A_2T^{\mathrm{I}}_{22}\varkappa_{22} \sim (A_1N_2)_{,2} \sim Et^3\varepsilon_{22},$$

if with the loss of stability

$$T^{\mathrm{I}}_{22} \sim \frac{Et^3\varkappa_{22}}{R^2\varkappa_{22}} \sim \frac{Et^3}{R^2}.$$

We also find that for the critical load, we must have

$$T^{\mathrm{I}}_{11} \sim \frac{Et^3}{R^3}\varepsilon_p^{2\lambda_1} \sim \frac{Et^2}{R},\quad T^{\mathrm{I}}_{12} \sim \frac{Et^2}{R}\left(\frac{t}{R}\right)^{1/4}. \tag{18.22}$$

On the basis of this analysis, the equations (18.6) and (18.7) may be simplified by neglecting the underlined terms like $T^{\mathrm{I}}_{22}\,\varepsilon_{22}$ and $T'_{22}\,\varepsilon^{\mathrm{I}}_{22}$. ★

The validity of the conclusions, which we arrived at for case B by analysis of the orders of magnitude, will be corroborated in the following section by a more precise examination of the problem of stability of a very long cylindrical shell, i. e., the special case for which, after neglecting the shear in comparison with unity, it is necessary to retain the squares of the elongations and the shearing forces in the equations for the tangential components. We therefore consider that the theory of shells, and especially the theory of stability proposed above on the basis of the assumption that in the analysis of the state of stress of the shell the shear may be neglected in comparison with unity is justified.

§ 19. The Stability of a Long Cylindrical Shell of Circular Cross-Section

Let the position of a point on the middle surface be specified by the axial coordinate $x = R\alpha_1$ and by the polar angle $\theta = \alpha_2$. Then the line element on the surface σ is given by

$$ds^2 = dx^2 + R^2 d\theta^2 = R^2(d\alpha_1^2 + d\alpha_2^2).$$

Therefore, in this case,

$$A_1 = A_2 = R;\ k_{11} = k_{12} = 0;\ k_{22} = 1/R. \tag{19.1}$$

We shall examine the stability of a shell under a normal external pressure p uniformly distributed over its lateral surface, an axial compressive force T^I and a shearing force S^I, being uniformly distributed over its end sections $\alpha_1 = 0$ and $\alpha_1 = 1/R$. We shall neglect the influence of the nature of securing the edges because of their remoteness. Evidently, from (17.8) and (17.9)

$$\begin{gathered} T^I_{11} = \text{const} = -T^I,\ T^I_{12} = T^I_{21} = S^I,\ T^I_{22} = -pR, \\ \varepsilon^I_{11} = u^I_{1,1}/R = \text{const},\ \varepsilon^I_{22} = (u^I_{2,2} + w^I) : R = \text{const}, \\ w^I = \text{const},\ u^I_{2,2} = 0,\ u^I_{1,2} = Re^I_{21} = 0, \\ e^I_{12} = 2\varepsilon^I_{12} = u^I_{2,1} : R,\ \omega^I_1 = w^I_{,1} : R = 0,\ \omega^I_2 = w^I_{,2} : R = 0 \end{gathered} \tag{19.2}$$

Consequently, according to (17.13) and (18.5)

$$\varepsilon^{\cdot}_{11} = 2\varepsilon^I_{12} e_{12} + e_{11};\ \varepsilon^{\cdot}_{22} = e_{22};\ 2\varepsilon^{\cdot}_{12} = 2\varepsilon_{12} + \varepsilon^I_{11} e_{21} + \varepsilon^I_{22} e_{12}; \tag{19.3}$$

$$Re_{22} = u_{2,2} + w,\qquad 2\varepsilon_{12}R = u_{2,1} + u_{1,2},\qquad Re_{11} = u_{1,1}. \tag{19.4}$$

From (17.16)

$$\varkappa_{11} = -w_{,11}/R^2,\ \varkappa_{12} = (u_{2,1} - w_{,12})/R^2,\ \varkappa_{22} = (u_{2,2} - w_{,22})/R^2 \tag{19.5}$$

Taking into account (17.6), (17.14), and (18.16), we introduce (19.3) in the equations (18.6) and (18.7). We shall neglect the underlined terms and also all the cubes of elongations and rotations and the infinitesimals of second order. Thus, from (18.6) we obtain the approximate equation:

$$\begin{gathered} T_{11,1} + T_{21,2} - T^I e_{22,1} + K[2\varepsilon^I_{12} e_{12} + \varepsilon^I_2 (e_{11} + \nu e_{22})]_{,1} + \\ + \frac{K(1-\nu)}{2} (\varepsilon^I_{11} e_{21} + \varepsilon^I_{22} e_{12})_{,2} = 0. \end{gathered}$$

But

$$Ke_{11} \sim Ke_{22} \sim T_{11},\qquad K\varepsilon^I_{12} \sim S^I << T^I,$$

therefore, in neglecting ε^I_{11} in comparison with unity, it is necessary to retain only the largest among the small terms of the equation, namely, the term $\frac{K(1-\nu)}{2} (\varepsilon^I_{11} e_{21} + \varepsilon^I_{22} e_{12})_{,2}$. In view of our considerations of § 18,

$$\begin{gathered} e_{21} = 2\varepsilon_{12} - e_{12} \approx -e_{12},\ T^I_{22} << T^I_{11},\ T_{22} = K(\varepsilon^I_{22} + \nu \varepsilon^I_{11}) \approx 0, \\ \varepsilon^I_{22} \approx -\nu \varepsilon^I_{11},\ T^I_{11} = K(\varepsilon^I_{11} + \nu \varepsilon^I_{22}) \approx K(1-\nu^2)\varepsilon^I_{11}, \end{gathered}$$

we find:

$$\frac{K(1-\nu)}{2}(\varepsilon^{\rm I}_{11}e_{21}+\varepsilon^{\rm I}_{22}e_{12})_{,2}\approx-\frac{K(1-\nu)}{2}(1+\nu)\varepsilon^{\rm I}_{11}e_{12,2}\approx\frac{T^{\rm I}}{2}e_{12,2}.$$

Consequently, the first equation of equilibrium becomes

$$T_{11,1}+T_{21,2}+\frac{1}{2}T^{\rm I}e_{12,2}=0. \tag{19.6}$$

In the same manner, we obtain the second equation

$$T_{12,1}+T_{22,2}-\frac{1}{2}T^{\rm I}e_{12,1}+N_2=0. \tag{19.7}$$

We replace (18.8) by the approximate equation

$$N_{1,1}+N_{2,2}+R(T^{\rm I}\varkappa_{11}-2S^{\rm I}\varkappa_{12}+pR\varkappa_{22})-T_{22}=0. \tag{19.8}$$

Introducing (19.4) and (19.5) in (19.6)-(19.8) we obtain a system of three linear differential equations for u_1, u_2, and w. We shall assume that the end sections of the shell are hinged to frames which prevent displacements u_2 and w in the plane of the section, but do not hinder the axial displacements u_1, i.e., we shall satisfy the boundary conditions:

$$u_2=w=w_{,11}=0 \quad \text{for} \quad \alpha_1=0,\ \alpha_1=l/R. \tag{19.9}$$

These conditions and the equations of neutral equilibrium may be satisfied by assuming, for $S^{\rm I}=0$,

$$\begin{gathered}u_1=A\cos m\alpha_1\cos n\alpha_2,\quad u_2=B\sin m\alpha_1\sin n\alpha_2,\\ w=C\sin m\alpha_1\cos n\alpha_2,\end{gathered} \tag{19.10}$$

where n is the integral number of waves that are formed on the periphery of the shell,

$m=k\pi R/l$ (k being an integer)

If $S^{\rm I}\neq 0$, the equations of equilibrium can be satisfied by assuming:

$$u_1=A\sin(m\alpha_1+n\alpha_2),\ u_2=B\sin(m\alpha_1+n\alpha_2),\ w=C\cos(m\alpha_1+n\alpha_2). \tag{19.11}$$

In this case the boundary conditions are not satisfied, which is not very important if l/R is large.

In both cases the introduction of (19.10) or (19.11) in the equations (19.6)-(19.8) leads to a system of linear homogeneous equations for A, B, and C. The condition of compatibility gives the required relation between $T^{\rm I}$, $S^{\rm I}$, p, m, n, for the determination of the critical load. A detailed study of this relation may be found, for example, in /0.13/ and /0.16/. It is shown there that the corresponding value of m for the minimal load is so small that $m^2 \ll n^2$, if the shell is long. Therefore, to simplify the calculations, we shall neglect m^2 in comparison with n^2 in the second order terms of the equations (19.6)-(19.8) containing the flexural rigidity D or $T^{\rm I}_{ij}$ as factors. We shall assume, in particular, that

$$\begin{gathered}M_{11,11}=D(\varkappa_{11}+\nu\varkappa_{22})_{,11}\approx0,\quad M_{12,12}=D(1-\nu)\varkappa_{12,12}\approx0,\\ N_{1,1}\approx0,\quad N_2\approx M_{22,2}/R\approx D\varkappa_{22,2}/R.\end{gathered}$$

All the same, we should have introduced these simplifications for the determining equation, which becomes, after these simplifications

$$\begin{gathered}Etm^4+Dn^4(n^2-1)^2/R^2=T^{\rm I}m^2n^2(n^2+1)-2S^{\rm I}mn^2(n^2-1)+\\ +pRn^4(n^2-1).\end{gathered} \tag{19.12}$$

In order to calculate the order of magnitude of the quantities ε_{ij} in the equations (19.6) and (19.7), retaining the principal terms we find the approximate equations

$$
\begin{aligned}
A(m^2+n^2)^2 &= -C(\nu m^3 \quad mn^2),\\
B(m^2+n^2)^2 &= -C[(2+\nu)m^2 n + n^3],
\end{aligned}
$$

where

$$
\begin{aligned}
&Re_{12} = u_{2,1} = B\,m\cos(m\alpha_1 + n\alpha_2) = B\,m\,w/C \simeq -w\,m/n,\\
&Re_{11} = u_{1,1} = A\,m\cos(m\alpha_1 + n\alpha_2) = A\,mw/C \sim m^2 w/n^2,\\
&Re_{22} = u_{2,2} + w = (Bn + C)\cos(m\alpha_1 + n\alpha_2) \sim m^2 w/n^2,\\
&2\varepsilon_{12}R = u_{2,1} + u_{1,2} = (An + Bm)\,w/C \simeq -(2+2\nu)\,m^3 w/n^3.
\end{aligned}
\tag{19.13}
$$

It can be seen from these that the shear is really small in comparison with the elongations, as we had assumed in the special case considered.

The elongations are also small in comparison with the rotations e_{11} and ω_i. However, a systematic neglect of these may lead to a considerable error because $e_{12,1} = (2\varepsilon_{12} \cdot e_{21})_{,1} \simeq -e_{21,1} = -u_{1,21} = -e_{11,2}$.

Let us examine a few special cases.

1. Crumpling of a circular cylindrical shell under an external normal pressure.

Let $T^I = S^I = 0$.

According to (19.12), m = 0 corresponds to the minimum value of the pressure p, i.e., in this case the approximate solution coincides with the exact solution of Maurice Levi /V.12/ for a circular ring:

$$p_k = 3D/R^3,\ n = 2. \tag{19.14}$$

2. Buckling of a shell*. Assuming S^I = p = 0 in (19.12), we obtain

$$T^I = \frac{Etm^2}{n^2(n^2+1)} + \frac{Dn^2(n^2-1)^2}{R^2m^2(n^2+1)}.$$

If $n > 1$ the minimal axial force will be reached for

$$n = 2,\quad m_k^2 = 6t/R\sqrt{3(1-\nu^2)}. \tag{19.15}$$

and will be

$$T_k^I = 0{,}2\sqrt{3}\,Et^2/R\sqrt{1-\nu^2}. \tag{19.16}$$

For very long pipes

$$n_k = 1,\ T^I = Et\pi^2R^2/2l^2. \tag{19.17}$$

i.e., we obtain Euler's well-known formula for he buckling of a pipe considered as a beam with hinged ends.

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* This problem has been examined by S. P. Timoshenko (see [V.6] or [0.16]).

We have generalized this solution for the case of a long shell of an elliptic cross-section of small eccentricity*. It followed that the critical compressive force for a shell with an elliptic cross-section of infinitesimal eccentricity is:

$$T^{\mathrm{I}}_{\kappa,3} = T^{\mathrm{I}}_{\kappa}\left(1 + \frac{1}{4}e^2\right), \tag{19.18}$$

where $T^{\mathrm{I}}_{\mathrm{K}}$ is the critical force for a shell of circular cross section with R = a,

$ae = \sqrt{a^2 - b^2}$ (a being the semi-major axis of the ellipse)

3. Twisting**. In this case $T^{\mathrm{I}} = p = 0$. From the condition of minimum load $\partial S^{\mathrm{I}}/\partial m = 0$ we obtain:

$$\begin{gathered} m_\kappa^2 = n^2(n^2-1)\,t/6R\sqrt{1-\nu^2}, \quad n = 2, \\ S^{\mathrm{I}}_\kappa = \frac{1}{3\sqrt{2}}\,Et^{5/2}/R^{3/2}(1-\nu^2)^{3/4}. \end{gathered} \tag{19.19}$$

For a long shell with an elliptic cross-section*** of small eccentricity

$$S^{\mathrm{I}}_{\kappa,\,\mathfrak{s}} = S^{\mathrm{I}}_\kappa\left(1 + \frac{1}{6}e^2\right), \tag{19.20}$$

where $S^{\mathrm{I}}_{\mathrm{K}}$ is the critical force for R = a.

As may be seen from (19.15) and (19.19),

$$m_\kappa = \frac{\pi R}{l} \sim \sqrt{\frac{t}{R}}. \tag{19.21}$$

The shells which satisfy this condition will be called "long shells".

★ Using the solutions obtained for the particular problems we shall convince ourselves of the correctness of the magnitudes of the critical forces (18.22) calculated earlier from qualitative considerations of the equations of equilibrium. We shall also use these solutions for a more complete elucidation of the peculiarity of the types of loss of stability of shells considered above.

According to (19.13)

$$\begin{gathered} T_{11} = \frac{Et}{1-\nu^2}(e_{11} + \nu e_{22}) \simeq \frac{Et}{R}\cdot\frac{m^2 w}{n^2}, \quad T_{11,1} \sim \frac{Et}{R}\, m^3 w, \\ T_{12} = \frac{Et}{1+\nu}\, \varepsilon_{12} \simeq -\frac{Et}{R}\cdot\frac{m^3 w}{n^3}, \quad T_{12,1} \sim \frac{Et}{R}\, m^4 w, \quad T_{12,2} \sim T_{12}, \\ T_{22} = \frac{Et}{1-\nu^2}(e_{22} + \nu e_{11}) \simeq \frac{Et}{R}\cdot\frac{m^4 w}{n^4} \sim T_{22,2}. \end{gathered} \tag{19.22}$$

Furthermore, by (19.13), (19.15), and (19.16)

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* See /V.3/.

** This problem has been examined by Shverin in article /V.8/.

*** See /V.3/.

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$$T^1 e_{12,1} \sim \frac{Et^2}{R}\cdot\frac{wm^3}{R} \sim \frac{Et}{R}\, m^4 w. \qquad T^1 e_{12,2} \sim \frac{Et}{R}\, m^4 w.$$

$$N_2 \sim \frac{D\varkappa_{22,2}}{R} \sim \frac{Et^3 w}{R^3} \sim \frac{Et}{R}\, m^4 w.$$

Consequently, all the terms in the equation (19.6) are quantities of the same order as $Etm^3 w/R$. Similarly, all the terms in the equation (19.7) are of the order of $Etm^4 w/R$ where $m \sim \sqrt{t/R}$. In this case it is therefore not possible to neglect the "small" quantities. We observe that, in addition to these, in the equations (19.6) and (19.7) one also often retains terms which are really small*. Such a term is, for example, $S^1 e_{12,1}$ in equation (19.6) which, compared with the principal terms of this equation, has a factor of an order of magnitude m^2 or t/R.

The terms dependent on the normal pressure in (19.6) and (19.7) are still less important. But they must be retained in (19.8) because according to (19.5):

$$\varkappa_{11} \sim m^2 \varkappa_{22}, \quad \varkappa_{12} \sim m\varkappa_{22}.$$

Consequently,

$$T^1 \varkappa_{11} \sim S^1 \varkappa_{12} \sim pR\varkappa_2.$$

It may be seen from this analysis that the theory of shells, based on the systematic neglect of the elongation and shears in comparison with unity, may lead to substantial error in the value of the critical load, only for the loss of stability of a long cylindrical shell under axial compression, because in this case the principal terms in the equations of equilibrium cancel each other, and the approximate equations lose their validity**. The validity of applying the equations of the theory of non-shallow shells to the study of the stability of long cylindrical shells has to be especially examined if one wants to express the elastic forces and moments in terms of the deformation by first approximation formulas of the type (17.6), as these equations are obtained by neglecting the quantities of order t/R in comparison with unity. On comparing these approximate formulas with the exact expressions (0.19) for moments and forces, or with the expressions derived on the basis of the Kirchhoff-Love hypothesis without any simplification (0.15), it may be seen that the expression for T_{12} contains, in addition to the principal term we have retained, also terms as small as mN_2, while T_{22} and T_{11} contain also terms as small as N_2. For the case under consideration, the principal terms cancel each other only in T_{22}. Besides, the small terms in the expressions $T_{11,1}$ and $T_{12,2}$ are of the same order as mN. In $T_{12,1}$ they are still smaller, being of the order of $m^2 N_2$.

Therefore, the errors arising from the use of first approximation formulae for these quantities are smaller than t/R as compared with unity. On the other hand, the error in $T_{22,2}$ in equation (19.7) is of the same order as the other terms in this equation. But this fact does not lead to an error in the equation for determining the critical loads, because the same error in the expression T_{22} also enters the equation (19.8). In order to avoid these errors, it is possible to eliminate T_{22} and T_{12} from the latter equation. In fact, after differentiating equation (19.8) twice with respect to α_2, adding to it the equation (19.7) differentiated once with respect to α_2, and afterwards subtracting (19.6) differentiated once with respect to α_1, we obtain, instead of (19.8) the equation:

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* See, for example, equation (63.1) in /0.13/, or equations (2.54) in /0.16/.

** We have pointed this out in /V.4/.

★
$$N_{2,222}+N_{2,2}+R\left[T^{1}\left(\varkappa_{11,22}-\frac{\varepsilon_{12,12}}{R}\right)-\right.$$
$$\left.-2S^{1}\varkappa_{12,12}+pR\varkappa_{22,22}\right]-T_{11,11}=0. \tag{19.23}$$

By substituting here for T_{11} its approximate expression (19.22), we may only introduce an error which is $1/m^2$ times the other terms of the equation. Thus, we obtain an equation for the determination of the critical loads which is accurate as far as t/R in comparison with unity. ★

In the following we shall examine the stability of shells of medium length for which $m = \pi R/l$ is not small in comparison with unity, so that the particular case considered above does not apply; therefore, equations (19.6) and (19.7) may be replaced by the approximate equations (18.14) which are simpler. Then a large number of waves are formed by buckling, so that a non-shallow shell is divided into shallow parts to which the theory of stability of shallow shells may be applied; it is set forth in detail in what follows.

Chapter VI

THE NON-LINEAR THEORY OF SHALLOW SHELLS

§ 20. Shallow Shells with Initial Deformations*

Let A_i and k_{ij} be the corresponding quantities which characterize the middle surface σ of the shell before deformation, referred to orthogonal curvilinear coordinates α_i. It is usually a surface of a regular geometrical form. During the manufacturing and erecting of the shell, initial deformations occur inevitably, leading to deviations from the surface σ. These irregularities of the middle surface may appear prior to the application of the surface load or contour load, or owing to a non-uniform temperature distribution in the shell. In this case, initial stresses may also appear (for example, thermal stresses). Irrespective of their origin, these deviations from σ will be called here initial deformations.

Let us consider that the surface σ transforms into the surface σ^0 under the initial deformation. We shall denote by u_1^0, u_i^0, and w^0 the components of the corresponding displcament. We assume that this displacement is at most of the same order of magnitude as the thickness of the shell.

We also assume either that the surface σ is shallow or that the initial deviations of σ are rapidly varying so that σ^0 divides into shallow parts.

Thus, the slowly varying irregularities are excluded from our considerations, as for instance the slight ellipticity of a cylindrical shell with circular cross-section. This limitation is not substantial from the point of view of the applications of the theory, the more so as it is seen from § 19 that the slowly varying irregularities have little effect on the state of stress of the shell. The limitation mentioned is, however, important for the simplification of the theory because it enables us to neglect the effect of the tangential displacements u_i on the initial rotations ω_i^0 and also the changes of curvature $\varkappa_{ij}^0$: consequently, these quantities and also the elongations may be determined by the formulas (15.5), (15.6), and (15.7) of the theory of shallow shells:

$$\begin{gathered}\omega_1^0 = w^0_{,1} : A_1, \quad \varepsilon^0_{11} = e^0_{11} + \frac{1}{2}\omega_1^{0\,2}, \quad 2\varepsilon^0_{12} = e^0_{12} + e^0_{21} + \omega_1^0\omega_2^0, \\ A_1A_2\varkappa^0_{11} = -A_2\omega^0_{1,1} - \omega^0_2 A_{1,2}, \\ A_1A_2\varkappa^0_{12} = -A_2\omega^0_{2,1} + \omega^0_1 A_{1,2}, \qquad \overset{\rightarrow}{\underset{\leftarrow}{1,2}},\end{gathered} \tag{20.1}$$

where e_{ij}^0 are quantities determined from formulas similar to (3.5). We note that initial bending displacements which cause initial membrane stresses are also admitted.

Let u_1^I, u_2^I, w^I be the projections of the displacement due to the load on the principal directions $\bar{e}_1$, $\bar{e}_2$, and $\overline{m}$ of the surface σ. Due to this displacement σ^0

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* The theory of such shells has been given in a rather different form in /VI.1/.

transforms into σ^1. We shall assume that there may be a loss of stability of this state and a transition to a new state σ^* with an additional displacement, whose projections u_1, u_2, w on the same axes are infinitesimals. Then the projections of the total displacement, the corresponding elongations and the changes in curvature will be:

$$u_1^* = u_1^0 + u_1^1 + u_1, \ldots, \quad \varepsilon_{11}^* = \varepsilon_{11}^0 + \varepsilon_{11}^1 + \varepsilon_{11}' = e_{11}^* + \frac{1}{2}\omega_1^{*2},$$
$$2\varepsilon_{12}^* = e_{12}^* + e_{21}^* + \omega_1^*\omega_2^* = 2(\varepsilon_{12}^0 + \varepsilon_{12}^1 + \varepsilon_{12}'),$$
$$\varkappa_{ij}^* = \varkappa_{ij}^0 + \varkappa_{ij}^1 + \varkappa_{ij}, \; e_{ij}^* = e_{ij}^0 + e_{ij}^1 + e_{ij}, \; \omega_1^* = \omega_1^0 + \omega_1^1 + \omega_1 \,. \; \overrightarrow{\underleftarrow{1, 2}} \tag{20.2}$$

The quantities e_{ij}^1 and e_{ij} may be expressed linearly in terms of the corresponding displacements according to (3.5). The elongations before the loss of stability and the rotations which depend on the load may be determined by the following formulas:

$$\varepsilon_{11}^1 = e_{11}^1 + \frac{1}{2}\omega_1^{1^2} + \omega_1^0\omega_1^1,$$
$$2\varepsilon_{12}^1 = e_{12}^1 + e_{21}^1 + \omega_1^1\omega_2^1 + \omega_1^0\omega_2^1 + \omega_2^0\omega_1^1,$$
$$\omega_1^1 = w_{,1}^1 : A_1 \qquad \overrightarrow{\underleftarrow{1, 2}}\,. \tag{20.3}$$

We also obtain the additional elongations $\varepsilon_{ij}' + \varepsilon_{ij}''$ for loss of stability, where

$$\varepsilon_{11}' = e_{11} + \omega_1(\omega_1^0 + \omega_1^1),$$
$$2\varepsilon_{12}' = e_{12} + e_{21} + \omega_1(\omega_2^0 + \omega_2^1) + \omega_2(\omega_1^0 + \omega_1^1) \tag{20.4}$$

are first order infinitesimals, and

$$\varepsilon_{11}'' = \frac{1}{2}\omega_1^2, \quad 2\varepsilon_{12}'' = \omega_1\omega_2$$

are second order infinitesimals. They may be neglected in setting up the equations of equilibrium, but are to be retained when setting up the energy criterion for stability.

We shall determine the changes in curvature $\varkappa_{ij}^1$ and $\varkappa_{ij}$ from linear formulas like (20.1) replacing ω_i^0 by ω_i^1 and ω_i respectively.

The membrane forces and bending moments due to the load before the loss of stability are respectively

$$T_{11}^1 = K(\varepsilon_{11}^1 + \nu\varepsilon_{22}^1), \quad T_{12}^1 = K(1-\nu)\varepsilon_{12}^1,$$
$$M_{11}^1 = D(\varkappa_{11}^1 + \nu\varkappa_{22}^1), \; M_{12}^1 = D(1-\nu)\varkappa_{12}^1 \quad \overrightarrow{\underleftarrow{1, 2}} \tag{20.5}$$

The additional forces and moments will be given by similar formulas:

$$T_{11}' = K(\varepsilon_{11}' + \nu\varepsilon_{22}'), \ldots, \quad M_{12}' = D(1-\nu)\varkappa_{12}. \tag{20.6}$$

It should be noted that when deriving these formulas we neglected the elongations in comparison with unity and assumed that the shell is shallow before and after the deformation, or that it divides during deformation into shallow parts.

The equations of equilibrium of the internal forces and moments for the initial state of deformation will be obtained from equations like (15.8) and (15.9), assuming that the surface forces equal zero:

$$(A_2 T_{11}^0)_{,1} + (A_1 T_{12}^0)_{,2} + T_{12}^0 A_{1,2} - T_{22}^0 A_{2,1} = 0 \qquad \overrightarrow{\underleftarrow{1,2}},$$

$$(A_2 N_1^0)_{,1} + (A_1 N_2^0)_{,2} - A_1 A_2 [T_{11}^0 k_{11}^0 + 2 T_{12}^0 k_{12}^0 + T_{22}^0 k_{22}^0] = 0;$$

$$k_{ij}^0 = k_{ij} + \varkappa_{ij}^0; \tag{20.7}$$

$$A_i N_i^0 = D(\varkappa_{11}^0 + \varkappa_{22}^0)_{,i} = -D(\Delta w^0)_{,i} \qquad i = 1,\ 2. \tag{20.8}$$

We introduce the force function ψ^0 by the formulas

$$A_2 T_{11}^0 = (\psi_{,2}^0 / A_2)_{,2} + A_{2,1} \psi_{,1}^0 / A_1^2 \qquad \overrightarrow{\underleftarrow{1,2}},$$

$$A_1 A_2 T_{12}^0 = -\psi_{,12}^0 + A_{2,1} \psi_{,2}^0 / A_2 + A_{1,2} \psi_{,1}^0 / A_1. \tag{20.9}$$

Then the equations (20.7) will be satisfied in the assumed approximation and the function ψ^0 must satisfy the equation of compatibility

$$\Delta\Delta\psi^0 - Et[\varkappa_{12}^{0^2} - \varkappa_{11}^0(\varkappa_{22}^0 + k_{22}) - \varkappa_{22}^0 k_{11} + 2\varkappa_{12}^0 k_{12}] = 0. \tag{20.10}$$

If the stresses appearing in the shell during the manufacture and assembly are removed by annealing or by other methods, but the initial deformation remains, then

$$T_{ij}^0 = 0, \quad N_i^0 = 0,$$

and the equations (20.7) are identically satisfied and (20.8)-(20.10) become meaningless. In this case, the irregularities are residual and it may be considered that they appeared after a single normal displacement w, assuming $u_1^0 = u_2^0 = 0$.

We shall determine the displacements due to the load by equations similar to (15.8) and (15.15) for the state of equilibrium σ^I, taking (20.7) into account

$$(A_2 T_{11}^I)_{,1} + (A_1 T_{12}^I)_{,2} + T_{12}^I A_{1,2} - T_{22}^I A_{2,1} + A_1 A_2 X_1^I = 0 \quad \overrightarrow{\underleftarrow{1,2}}; \tag{20.11}$$

$$D\Delta\Delta w^I + T_{11}^0 \varkappa_{11}^I + 2T_{12}^0 \varkappa_{12}^I + T_{22}^0 \varkappa_{22}^I +$$

$$+ T_{11}^I k_{11}^I + 2T_{12}^I k_{12}^I + T_{22}^I k_{22}^I - X_3^I = 0,$$

$$k_{ij}^I = k_{ij} + \varkappa_{ij}^0 + \varkappa_{ij}^I, \quad A_i N_i^I = -D(\Delta w^I)_{,i}, \tag{20.12}$$

where T_{ij}^I are given by (20.3) and (20.5).

The additional displacements u_i and w, which appear with the loss of stability, will be determined from (15.8) and (15.9) assuming

$$T_{ij}^* = T_{ij}^0 + T_{ij}^I + T_{ij}', \quad k_{ij}^* = k_{ij} + \varkappa_{ij}^0 + \varkappa_{ij}^I + \varkappa_{ij}. \tag{20.13}$$

Thus, from equations (20.7)-(20.12) we obtain the equations of neutral equilibrium

$$(A_2 T_{11}')_{,1} + (A_1 T_{21}')_{,2} + T_{12}' A_{1,2} - T_{22}' A_{2,1} +$$

$$+ A_1 A_2 (X_1^* - X_1^I) = 0 \quad \overrightarrow{\underleftarrow{1,2}}, \tag{20.14}$$

$$D\Delta\Delta w + (T_{11}^0 + T_{11}^I)\varkappa_{11} + 2(T_{12}^0 + T_{12}^I)\varkappa_{12} + (T_{22}^0 + T_{22}^I)\varkappa_{22} +$$
$$+ T_{11}^{\prime} k_{11}^I + 2T_{12}^{\prime} k_{12}^I + T_{22}^{\prime} k_{22}^I - X_3^{*} + X_3^I = 0, \qquad (20.15)$$
$$A_i N_i^{\prime} = -D(\Delta w)_{,i} \qquad i = 1,\ 2.$$

If the shell has no saddle points, it is more convenient for applications to consider the state of stress σ^I as the sum of a membrane state and a state of stress of the same type as the edge effect. We shall denote by u_1^{IB}, u_2^{IB}, w^{IB} and $u_1^{I\kappa}$, $u_2^{I\kappa}$, $w^{I\kappa}$ the components of the corresponding displacements. Thus,

$$u_1^I = u_1^{IB} + u_1^{I\kappa}, \quad T_{ij}^I = T_{ij}^{IB} + T_{ij}^{I\kappa}, \quad T_{11}^{IB} = K(\epsilon_{11}^{IB} + \nu\epsilon_{22}^{IB}). \qquad (20.16)$$

We shall determine the elongations corresponding to the membrane state from the usual linear theory. The elongations due to the edge effect will be given by formulas like (20.3), replacing $u_1^I, \ldots$ by u_1^{IK} $, \ldots.$

We shall neglect the changes in curvature in the membrane state. Accordingly, $\varkappa_{ij}^I$ will be given in terms of w^{IK} by formulas similar to (20.1). For the membrane deformation, we have the equations

$$(A_2 T_{11}^{IB})_{,1} + (A_1 T_{12}^{IB})_{,2} + T_{12}^{IB} A_{1,2} - T_{22}^{IB} A_{2,1} + A_1 A_2 X_1^I = 0 \quad \overleftrightarrow{1,2},$$
$$T_{11}^{IB} k_{11} + 2T_{12}^{IB} k_{12} + T_{22}^{IB} k_{22} - X_3^I = 0. \qquad (20.17)$$

Using (20.16) and (20.17) we may derive from (20.11) and (20.12) the equations

$$(A_2 T_{11}^{I\kappa})_{,1} + (A_1 T_{12}^{I\kappa})_{,2} + T_{12}^{I\kappa} A_{1,2} - T_{22}^{I\kappa} A_{2,1} = 0 \quad \overleftrightarrow{1,2}; \qquad (20.18)$$

$$D\Delta\Delta w^{I\kappa} + T_{11}^0 \varkappa_{11}^I + 2T_{12}^0 \varkappa_{12}^I + T_{22}^0 \varkappa_{22}^I + T_{11}^{IB}(\varkappa_{11}^0 + \varkappa_{11}^I) +$$
$$+ 2T_{12}^{IB}(\varkappa_{12}^0 + \varkappa_{12}^I) + T_{22}^{IB}(\varkappa_{22}^0 + \varkappa_{22}^I) + \qquad (20.19)$$
$$+ T_{11}^{I\kappa} k_{11}^I + 2T_{12}^{I\kappa} k_{12}^I + T_{22}^{I\kappa} k_{22}^I = 0.$$

The equation (20.18) may be satisfied by introducing the force function $\psi^{I\kappa}$ according to formulas like (15.11), i.e., by assuming

$$A_2 T_{11}^{I\kappa} = (\psi_{,2}^{I\kappa}/A_2)_{,2} + A_{2,1}\psi_{,1}^{I\kappa}/A_1^2,$$
$$A_1 A_2 T_{12}^{I\kappa} = -\psi_{,12}^{I\kappa} + A_{2,1}\psi_{,2}^{I\kappa}/A_2 + A_{1,2}\psi_{,1}^{I\kappa}/A_1 \quad \overleftrightarrow{1,2}. \qquad (20.20)$$

From the condition of compatibility, similar to (15.16), after replacing k_{ij} by $k_{ij} + \varkappa_{ij}^0$ the equation

$$\Delta\Delta\psi^{I\kappa} - Et\left[\varkappa_{12}^{I2} + 2\varkappa_{12}^I k_{12}^0 - \varkappa_{11}^I(k_{22}^0 + \varkappa_{22}^I) - \varkappa_{22}^I k_{11}^0\right] = 0. \qquad (20.21)$$

The same equation may also be obtained from (15.16) after replacing ψ by $\psi^0 + \psi^{I\kappa}$ and $\varkappa_{ij}$ by $\varkappa_{ij}^0 + \varkappa_{ij}^I$ and subtracting the corresponding equation for the initial state*.

The equations (20.19) and (20.21) form, with (20.20) and (20.1), a system of two equations for $\psi^{I\kappa}$ and $w^{I\kappa}$.

If the shell is shallow and $X_1^I = X_2^I = 0$, we may introduce the force function without resolving the state of stress σ^I into a membrane part and a part similar to the edge effect. In this case, we may satisfy the equations (20.11) by introducing a force function ψ^I according to formulas like (20.20). In the equation (20.21) we

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* The equations (20.12) and (20.21) for shells without initial irregularities have been obtained in /VI.12/. See also /VI.13/.

assume $\psi^{Ik} = \psi^{I}$ and instead of (20.19) we use the initial equation (20.12). In particular, with $u_1^0 = u_2^0 = w^0 = 0$ we may derive from these the equations (15.15) and (15.16). The equations of neutral equilibrium (20.14)-(20.15) may also be simplified if the surface load is self-adjusting, i. e., if the conditions

$$X_i^* = X_i^I, \quad i = 1, 2, 3, \tag{20.22}$$

are satisfied. In this case the equations (20.14) may be satisfied by expressing the additional membrane forces T'_{ij} in terms of ψ' by equations similar to (20.20):

$$\begin{gathered} A_2 T'_{11} = (\psi'_{,2}/A_2)_{,2} + A_{2,1}\psi'_{,1}/A_1^2, \\ A_1 A_2 T'_{12} = -\psi'_{,12} + A_{2,1}\psi'_{,2}/A_2 + A_{1,2}\psi'_{,1}/A_1 \quad \overleftarrow{\overline{1,2}}. \end{gathered} \tag{20.23}$$

Here the condition of compatibility deriving from (20.21) must be satisfied after replacing ψ^{Ik} by $\psi^{Ik} + \psi'$ and $\varkappa^{I}_{ij}$ by $\varkappa^{I}_{ij} + \varkappa_{ij}$ and subtracting (20.21) from the equation thus obtained. We find

$$\begin{gathered} \Delta\Delta\psi' - Et\left(2\varkappa_{12}k_{12}^{I} - \varkappa_{11}k_{22}^{I} - \varkappa_{22}k_{11}^{I}\right) = 0 \\ k_{ij}^{I} = k_{ij} + \varkappa_{ij}^{0} + \varkappa_{ij}^{I}. \end{gathered} \tag{20.24}$$

Introducing (20.22), (20.23), and expressions like (20.21) in (20.15) we obtain one more equation for w.

If the initial stresses corresponding to the initial deformations are zero, it must be assumed that $T^{0}_{ij} = 0$.

Let us formulate the static boundary conditions. Just as at the end of § 15, we shall consider two possible cases.

A. Let us assume that the edge contour is free. In the state σ^{I} the contour will be under the action of external forces and moments Φ_n^I, Φ_τ^I, Φ_3^I and $\tilde{G}{}^{I}$. At the contour the elastic forces and the moments must satisfy the conditions obtained from (15.17)-(15.19), after replacing there the quantities marked by an asterisk by quantities with the index I.

Let the contour load and the surface load be self-adjusting so that their projections on the principal directions of the deformed shell remain the same after the loss of stability. Then

$$\Phi_n^* = \Phi_n^I + \Phi_n', \quad \Phi_n' = 0, \ldots, \tag{20.25}$$

where Φ_n' are additional external loads.

Introducing these, and also

$$T_{ij}^* = T_{ij}^I + T_{ij}', \quad M_{ij}^* = M_{ij}^I + M_{ij}'$$

in equations (15.17)-(15.19) and taking into account the static boundary conditions for the state σ^{I}, we find the required boundary conditions for the additional forces and moments. If the edge contour is given by the line $\alpha_1 =$ const, the static boundary conditions assume the form of the corresponding conditions in the linear theory

$$T_{11}' = 0, \quad T_{12}' = 0, \quad N_1' = 0, \quad M_{11}' = 0. \tag{20.26}$$

It has however, to be remembered that the additional forces depend both on the initial deformation and the initial stresses, according to (20.4) and (20.6).

B. Let us assume that the external contour load is given in terms of forces and moments along the principal directions of the undeformed shell, which will be marked with the superscript "H". In other words, we shall assume that there are given the quantities Φ_n^H, Φ_τ^H, Φ_3^H and $\vec{G}^H$ which remain constant for an additional deformation. Then we shall also refer the internal forces and moments to the undeformed state, using formulas similar to (15.20). Thus we have for the state σ^0:

$$T_{ij}^{0H} \approx T_{ij}^0,\ M_{ij}^{0H} \approx M_{ij}^0,\ N_1^{0H} = N_1^0 + T_{11}^0\omega_1^0 + T_{12}^0\omega_2^0,\ \dots \tag{20.27}$$

Taking this into account, we find for the state σ^1

$$\begin{gathered} T_{ij}^{1H} \approx T_{ij}^1;\ M_{ij}^{1H} \approx M_{ij}^1,\ N_1^{1H} = N_1^1 + T_{11}^0\omega_1^1 + T_{12}^0\omega_2^1 + \\ + T_{11}^1(\omega_1^0 + \omega_1^1) + T_{12}^1(\omega_2^0 + \omega_2^1). \end{gathered} \tag{20.28}$$

The state of equilibrium after buckling will finally be characterized by the relations

$$\begin{gathered} (T_{ij}')^H \approx T_{ij}',\ (M_{ij}')^H \approx M_{ij}',\ (N_1')^H = N_1' + T_{11}'(\omega_1^0 + \omega_1^1) + \\ + (T_{11}^0 + T_{11}^1)\omega_1 + T_{12}'(\omega_2^0 + \omega_2^1) + (T_{12}^0 + T_{12}^1)\omega_2 \quad \overleftrightarrow{1,2}. \end{gathered} \tag{20.29}$$

We introduce in the right-hand side of (15.17)-(15.19) the quantities T_{ij}', M_{ij}' and $(N_i')^H$ instead of T^*_{ij} ...... Equating the expressions obtained to zero, we obtain the boundary conditions. In the particular case when the edge contour coincides with the section α_1= const, the boundary conditions

$$T_{11}' = 0,\ T_{12}' = 0,\ (N_1')^H + M'_{12,2}/A_2 = 0,\ M_{11}' = 0. \tag{20.30}$$

must be fulfilled.

If there are no initial deformations and we consider the loss of stability of the membrane state of equilibrium σ^I, the conditions (20.26) and (20.30) will be identical and will not contain quantities depending on the state σ^I.

If the initial deformations and stresses are the result of a non-uniform temperature τ of the shell, the elastic forces and moments for the state σ^0 must be determined from a modified version of Hooke's law, by subtracting the overall free expansion due to heating from the total deformation of the shell. Let $(\varepsilon_{11}^0)^z$, $(\varepsilon_{22}^0)^z$, $2(\varepsilon_{12}^0)^z$ be the elongations and shear of the elements of the surface σ^z parallel to σ, resulting from the displacements u_1^0, u_2^0, w^0 due to the non-uniformity of the temperature field. Let λ be the coefficient of linear expansion of the material of the shell. Then the parts of the elongations which produce thermo-elastic stresses will be respectively

$$(\varepsilon_{11}^0)^z - \lambda\tau,\ (\varepsilon_{22}^0)^z - \lambda\tau,\ (\varepsilon_{33}^0)^z - \lambda\tau.$$

The elastic shear is $2(\varepsilon_{12}^0)^z$. Introducing these in (9.3) in place of ε_{ii}^z we find the components of the thermal stress:

$$\sigma_{11} = \frac{E}{1-\nu^2}\left[(\varepsilon_{11}^0)^z + \nu(\varepsilon_{22}^0)^z - (1+\nu)\lambda\tau\right],\ \sigma_{12} = \frac{E}{1+\nu}(\varepsilon_{12}^0)^z. \tag{20.31}$$

Furthermore, taking into account that

$$(\varepsilon_{11}^0)^z = \varepsilon_{11}^0 + z\varkappa_{11}^0,\ 2(\varepsilon_{12}^0)^z = 2\varepsilon_{12}^0 + z\varkappa_{12}^0,$$

and introducing (20.31) in (6.10), we obtain the first approximation formulas for determining the thermo-elastic forces

$$T_{11}^0 = T_{11}^f - T^\tau,\ T_{12}^0 = T_{12}^f \quad \overleftrightarrow{1,2} \tag{20.32}$$

and of the thermo-elastic moments

$$M_{11}^0 = M_{11}^f - M^\tau,\ M_{12}^0 = M_{12}^f;\quad \overrightarrow{\underleftarrow{1,2,}} \tag{20.33}$$

where

$$T_{11}^f = K(\varepsilon_{11}^0 + \nu\varepsilon_{22}^0),\ T_{12} = K(1-\nu)\varepsilon_{12}^0,\ M_{11}^f = D(\varkappa_{11}^0 + \nu\varkappa_{22}^0),$$

$$M_{12}^f = D(1-\nu)\varkappa_{12}^0,\ T^\tau = \frac{E\lambda}{1-\nu}\int_{-t/2}^{t/2}\tau dz,\ M^\tau = \frac{E\lambda}{1-\nu}\int_{-t/2}^{t/2}\tau z dz. \tag{20.34}$$

For a thin shell it may be assumed that τ is a linear function of z. Then if τ_2 and τ_1 are the temperatures of the surfaces $z = \pm t/2$,

$$2\tau^0 = \tau_1 + \tau_2,\ 2\Delta\tau = \tau_2 - \tau_1, \tag{20.35}$$

we will have

$$T^\tau = Et\lambda\tau^0/1-\nu;\quad M^\tau = Et^2\lambda\Delta\tau/6(1-\nu). \tag{20.36}$$

From (20.1), (20.35), and (20.34) we obtain, using (7.5)

$$N_i^0 = N_i^f - N^\tau,\ A_iN_i^f = D(\varkappa_{11}^0 + \varkappa_{22}^0)_{,i} = -D(\Delta w)_{,i}; \tag{20.37}$$

$$A_iN^\tau = 2M^\tau_{,i}/1+\nu,\quad i = 1,\ 2. \tag{20.38}$$

Introducing these expressions, (20.32) and (20.33) in (20.7) and (20.8) we obtain

$$(A_2T_{11}^f)_{,1} + (A_1T_{12}^f)_{,2} + T_{12}^fA_{1,2} - T_{22}^fA_{2,1} + A_1A_2X_1^\tau = 0\ \overrightarrow{\underleftarrow{1,2;}} \tag{20.39}$$

$$(A_2N_1^f)_{,1} + (A_1N_2^f)_{,2} - A_1A_2[T_{11}^f(k_{11}+\varkappa_{11}^0) + 2T_{12}^f(k_{12}+\varkappa_{12}^0) + T_{22}^f(k_{22}+\varkappa_{22}^0) - X_3^\tau] = 0. \tag{20.40}$$

Here

$$A_1A_2X_1^\tau = -(A_2T^\tau)_{,1} + A_{2,1}T^\tau = -A_2T^\tau_{,1}\quad \overrightarrow{\underleftarrow{1,2;}}$$

$$A_1A_2X_3^\tau = -\frac{2}{1+\nu}\left[\left(\frac{A_2M^\tau_{,1}}{A_1}\right)_{,1} + \left(\frac{A_1M^\tau_{,2}}{A_2}\right)_{,2}\right] + A_1A_2T^\tau(k_{11}+k_{22}+\varkappa_{11}^0+\varkappa_{22}^0). \tag{20.41}$$

If we consider T_{ij}^f and M_{ij}^f as fictitious forces and moments, expressed in terms of initial elongations and initial displacements u_i^0 and w^0 by the usual formulas (20.34) then X_i^τ will be the projections of the "thermal load" on a unit area of the surface of the shell before deformation.

Thus the problem of determination of thermal stresses in a shallow shell turns into an ordinary problem of the theory of shallow shells. By solving the latter we find all quantities characterizing the state σ^0; afterwards one may use the above-mentioned relations, without any transformations, to determine the states σ^I and σ^*. This problem has been examined in more detail for a number of particular cases in the works /VI.2-VI.4/.

§ 21. The Principle of Virtual Displacements and the Energy Criterion of Stability for Shallow Shells

In the previous sections considerable simplifications were obtained in the equations of equilibrium, the conditions of compatibility, the boundary conditions, and other relations of the general non-linear theory of shells for the case of shallow shells. Despite that, their non-linearity in the projections of the displacement has been preserved, and the analysis of finite bending of shells amounts, in general, to a boundary value problem for a system of non-linear partial differential equations, for which exact methods of solution have not yet been found. Therefore, approximate methods of solution based on the application of the variational equations of the theory of shells (which have been derived for the general case in § 10-11) are of great importance. For shallow shells these equations become considerably simpler.

We shall consider at first the equation (17.45) which expresses the principle of virtual displacements for the state of equilibrium σ^1 before the loss of stability.

Let the shell be under the action of a surface force

$$\overline{X} = \overline{X}_1 + \overline{X}_2^1,$$

where $\overline{X}_1$ is independent of the deformation, and $\overline{X}_2^1$ is a hydrostatic pressure always directed normal to the deformed middle surface and is numerically equal to the quantity p which does not vary with changing deformations. Therefore in (17.40), and also in the other formulas in § 17, one has to assume that

$$X_{21} = X_{22} = 0, \quad X_{23} = -p, \qquad (21.1)$$

and, for an external pressure, $p > 0$.

Let us assume that the external contour force $\overline{\Phi}^1$ may be also resolved into the components $\overline{\Phi}_1$ and $\overline{\Phi}_2^1$, the former being independent of deformations and the latter varying with the deformation so that its projections on the deformed axes retain their given values.

Thus,

$$\overline{\Phi}^1 = \Phi_{11}\overline{e}_1 + \Phi_{12}\overline{e}_2 + \Phi_{13}\overline{m} + \Phi_{21}\overline{e}_1^1 + \Phi_{22}\overline{e}_2^1 + \Phi_{23}\overline{m}^1. \qquad (21.2)$$

Besides, we shall assume that even before the application of the external load, the shell had initial irregularities w^0, representing deviations from the regular surface σ that was taken as the system of reference. We shall, however, consider that the initial (residual) stresses are negligibly small.

★ The relevant quantities, characterizing the elastic deformation, the forces, and the moments, may be determined from (20.3)-(20.6); in (17.41)-(17.54) the quantities e_{ij}^1 and ω_i^1 must be replaced by $e_{ij}^0 + e_{ij}^1$ and $\omega_i^0 + \omega_i^1$ respectively, expressing there e_{ij}^0 and ω_i^0 in terms of u_i^0 and ω^0 according to (3.5).

For a shallow shell, $e_{ij}^0 << 1$, $e_{ij}^1 << 1$, and according to (17.3) $E_1^1 \approx -\omega_1^0 - \omega_1^1$, $E_3^1 \approx 1$. According to (17.41), using (21.1)

$$\overline{X}^1 \delta\overline{v}^1 \approx \left[X_{11} + p\left(\omega_1^0 + \omega_1^1\right)\right]\delta u_1^1 + \left[X_{12} + p\left(\omega_2^0 + \omega_2^1\right)\right]\delta u_2^1 + (X_{13} - p)\,\delta w^1.$$

★ Owing to the smallness of u_i^{I} in comparison with w^{I} and the conditions $\omega_i^{\mathrm{I}2} \ll 1$ and $\omega_i^{\mathrm{I}2} \ll 1$, the quantities $(\omega_i^0 + \omega_i^{\mathrm{I}})\,\delta u_i^{\mathrm{I}}$ may be neglected in comparison with δw^{I}. **Therefore**

$$\bar{X}^{\mathrm{I}}\,\delta\bar{v}^{\mathrm{I}} = X_{11}\delta u_1^{\mathrm{I}} + X_{12}\delta u_2^{\mathrm{I}} + (X_{13} - p)\,\delta w^{\mathrm{I}}. \tag{21.3}$$

The elementary work of the contour forces may be determined from formulas **similar** to (17.41):

$$\bar{\Phi}^{\mathrm{I}}\,\delta\bar{v}^{\mathrm{I}} = \left[\Phi_{11} + \Phi_{21} - \Phi_{23}(\omega_1^0 + \omega_1^{\mathrm{I}})\right]\delta u_1^{\mathrm{I}} + \left[\Phi_{12} + \Phi_{22} - \Phi_{23}(\omega_2^0 + \omega_2^{\mathrm{I}})\right]\delta u_2^{\mathrm{I}} +$$
$$+ \left[\Phi_{13} + \Phi_{21}(\omega_1^0 + \omega_1^{\mathrm{I}}) + \Phi_{22}(\omega_2^0 + \omega_2^{\mathrm{I}}) + \Phi_{23}\right]\delta w^{\mathrm{I}}.$$

This expression may be considerably simplified by taking into account that the **projections** of the contour force Φ_{21}, Φ_{22} and Φ_{23} are the result of the change in **direction** of the normal load, of the order of pL^2. Therefore, at the free edge $\Phi_{23} \sim pL$, $\Phi_{21} \sim \Phi_{22} \sim pL\omega_i^0 \sim pL\omega_i^{\mathrm{I}}$, while Φ_{11} and Φ_{12} may be considerably larger.

For a fixed edge $\delta u_i^{\mathrm{I}} = 0$, $\delta w^{\mathrm{I}} = 0$.

Consequently, in the same degree of approximation as in the theory of shallow shells, it may be assumed that:

$$\bar{\Phi}^{\mathrm{I}}\,\delta\bar{v}^{\mathrm{I}} = \Phi_{11}\delta u_1^{\mathrm{I}} + \Phi_{12}\delta u_2^{\mathrm{I}} + (\Phi_{13} + \Phi_{23})\,\delta w^{\mathrm{I}}. \tag{21.4}$$

Furthermore, according to (17.43) and (17.44), and remembering that ω_i^0 does not change, we find

$$\delta E_1^{\mathrm{I}} = -\delta\omega_1^{\mathrm{I}} \quad \delta E_3^{\mathrm{I}} = \delta\left(e_{11}^{\mathrm{I}} + e_{22}^{\mathrm{I}}\right), \quad \bar{n}^{\mathrm{I}}\,\delta\bar{m}^{\mathrm{I}} = -n_1\delta\omega_1^{\mathrm{I}} - n_2\delta\omega_2^{\mathrm{I}}, \tag{21.5}$$

where n_1 and n_2 are, as before, the projections along the directions $\bar{e}_1$ and $\bar{e}_2$ of the unit normal to the edge contour before deformation.

The quantity δW^{I} may be calculated from (17.37) where it has to be assumed that

$$\delta\varepsilon_{11}^{\mathrm{I}} = \delta e_{11}^{\mathrm{I}} + (\omega_1^0 + \omega_1^{\mathrm{I}})\,\delta\omega_1^{\mathrm{I}}, \qquad \overleftrightarrow{1,2}.$$
$$2\delta\varepsilon_{12}^{\mathrm{I}} = \delta e_{12}^{\mathrm{I}} + \delta e_{21}^{\mathrm{I}} + (\omega_1^0 + \omega_1^{\mathrm{I}})\,\delta\omega_2^{\mathrm{I}} + (\omega_2^0 + \omega_2^{\mathrm{I}})\,\delta\omega_1^{\mathrm{I}}.\ \bigstar \tag{21.6}$$

Introducing (21.2)-(21.5) in the equation (17.45) we obtain the variational equation for **the** principle of virtual displacements of the state I

$$\delta\vartheta^{\mathrm{I}} = 0, \tag{21.7}$$

where ϑ^{I}, the sum of the potential energy of deformation and the potential energy **of the** field of external load, is the total potential energy of the shell, i.e.,

$$\vartheta^{\mathrm{I}} = \int_c \left[\bar{G}(n_1\omega_1^{\mathrm{I}} + n_2\omega_2^{\mathrm{I}}) - \Phi_{11}u_1^{\mathrm{I}} - \Phi_{12}u_2^{\mathrm{I}} - (\Phi_{13} + \Phi_{23})\,w^{\mathrm{I}}\right] ds +$$
$$+ \iint_{(s)} \left[W^{\mathrm{I}} - X_{11}u_1^{\mathrm{I}} - X_{12}u_2^{\mathrm{I}} - (X_{13} - p)\,w^{\mathrm{I}}\right] A_1A_2\,d\alpha_1 d\alpha_2. \tag{21.8}$$

The quantity W^{I} **may be calculated from (17.37) in which the expressions for** $\varepsilon_{ij}^{\mathrm{I}}$ **must be obtained from (20.3),**

$$A_1A_2\,\varkappa_{11}^{\mathrm{I}} = -A_2\omega_{1,1}^{\mathrm{I}} - \omega_2^{\mathrm{I}} A_{1,2},\ \ A_1A_2\,\varkappa_{12}^{\mathrm{I}} = -A_2\omega_{2,1}^{\mathrm{I}} + \omega_1^{\mathrm{I}} A_{1,2}. \tag{21.9}$$

This equation means that in the state of equilibrium the total energy of the shell has a stationary value. For an approximate determination of this state, we shall write the expressions (17.55), which satisfy the geometric boundary conditions, and shall find the quantity $\mathcal{E}^{\mathrm{I}}$ as a function of the undetermined parameters C_{ij}^{I}, m, n, ..., which we introduced. The condition (21.8) gives a system of equations for determining these parameters:

$$\frac{\partial \mathcal{E}^{\mathrm{I}}}{\partial C_{ij}^{\mathrm{I}}} = 0\ (i, j = 1, 2, 3 \ldots),\ \ \frac{\partial \mathcal{E}^{\mathrm{I}}}{\partial m} = 0, \ldots \tag{21.10}$$

For small displacements, $\mathcal{E}^{\mathrm{I}}$ is a quadratic function of C_{ij}^{I} and the equations (21.10) are linear in these quantities. For medium bending, $\mathcal{E}^{\mathrm{I}}$ is a function of fourth order and the equations (21.10) are cubic. The equations obtained for the parameters m, n, ..., may be still more complicated; therefore, it is necessary to confine oneself to solving the non-linear problems with a small number of variable parameters. It is very important to take into account experimental data and other considerations in order to obtain the best approximation, with a small number of parameters, of the required displacement function.

Let us now consider the energy criterion for stability (17.54) which may also be considerably simplified for shallow shells.

★ Above all, taking into account that

$$e_{ij}^{\mathrm{I}} << 1,\ e_{ij} << \omega_{i,j},\ A_i \sim L,\ L^2 << R^2,\ \varkappa_{11}^{\mathrm{I}} \lesssim 1/R, \tag{21.11}$$

we find, from (17.17) and (17.18)

$$\begin{gathered} L\,\varkappa_{11}^{\mathrm{II}} \sim e_{11}\,\omega_{1,1} \sim e_{22}\,\omega_{1,1},\ \ D\,\varkappa_{11}^{\mathrm{I}}\,\varkappa_{11}^{\mathrm{II}} \lesssim Et^3 e_{22}\,\omega_{1,1}/LR, \\ L\,\varkappa_{11}^{\prime} \sim \omega_{1,1},\ \ D\,\varkappa_{11}^{\prime 2} \sim Et^3\omega_{1,1}^2/L^2. \end{gathered} \tag{21.12}$$

According to (20.3) and (20.4)

$$K\,\epsilon_{11}^{\mathrm{I}}\,\epsilon_{11}^{\mathrm{II}} \sim Et\,\epsilon_{11}^{\mathrm{I}}\,\omega_1^2,\ \ K\,\epsilon_{11}^{\prime 2} \sim Ete_{11}^2. \tag{21.13}$$

Consequently,

$$\frac{\varkappa_{11}^{\mathrm{I}}\,\varkappa_{11}^{\mathrm{II}}}{\varkappa_{11}^{\prime 2}} \lesssim \frac{e_{22}}{\omega_{1,1}} \cdot \frac{L}{R} << 1,\ \ \frac{D\varkappa_{11}^{\mathrm{I}}\,\varkappa_{11}^{\mathrm{II}}}{K\epsilon_{11}^{\mathrm{I}}\,\epsilon_{11}^{\mathrm{II}}} \lesssim \frac{t^2 e_{22}\omega_{1,1}}{RL\,\epsilon_{11}^{\mathrm{I}}\,\omega_1^2} << 1$$

if $\frac{t}{R} \sim \epsilon_p$, even if $\epsilon_{11}^{\mathrm{I}}$ is very small, i.e., $\epsilon_{11}^{\mathrm{I}} \sim \epsilon_p{}^2$. Thus, the terms $D\varkappa_{11}^{\mathrm{I}}\,\varkappa_{11}$ in the expression for W" are very small in comparison with the terms $K\epsilon_{11}^{\mathrm{I}}\,\epsilon_{11}^{\prime}$ or $D\varkappa_{11}^{\prime 2}$. The terms containing $\widetilde{G}$ in (17.54) are also negligibly small because the most important among them are

$$L\,\widetilde{G}\,(n_1\,\omega_1 + n_2\,\omega_2)\,\delta e_{11} \sim L\,D\,\varkappa_{11}^{\mathrm{I}}\,\omega_1\,\delta e_{11},$$

i.e., in view of (21.12), they are at most of the same order of magnitude as the neglected term

$$\delta \int\!\!\int_{(\sigma)} D\varkappa_{11}^{\mathrm{I}}\,\varkappa_{11}^{\mathrm{II}}\,A_1 A_2\,d\alpha_1\,d\alpha_2 \sim L^2 D\,\delta\,(\varkappa_{11}^{\mathrm{I}}\,\varkappa_{11}^{\mathrm{II}}) \sim L\,D\,\varkappa_{11}^{\mathrm{I}}\,\omega_{1,1}\,\delta e_{11}.$$

★ The term depending on the external surface load is of the order

$$\iint_{(\sigma)} X_{33} E_3^{\prime} \delta w \, d\sigma \sim p L^2 e_{11} \delta w.$$

But, as we know,

$$pR \sim T_2^{\mathrm{I}}, \; p \sim E t \epsilon_{22}^{\mathrm{I}} / R.$$

Consequently, the term considered is of the order of

$$E t \epsilon_{22}^{\mathrm{I}} L^2 e_{11} \delta w / R \sim E t \epsilon_{22}^{\mathrm{I}} L^2 e_{11} \delta e_{ij}.$$

It is small, for instance, in comparison with the term

$$\delta \iint_{(\sigma)} K \epsilon_{11}^{\mathrm{I}} \epsilon_{11}^{\prime\prime} d\sigma \sim L^2 E t \epsilon_{11}^{\mathrm{I}} \omega_i \delta \omega_i.$$

One may also neglect the term depending on the external contour load, the major part of which is

$$\int_C [\Phi_{31} \omega_1 + \Phi_{32} \omega_2 + \Phi_{33} (e_{11} + e_{22})] \delta w \, ds,$$

where, as stated above, $\Phi_{33} \sim Lp$, $[\Phi_{31} \sim \Phi_{32} \sim Lp \, \omega_i^{\circ} \sim Lp \, \omega_i^{\mathrm{I}}$. ★

Thus, the condition (17.54) which together with equation (21.7) determines the limit of stability of the state σ^{I}, is reduced to the simple form

$$\delta \iint_{(\sigma)} W^{\prime\prime} A_1 A_2 \, d\alpha_1 \, d\alpha_2 = 0, \tag{21.14}$$

where

$$\begin{aligned} W^{\prime\prime} = K \Big[& \frac{1}{2} (\epsilon_{11}^{\prime} + \epsilon_{22}^{\prime})^2 + (\epsilon_{11}^{\mathrm{I}} + \epsilon_{22}^{\mathrm{I}}) (\epsilon_{11}^{\prime\prime} + \epsilon_{22}^{\prime\prime}) - (1 - \nu) (\epsilon_{11}^{\mathrm{I}} \epsilon_{22}^{\prime\prime} + \\ & + \epsilon_{22}^{\mathrm{I}} \epsilon_{11}^{\prime\prime} + \epsilon_{11}^{\prime} \epsilon_{22}^{\prime} - \epsilon_{12}^{\prime 2} - 2 \epsilon_{12}^{\mathrm{I}} \epsilon_{12}^{\prime\prime}) \Big] + \\ & + D \Big[\frac{1}{2} (\varkappa_{11} + \varkappa_{22})^2 - (1 - \nu) (\varkappa_{11} \varkappa_{22} - \varkappa_{12}^2) \Big], \end{aligned} \tag{21.15}$$

The quantities $\epsilon_{ij}^{\mathrm{I}}, \epsilon_{ij}^{\prime}, \epsilon_{ij}^{\prime\prime}$ may be determined from (20.3) and (20.4), and the other quantities are:

$$A_1 A_2 \varkappa_{11} = - A_2 \omega_{1,1} - \omega_2 A_{1,2}, \quad A_1 A_2 \varkappa_{12} = - A_2 \omega_{2,1} + \omega_1 A_{1,2}, \quad \omega_1 = w_{,1} : A_1; \tag{21.16}$$

$$\begin{aligned} e_{11} &= A_1^{-1} u_{1,1} + u_2 (A_1 A_2)^{-1} A_{1,2} + w k_{11}, \\ e_{12} &= A_1^{-1} u_{2,1} - u_1 (A_1 A_2)^{-1} A_{1,2} + w k_{12} \qquad \overrightarrow{\underleftarrow{1,2}}, \\ K &= Et/1 - \nu^2, \quad D = Et^3/12 (1 - \nu^2). \end{aligned} \tag{21.17}$$

Using (20.5), the expression (21.15) becomes

$$\begin{aligned} W^{\prime\prime} = {} & T_{11}^{\mathrm{I}} \epsilon_{11}^{\prime\prime} + 2 T_{12}^{\mathrm{I}} \epsilon_{12}^{\prime\prime} + T_{22}^{\mathrm{I}} \epsilon_{22}^{\prime\prime} + \frac{K}{2} [\epsilon_{11}^{\prime 2} + \epsilon_{22}^{\prime 2} + 2\nu \epsilon_{11}^{\prime} \epsilon_{22}^{\prime} + \\ & + 2 (1 - \nu) \epsilon_{12}^{\prime 2}] + \frac{D}{2} [\varkappa_{11}^2 + \varkappa_{22}^2 + 2\nu \varkappa_{11} \varkappa_{22} + 2 (1 - \nu) \varkappa_{12}^2], \end{aligned} \tag{21.18}$$

where $\epsilon_{11}^{\prime\prime}, \epsilon_{11}^{\prime}, \ldots$ are given by (20.4).

In the particular case $w^0 = 0$, i.e., when there are no initial deviations from the system of reference σ, the integral of this over the area σ actually turns into the functional (44_2) given in our previous work /0.13/*.

If the tangential surface load is zero, the equations of equilibrium of the tangential forces may be satisfied by expressing the tangential forces of the state z^1 according to (20.20) in terms of the force function ψ', and the additional tangential forces in terms of ψ^1 according to (20.23).

Introducing (20.20) in (21.7) we obtain Lagrange's variational equation with the modification introduced by P. F. Papkovich /0.17/. The function ψ^1 has to satisfy the condition of compatibility (20.21), whereas the tangential displacements u_i^1, expressed in terms of it, must satisfy the geometrical boundary conditions. One may also introduce (20.23) in (21.14) whereby ψ' has the satisfy the equation (20.24) and the corresponding geometrical conditions.

In the exact solution of this variational problem, the static boundary conditions will be satisfied automatically, i. e., in this case they will be the natural boundary conditions, while the geometrical boundary conditions are essential. When solving this problem approximately, the natural boundary conditions will also be satisfied approximately, a higher degree of approximation rendering these conditions more readily satisfiable.

----

* We note that in /0.13/ we retained in W" certain small terms of the same order of magnitude, which should have been neglected when assuming $\varkappa'_{11} \approx \varkappa_{11}$, $\varkappa'_{12} \approx \varkappa_{12}$.

§ 22. Equations of the Bubnov-Galerkin Method for the Lagrange Variational Principle

In the Bubnov-Galerkin method one writes the variational equations (10.26), which turn into the (11.12) or (11.7) upon expressing the virtual displacement vector $\delta\bar{v}$ in terms of its components along the principal directions of σ^* or of σ.

For shallow shells, these equations may be considerably simplified. In fact

$$\delta\bar{v} = \bar{e}_1 \delta u_1 + \bar{e}_2 \delta u_2 + \bar{m}\,\delta w = \bar{e}_1^* (\delta \bar{v})_1 + \bar{e}_2^* (\delta \bar{v})_2 + \bar{m}^* (\delta\bar{v})_3.$$

Substituting here the expressions for $\bar{e}_1^*$, $\bar{e}_2^*$, $\bar{m}^*$ in terms of $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$ from (3.19) and (3.20), we obtain

$$(\delta \bar{v})_1 = (1 + e_{11})\,\delta u_1 + e_{12}\,\delta u_2 + \omega_1\,\delta w, \qquad \overrightarrow{1,2}$$
$$(\delta\bar{v})_3 = E_1\,\delta u_1 + E_2 u_2 + E_3\,\delta w$$

But for shallow shells, $\omega_i^2 \sim e_{ij} \ll 1$, $E_i = -\omega_i$, $E_3 = 1$, $u_i \sim w\omega_i$; therefore, the quantities $e_{11}\delta u_1$, $e_{12}\delta u_2$ may be neglected in comparison with $\omega_i\,\delta w$ and

$$(\delta \bar{v})_i = \delta u_i + \omega_i\,\delta w, \quad (\delta \bar{v})_3 = \delta w. \tag{22.1}$$

Furthermore, the internal force on the contour C with normal $\bar{n}$ is, according to (8.19)

$$\bar{K}_n - \frac{\partial H^* \bar{m}^*}{\partial s} = (T_{11}^* n_1 + T_{21}^* n_2)\,\bar{e}_1^* + (T_{12}^* n_1 + T_{22}^* n_2)\,\bar{e}_2^* +$$
$$+ \bar{m}^* \left(N_1^* n_1 + N_2^* n_2 + \frac{n_2}{A_1} \cdot \frac{\partial H^*}{\partial \alpha_1} - \frac{n_1}{A_2} \cdot \frac{\partial H^*}{\partial \alpha_2}\right).$$

where H^* may be calculated from (8.12), $n_1 = \sin\varphi$, $n_2 = -\cos\varphi$ and φ is the angle between $\bar{e}_1$ and the positive direction of the tangent to C before deformation. The external contour load is

$$\bar{\Phi} = \Phi_1^* \bar{e}_1^* + \Phi_2^* \bar{e}_2^* + \Phi_3^* \bar{m}^*.$$

According to (22.1)

$$\delta \bar{v} = (\delta u_1 + \omega_1\,\delta w)\,\bar{e}_1^* + (\delta u_2 + \omega_2\,\delta w)\,\bar{e}_2^* + \bar{m}^*\,\delta w.$$

From (8.12), (8.32), and (11.5)

$$(\tilde{G}^* - G^*)\,\bar{n}^*\,\delta\bar{m}^* =$$
$$= -(\tilde{G}^* - M_{11}^* n_1^2 - 2 M_{12}^* n_1 n_2 - M_{22}^* n_2^2)(n_1\,\delta\omega_1 + n_2\,\delta\omega_2).$$

In the case considered, the equations (7.3) and (7.4) were transformed into (15.8) and (15.9). Thus, from (11.2) we obtain the variational equation

$$\begin{gathered}
\int_C \{(\Phi_1^* - T_{11}^* n_1 - T_{21}^* n_2)(\delta u_1 + \omega_1 \delta w) + \\
+ (\Phi_2^* - T_{12}^* n_1 - T_{22}^* n_2)(\delta u_2 + \omega_2 \delta w) + (\Phi_3^* - N_1^* n_1 - \\
- N_2^* n_2 - n_2 H_{,1}^*/A_1 + n_1 H_{,2}^*/A_2)\delta w + \\
+ (G^* - \tilde{G}^*)(n_1 \delta\omega_1 + n_2 \delta\omega_2)\} ds + \iint_{(\Omega)} \{[(A_2 T_{11}^*)_{,1} + (A_1 T_{21}^*)_{,2} + \\
+ T_{12}^* A_{1,2} - T_{22}^* A_{2,1} + A_1 A_2 X_1^*](\delta u_1 + \omega_1 \delta w) + [(A_1 T_{22}^*)_{,2} + \\
+ (A_2 T_{12}^*)_{,1} + T_{21}^* A_{2,1} - T_{11}^* A_{1,2} + A_1 A_2 X_2^*](\delta u_2 + \omega_2 \delta w) + \\
+ [(A_2 N_1^*)_{,1} + (A_1 N_2^*)_{,2} - A_1 A_2 (T_{11}^* k_{11}^* + 2T_{12}^* k_{12}^* + \\
+ T_{22}^* k_{22}^* - X_3^*)]\delta w\} d\alpha_1 d\alpha_2 = 0,
\end{gathered} \tag{22.2}$$

where

$$\begin{gathered}
A_i N_i^* = - D(\Delta w)_{,i}, \quad k_{ij}^* = k_{ij} + \varkappa_{ij}, \quad \omega_i = w_{,i}/A_i, \quad \delta\omega_i = (\delta w)_{,i}/A_i, \\
G^* = M_{11}^* n_1^2 + 2M_{12}^* n_1 n_2 + M_{22}^* n_2^2, \\
H^* = (M_{11}^* - M_{22}) n_1 n_2 - M_{12}^* (n_1^2 - n_2^2).
\end{gathered} \tag{22.3}$$

If the displacement at the contour is constant, or the trial functions are chosen so that not only the geometrical boundary conditions but also the static ones are satisfied, the contour integral will vanish. In general, equation (22.2) will be applicable if the external contour load is given by its projections along the principal directions of the shell after deformation. It may be considerably simplified if the edge contour is composed of parts directed along the coordinate lines after deformation. For example, at those parts of the contour which coincide with the line α_1 = const we have $\varphi = \pi/2$, $n_1 = 1$, $n_2 = 0$.

Similarly, for shallow shells the equation (11.7) may be simplified as follows:

According to (8.30) $N_1 = T_{11}^* \omega_1 + T_{12}^* \omega_2 + N_1^*$;

From (8.32) the static boundary conditions may be expressed by the equations

$$\begin{gathered}
\Phi_1^{\text{н}} = T_{11}^* n_1 + T_{21}^* n_2, \quad \Phi_2^{\text{н}} = T_{12}^* n_1 + T_{22}^* n_2, \quad \tilde{G} = G^*, \\
\Phi_3^{\text{н}} = N_1^{\text{н}} n_1 + N_2^{\text{н}} n_2 + n_2 H_{,1}^*/A_1 - n_1 H_{,2}^*/A_2, \\
N_1^{\text{н}} = T_{11}^* \omega_1 + T_{12}^* \omega_2 + N_1^* \qquad \overleftarrow{\overrightarrow{1,2}}.
\end{gathered} \tag{22.4}$$

where $\Phi_1^{\text{н}}$, $\Phi_2^{\text{н}}$, $\Phi_3^{\text{н}}$ are the projections of the external contour load along the principal directions $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$ before deformation; in the first two of the equations (8.29) we neglect the terms containing N_1 and N_2. We thus obtain the equation

$$\begin{gathered}
\int_C \{(\Phi_1^{\text{н}} - T_{11}^* n_1 - T_{21}^* n_2)\delta u_1 + (\Phi_2^{\text{н}} - T_{12}^* n_1 - T_{22}^* n_2)\delta u_2 + \\
+ (\Phi_3^{\text{н}} - N_1^{\text{н}} n_1 - N_2^{\text{н}} n_2 - n_2 H_{,1}^*/A_1 + n_1 H_{,2}^*/A_2)\delta w + \\
+ (G^* - G)(n_1 \delta\omega_1 + n_2 \delta\omega_2)\} ds + \\
+ \iint_{(\Omega)} \{[(A_2 T_{11}^*)_{,1} + (A_1 T_{21}^*)_{,2} + T_{12}^* A_{1,2} - T_{22}^* A_{2,1} + A_1 A_2 X_1]\delta u_1 + \\
+ [(A_1 T_{22}^*)_{,2} + (A_2 T_{12}^*)_{,1} + T_{21}^* A_{2,1} - T_{11}^* A_{1,2} + A_1 A_2 X_2]\delta u_2 + \\
+ [(A_2 N_1^{\text{н}})_{,1} + (A_1 N_2^{\text{н}})_{,2} - A_1 A_2 (k_{11} T_{11}^* + 2k_{12} T_{12}^* + \\
+ k_{22} T_{22}^* - X_3)]\delta w\} d\alpha_1 d\alpha_2 = 0.
\end{gathered} \tag{22.5}$$

If, in order to solve this equation approximately we are given the displacements in series form (17.55), in which every term satisfies not only the geometrical boundary conditions but also the static ones, the contour integral in the expression for these conditions will vanish as in equation (22.2).

In nearly all cases which are of practical importance, we have $\delta w = 0$ at the contour; therefore, the corresponding contour integrals in equations (22.2) and (22.5) vanish. Finally, if the edge is clamped, $u_i = 0$, $\delta w = 0$, $\delta\omega_i = 0$ at the contour and the entire contour integral in these equations vanishes.

If $X_1 = X_2 = 0$ it is possible to introduce a stress function according to (15.11). Then the terms containing δu_1 and δu_2 in the surface integral in equation (22.5) will vanish and the trial functions must satisfy the condition of compatibility (15.16) and the geometric boundary conditions. We thus obtain Galerkin's variational equation in a generalized form, including the modification introduced by Papkovich. It should be noted that the requirement of satisfying the indicated essential conditions makes the choice of trial functions for ψ very difficult. These difficulties are increased by the fact that the stress function expresses explicitly only the linear combinations of the projections of the tangential displacement and their first order derivatives with respect to α_i. In the particular case where the displacements at the contour C are not limited by constraints, the functions ψ must satisfy the equation (15.16) and the static boundary conditions imposed on the tangential forces if we want to get rid of that part of the contour integral which contains δu_i. Otherwise δu_i must be expressed in terms of ψ.

§ 23. The Variational Equation of the Mixed Method

Taking into account the above-mentioned difficulties connected with the choice of the trial stress functions in applying the principle of virtual displacement, N. A. Alumyae has proposed a new variational equation of equilibrium for shallow shells. In his work /VI.5/ he has proved that when approximating a function w, which satisfies the geometrical boundary conditions for w, and a function ψ, which satisfies the static boundary conditions for the tangential forces, the solution of the boundary problem of equilibrium for a shallow shell will be built up from such admissible functions w and ψ which satisfy the equation*:

$$
\begin{aligned}
\delta \iint_{(\sigma)} \Big\{ & \frac{D}{2} \left[(\varkappa_{11} + \varkappa_{22})^2 - 2(1-\nu)(\varkappa_{11}\varkappa_{22} - \varkappa_{12}^2) \right] + pw + \\
& + T_{11}\left(wk_{11} + \frac{1}{2}\omega_1^2 + \omega_1^0 \omega_1 - \frac{1}{2}\varepsilon_{11} \right) + \\
& + T_{22}\left(wk_{22} + \frac{1}{2}\omega_2^2 + \omega_2^0 \omega_2 - \frac{1}{2}\varepsilon_{22} \right) + \\
& + T_{12}\left(2wk_{12} + \omega_1\omega_2 + \omega_1^0\omega_2 + \omega_2^0\omega_1 - \varepsilon_{12} \right) \Big\} d\sigma + \\
+ \int_C \Big\{ & \widehat{u}_n \delta T + \widehat{u}_\tau \delta S - \Phi_3 \delta w + \overline{G}(n_1 \delta\omega_1 + n_2 \delta\omega_2) \Big\} ds = 0
\end{aligned}
\tag{23.1}
$$

$$(d\sigma = A_1 A_2 \, d\alpha_1 \, d\alpha_2).$$

Here $\widehat{u}_n$ and $\widehat{u}_\tau$ are the contour values of the projections of the tangential displacement on the normal and the tangent to the contour

$$
\begin{gathered}
A_2 T_{11} = (\psi_{,2}/A_2)_{,2} + A_{2,1}\psi_{,1}/A_1^2, \\
A_1 A_2 T_{12} = -\psi_{,12} + A_{2,1}\psi_{,2}/A_2 + A_{1,2}\psi_{,1}/A_1; \\
T = T_{11} n_1^2 + 2T_{12} n_1 n_2 + T_{22} n_2^2, \\
S = (T_{22} - T_{11}) n_1 n_2 + T_{12}(n_1^2 - n_2^2), \\
A_1 \varkappa_{11} = -\omega_{1,1} - A_{1,2}\omega_2/A_2, \quad A_1 \varkappa_{12} = -\omega_{2,1} + A_{1,2}\omega_1/A_2, \\
\varepsilon_{11} = (T_{11} - \nu T_{22}) : Et = e_{11} + \frac{1}{2}\omega_1^2 + \omega_1^0 \omega_1, \quad \omega_1^0 = w^0_{,1} : A_1, \\
\omega_1 = w_{,1} : A_1, \\
2\varepsilon_{12} = e_{12} + e_{21} + \omega_1(\omega_2^0 + \omega_2) + \omega_2(\omega_1^0 + \omega_1) \quad \overrightarrow{\underleftarrow{1,2}}.
\end{gathered}
\tag{23.2}
$$

It is not essential here that the admissible functions ψ should satisfy the condition of compatibility and the geometrical boundary conditions.

We shall prove that equation (23.1) will be satisfied for the actual state of equilibrium, i. e., when all the boundary conditions, the equations of equilibrium

$$(A_2 T_{11})_{,1} + (A_1 T_{21})_{,2} + T_{12} A_{1,2} - T_{22} A_{2,1} = 0 \quad \overrightarrow{\underleftarrow{1,2}}, \tag{23.3}$$

----

* N. A. Alumyae has derived, in the work mentioned, a variational equation for a shell without initial deviations from the surface of reference σ. Here we shall assume the more general case, where there is an initial deviation w^0 from the surface σ, but there are no initial stresses. This generalization of Alumyae's equation may be obtained by substituting in his equation $w + w^0$ for w and assuming $\delta(\omega_i + \omega_i^0) = \delta\omega_i$, $\delta(w + w_0) = \delta w$ whereby $\varkappa_{ij}$ is left unchanged.

$$-D\Delta\Delta w+T_{11}(k_{11}+\overset{0}{\varkappa}_{11}+\varkappa_{11})+2T_{12}(k_{12}+\overset{0}{\varkappa}_{12}+\varkappa_{12})+$$
$$+T_{22}(k_{22}+\overset{0}{\varkappa}_{22}+\varkappa_{22})+q=0 \tag{23.4}$$

and the conditions of compatibility of deformations are satisfied.

★ First of all, it can easily be shown, in view of (23.2), that

$$\frac{1}{2}\delta(T_{11}\varepsilon_{11}+2T_{12}\varepsilon_{12}+T_{22}\varepsilon_{22})=\varepsilon_{11}\delta T_{11}-2\varepsilon_{12}\delta T_{12}+\varepsilon_{22}\delta T_{22}. \tag{23.5}$$

Further, let us consider the expression

$$\int_C[(T_{11}n_1+T_{21}n_2)u_1+(T_{12}n_1+T_{22}n_2)u_2]\,ds=\int_C(Tu_n+Su_\tau)\,ds,$$

where

$$u_n=n_1u_1+n_2u_2,\quad u_\tau=n_1u_2-n_2u_1 \tag{23.6}$$

are the projections of the tangential displacements on the normal and the tangent to the contour.

Using Green's formula

$$\int_C fFA_i r_i\,ds=\iint_{(\sigma)}F(fA_1A_2)_{,i}\,d\alpha_1\,d\alpha_2+\iint_{(\sigma)}fA_1A_2F_{,i}\,d\alpha_1\,d\alpha_2, \tag{23.7}$$

where F and f are functions of α_1 and α_2, we find

$$\int_C T_{11}n_1u_1\,ds=\iint_{(\sigma)}\{(A_2T_{11})_{,1}u_1+A_2T_{11}u_{1,1}\}\,d\alpha_1\,d\alpha_2,$$
$$\int_C T_{21}n_2u_1\,ds=\iint_{(\sigma)}\{(A_1T_{21})_{,2}u_1+A_1T_{21}u_{1,2}\}\,d\alpha_1\,d\alpha_2\quad \overleftarrow{\overrightarrow{1,2}}.$$

Thus

$$\int_C(Tu_n+Su_\tau)\,ds=\iint_{(\sigma)}\{[(A_2T_{11})_{,1}+(A_1T_{21})_{,2}]u_1+A_2T_{11}u_{1,1}+A_1T_{21}u_{1,2}+$$
$$+[(A_1T_{22})_{,2}+(A_2T_{12})_{,1}]u_2+A_1T_{22}u_{2,2}+A_2T_{12}u_{2,1}\}\,d\alpha_1\,d\alpha_2.$$

Using (23.3) and (3.5) we obtain

$$\int_C(Tu_n+Su_\tau)\,ds=\iint_{(\sigma)}\{T_{11}(e_{11}-wk_{11})+T_{12}(e_{12}-wk_{12})+\overleftarrow{\overrightarrow{1,2}}\}\,d\sigma.$$

If only tangential forces which satisfy the equation of equilibrium (23.3) in the tangential plane are allowed, as it occurs when varying the force functions, the former equations will be valid for the quantities $T_{ij}+\delta T_{ij}$ and, owing to their linearity, also for δT_{ij}. Therefore,

$$\int_C(u_n\delta T+u_\tau\delta S)\,ds=\iint_{(\sigma)}\{(e_{11}-wk_{11})\delta T_{11}+(e_{12}-wk_{12})\delta T_{21}+\overleftarrow{\overrightarrow{1,2}}\}\,d\sigma. \tag{23.8}$$

Using this equation, (23.2) and (23.5), we obtain

★

$$
\begin{aligned}
&\delta \iint_{(\sigma)} \Big\{ T_{11} \left(w k_{11} + \frac{1}{2} \omega_1^2 + \omega_1^0 \omega_1 - \frac{1}{2} \varepsilon_{11} \right) + \\
&\quad + T_{22} \left(w k_{22} + \frac{1}{2} \omega_2^2 + \omega_2^0 \omega_2 - \frac{1}{2} \varepsilon_{22} \right) + \\
&\quad + T_{12} (2 w k_{12} + \omega_1 \omega_2 + \omega_1^0 \omega_2 + \omega_2^0 \omega_1 - \varepsilon_{12}) \Big\} d\sigma = \\
&= \iint_{(\sigma)} \{ T_{11} [k_{11} \delta w + (\omega_1 + \omega_1^0) \delta \omega_1] + T_{21} [k_{12} \delta w + (\omega_1 + \omega_1^0) \delta \omega_2] + \overleftrightarrow{1,2} \} d\sigma - \\
&\quad - \int_C (u_n \delta T + u_\tau \delta S) \, ds.
\end{aligned}
\tag{23.9}
$$

Here $\delta\omega_1 = (\delta w)_{,1}$; therefore, using (23.7)

$$
\begin{aligned}
&\iint_{(\sigma)} T_{11} (\omega_1 + \omega_1^0) \delta \omega_1 \, d\sigma = \iint_{(\sigma)} T_{11} (\omega_1 + \omega_1^0) A_2 (\delta w)_{,1} \, d\alpha_1 \, d\alpha_2 = \\
&= \int_C T_{11} (\omega_1 + \omega_1^0) n_1 \delta w \, ds - \iint_{(\sigma)} [T_{11} (\omega_1 + \omega_1^0) A_2]_{,1} \delta w \, d\alpha_1 \, d\alpha_2, \\
&\iint_{(\sigma)} T_{21} (\omega_1^0 + \omega_1) \delta \omega_2 \, d\sigma = \int_C T_{21} (\omega_1 + \omega_1^0) n_2 \delta w \, ds - \\
&\quad - \iint_{(\sigma)} [T_{21} (\omega_1 + \omega_1^0) A_1]_{,2} \delta w \, d\alpha_1 \, d\alpha_2 \quad \overleftrightarrow{1,2}.
\end{aligned}
$$

But, from (23.2)

$$
\omega_{1,1} = - A_1 \varkappa_{11} - \frac{A_{1,2} \omega_2}{A_2}, \quad \omega_{2,1} = - A_1 \varkappa_{12} + \frac{A_{1,2} \omega_1}{A_2} \quad \overleftrightarrow{1,2}.
$$

Consequently, taking the equations of equilibrium (23.3) into account, we have

$$
\begin{aligned}
&[T_{11} (\omega_1 + \omega_1^0) A_2]_{,1} + [T_{21} (\omega_1 + \omega_1^0) A_1]_{,2} + \overleftrightarrow{1,2} = \\
&= [(T_{11} A_2)_{,1} + (T_{21} A_1)_{,2} - T_{22} A_{2,1} + T_{12} A_{1,2}] (\omega_1 + \omega_1^0) - \\
&\quad - [T_{11} (\varkappa_{11} + \varkappa_{11}^0) + T_{21} (\varkappa_{12} + \varkappa_{12}^0)] A_1 A_2 + \overleftrightarrow{1,2} = \\
&= - A_1 A_2 [T_{11} (\varkappa_{11} + \varkappa_{11}^0) + 2 T_{12} (\varkappa_{12} + \varkappa_{12}^0) + T_{22} (\varkappa_{22} + \varkappa_{22}^0)].
\end{aligned}
\tag{23.10}
$$

The derivatives along the normal and the tangent may be expressed in terms of the derivatives along the orthogonal coordinates, by the formulas

$$
\frac{\partial}{\partial n} (\ ) = \frac{n_1}{A_1} (\ )_{,1} + \frac{n_2}{A_2} (\ )_{,2}, \quad \frac{\partial}{\partial s} (\ ) = - \frac{n_2}{A_1} (\ )_{,1} + \frac{n_1}{A_2} (\ )_{,2},
$$

or

$$
\frac{1}{A_1} (\ )_{,1} = n_1 \frac{\partial (\ )}{\partial n} - n_2 \frac{\partial (\ )}{\partial s}, \quad \frac{1}{A_2} (\ )_{,2} = n_2 \frac{\partial (\ )}{\partial n} + n_1 \frac{\partial (\ )}{\partial s}.
\tag{23.11}
$$

Consequently,

$$
\begin{aligned}
&(T_{11} n_1 + T_{21} n_2) (\omega_1 + \omega_1^0) + \overleftrightarrow{1,2} = \frac{1}{A_1} (T_{11} n_1 + T_{21} n_2) (w + w^0)_{,1} + \overleftrightarrow{1,2} = \\
&= T \frac{\partial}{\partial n} (w + w^0) + S \frac{\partial}{\partial s} (w + w^0).
\end{aligned}
\tag{23.12}
$$

In view of (23.9)-(23.12) we have

★

$$
\begin{aligned}
&\delta \iint_{(\sigma)} \Big\{ T_{11}\left(wk_{11} + \frac{1}{2}\omega_1^2 + \omega_1^0\omega - \frac{1}{2}\varepsilon_{11}\right) + \\
&\quad + T_{12}(2wk_{12} + \omega_1\omega_2 + \omega_1^0\omega_2 + \omega_2^0\omega_1 - \varepsilon_{12}) + \\
&\quad + T_{22}\left(wk_{22} + \frac{1}{2}\omega_2^2 + \omega_2\omega_2^0 - \frac{1}{2}\varepsilon_{22}\right)\Big\} d\sigma = \\
&= -\int_C (u_n\delta T + u_\tau \delta S)\, ds + \int_C \left\{T\frac{\partial}{\partial n}(w + w^0) + S\frac{\partial}{\partial s}(w + w^0)\right\}\delta w\, ds + \\
&\quad + \iint_{(\sigma)} \{T_{11}(k_{11} + \varkappa_{11} + \varkappa_{11}^0) + 2T_{12}(k_{12} + \varkappa_{12} + \varkappa_{12}^0) + \\
&\quad + T_{22}(k_{22} + \varkappa_{22} + \varkappa_{22}^0)\}\,\delta w\, d\sigma.
\end{aligned} \tag{23.13}
$$

Let us, furthermore, consider the variation of the energy of bending. Using as before (23.7), (23.2), (23.11) and (7.5), we obtain

$$
\begin{aligned}
&\delta \iint_{(\sigma)} \frac{D}{2}\{(\varkappa_{11} + \varkappa_{22})^2 - 2(1 - \nu)(\varkappa_{11}\varkappa_{22} - \varkappa_{12}^2)\}\, d\sigma = \\
&= \iint_{(\sigma)} (M_{11}\delta\varkappa_{11} + M_{12}\delta\varkappa_{12})\, d\sigma + \overset{\rightarrow}{\underset{\leftarrow}{1,2}} = \\
&= \int_C (-M_{11}A_2 n_1\delta\omega_1 - M_{12}A_2 n_1\delta\omega_2)\, ds + \iint_{(\sigma)} \{[(M_{11}A_2)_{,1} + M_{12}A_{1,2}]\delta\omega_1 + \\
&\quad + [-M_{11}A_{1,2} + (M_{12}A_2)_{,1}]\delta\omega_2\}\, d\alpha_1 d\alpha_2 + \overset{\rightarrow}{\underset{\leftarrow}{1,2}} = -\int_C \Big[G\delta\left(\frac{\partial w}{\partial n}\right) + \\
&\quad + \left(\frac{\partial H}{\partial s} - N_1 n_1 - N_2 n_2\right)\delta w\Big]\, ds - \iint_{(\sigma)} [(A_2 N_1)_{,1} + (A_1 N_2)_{,2}]\,\delta w\, d\alpha_1 d\alpha_2,
\end{aligned} \tag{23.14}
$$

where

$$
\begin{aligned}
&G = M_{11}n_1^2 + M_{22}n_2^2 + 2M_{12}n_1 n_2, \quad H = (M_{11} - M_{22})n_1 n_2 - M_{12}(n_1^2 - n_2^2), \\
&N_i = -D\,\Delta w / A_i, \quad (A_2 N_1)_{,1} + (A_1 N_2)_{,2} = -A_1 A_2 D\,\Delta\Delta w \;\bigstar
\end{aligned} \tag{23.15}
$$

Using the equation (23.12)-(23.15), we may convince ourselves that equation (23.1) becomes

$$
\begin{aligned}
&\iint_{(\sigma)} \{D\,\Delta\Delta w + T_{11}(k_{11} + \varkappa_{11} + \varkappa_{11}^0) + 2T_{12}(k_{12} + \varkappa_{12} + \varkappa_{12}^0) + \\
&\quad + T_{22}(k_{22} + \varkappa_{22} + \varkappa_{22}^0) + p\}\,\delta w\, d\sigma - \\
&\int_C (G - \tilde{G})\,\delta\left(\frac{\partial w}{\partial n}\right) ds + \int_C \Big[N_1 n_1 + N_2 n_2 - \frac{\partial H}{\partial s} + T\frac{\partial}{\partial n}(w + w^0) + \\
&\quad + S\frac{\partial}{\partial s}(w + w_0) - \Phi_3\Big]\delta w\, ds + \int \{(\tilde{u}_n - u_n)\,\delta T + \\
&\quad + (\tilde{u}_\tau - u_\tau)\,\delta S\}\, ds = 0.
\end{aligned} \tag{23.16}
$$

It will be satisfied if the equations of equilibrium (23.4) and the equations (23.3) are satisfied together with the following boundary conditions:

a. the geometrical condition

$$
w = \tilde{w}, \quad \delta w = 0 \tag{23.17}
$$

on those parts of the contour where the normal component of the displacement is given;

b. the geometrical condition

$$
\partial w / \partial n = \partial \tilde{w} / \partial n, \quad \delta(\partial w / \partial n) = 0 \tag{23.18}
$$

on those parts of the contour where the rotation about the tangent to the contour is given;

c. the static condition

$$T = \tilde{T}, \ \delta T = 0 \tag{23.19}$$

on those parts of the contour where the normal component of the tangential force is given;

d. the static condition

$$S = \tilde{S}, \ \delta S = 0 \tag{23.20}$$

on those parts of the contour where the projection of the tangential force along the tangent to the contour is given;

e. the static condition

$$N_1 n_1 + N_2 n_2 - \frac{\partial H}{\partial s} + T \frac{\partial}{\partial n}(w + w^0) + S \frac{\partial}{\partial s}(w + w^0) = \Phi_3 \tag{23.21}$$

on those parts of the contour where the condition (23.17) is not satisfied;

f. the static condition

$$G = \tilde{G} \tag{23.22}$$

on those parts of the contour where the condition (23.18) is not satisfied;

g. the geometrical condition

$$u_n = \tilde{u}_n \tag{23.23}$$

on those parts of the contour where the condition (23.19) is not satisfied;

h. the geometrical condition

$$u_\tau = \tilde{u}_\tau \tag{23.24}$$

on those parts of the contour where the condition (23.20) is not satisfied. In the actual state of equilibrium we have a combination of these conditions. Consequently, the validity of the variational equation (23.1) has been proved.

For approximate solution of this problem by means of this variational equation it is admissible to take as trial functions, as it has been shown above, the function w which satisfies the geometric conditions (23.17)-(23.18), and the function ψ which satisfies the static conditions (23.19)-(23.20).

These boundary conditions are essential. The boundary conditions are natural for the variational equation (23.1). They will be identically satisfied when solving the variational problem. The higher the degree of the approximation, the more exact will be the solution.

All the quantities contained in equation (23.1) and in the essential boundary conditions may be expressed in terms of w and ψ according to (23.2).

After convincing ourselves of the validity of the variational equation (23.1) we shall transform it, without using the equation (23.9) which has been obtained by expressing the elongations in terms of displacement. To that end, it is sufficient to substitute in (23.16) the right-hand member of equation (23.8) instead of its left-hand member. We shall confine ourselves to the case where ψ is a single-valued function; by repeated use of (23.2) and (23.7), and after laborious calculations—which we have omitted here—we obtain

$$\iint_{(\sigma)} \left\{ (e_{11} - wk_{11})\,\delta T_{11} + (e_{12} - wk_{12})\,\delta T_{12} + \overrightarrow{\underleftarrow{1,2}} \right\} d\sigma =$$
$$= \iint_{(\sigma)} \left\{ \frac{1}{Et}\Delta\Delta\psi + \varkappa_{11}(k_{22} + \varkappa^0_{22} + \varkappa_{22}) + \varkappa_{22}(k_{11} + \varkappa^0_{11} + \varkappa_{11}) - \right.$$
$$\left. - \varkappa^2_{12} - 2\varkappa_{12}(k_{12} + \varkappa^0_{12}) \right\} \delta\psi\, d\sigma + I. \tag{23.25}$$

where

$$I = \int_C \left\{ \left(\eta_1 \frac{n_2}{A_2} - \frac{1}{2}\eta_{12}\frac{n_1}{A_2} \right) \delta\psi_{,2} + \frac{1}{A_1 A_2} \left[A_{2,1} n_1 \eta_1 + A_{1,2} n_1 \eta_{12} - \right. \right.$$
$$\left. \left. - n_2 (A_1\eta_1)_{,2} + \frac{1}{2} A_2 n_2 \eta_{12,1} \right] \delta\psi + \overrightarrow{\underleftarrow{1,2}} \right\} ds. \tag{23.26}$$
$$\eta_1 = \varepsilon_{11} - wk_{11} - \frac{1}{2}\omega_1^2 - \omega_1^0\omega_1,$$
$$\eta_{12} = 2\varepsilon_{12} - 2wk_{12} - \omega_1(\omega_2 + \omega_2^0) - \omega_2\omega_1^0 \quad \overrightarrow{\underleftarrow{1,2}},$$

Here we have assumed that ε_{ij} are expressed in terms of ψ, and

$$e_{11} - \frac{1}{2}\omega_1^2 - \omega_1^0\omega_1,\ e_{12},\ e_{22}$$

have been considered here only as expressions for quantities which are not necessarily related to the displacements.

The derivation of (23.25) may be considerably simplified by taking into account that the first term of the surface integral is the well-known invariant of coordinate transformation, and the second term, which expresses the variation of the Gaussian curvature of the surface during deformation, is also an invariant. Consequently, one may find at first the expression of this integral in Cartesian coordinates and dfterwards express the invariants in terms of arbitrary curvilinear coordinates.

Introducing (23.25) we obtain the following variational equation equivalent to (23. 1):

$$\iint_{(\sigma)} \left\{ D\Delta\Delta w + T_{11}(k_{11} + \varkappa^0_{11} + \varkappa_{11}) + 2T_{12}(k_{12} + \varkappa^0_{12} + \varkappa_{12}) + \right.$$
$$\left. + T_{22}(k_{22} + \varkappa^0_{22} + \varkappa_{22}) + p \right\} \delta w\, d\sigma - \iint_{(\sigma)} \left\{ \frac{1}{Et}\Delta\Delta\psi + \varkappa_{11}(k_{22} + \varkappa^0_{22} + \varkappa_{22}) + \right.$$
$$\left. + \varkappa_{22}(k_{11} + \varkappa^0_{11}) - \varkappa^2_{12} - 2\varkappa_{12}(k_{12} + \varkappa^0_{12}) \right\} \delta\psi\, d\sigma + \tag{23.27}$$
$$+ \int_C \left\{ \left[N_1 n_1 + N_2 n_2 - \frac{\partial H}{ds} + T\frac{\partial}{\partial n}(w + w^0) + S\frac{\partial}{\partial s}(w + w^0) - \Phi_3 \right] \delta w + \right.$$
$$\left. + (\widehat{G} - G)\,\delta\left(\frac{\partial w}{\partial n}\right) + \widetilde{u}_n \delta T + \widetilde{u}_s \delta S \right\} ds - I = 0.$$

If all contour integrals in this equation vanish, we obtain from it--since δw and $\delta\psi$ are independent--the equation

$$\iint\limits_{(\sigma)} \Big\{ D\Delta\Delta w + T_{11}(k_{11} + \varkappa_{11}^{0} + \varkappa_{11}) + 2T_{12}(k_{12} + \varkappa_{12}^{0} + \varkappa_{12}) + T_{22}(k_{22} + \varkappa_{22}^{0} + \varkappa_{22}) + p \Big\} \delta w d\sigma = 0; \tag{23.28}$$

$$\iint\limits_{(\sigma)} \Big\{ \frac{1}{Et} \Delta\Delta\psi + \varkappa_{11}(k_{22} + \varkappa_{22}^{0} + \varkappa_{22}) + \varkappa_{22}(k_{11} + \varkappa_{11}^{0}) - \varkappa_{12}^{2} - 2\varkappa_{12}(k_{12} + \varkappa_{12}^{0}) \Big\} \delta\psi d\sigma = 0. \tag{23.29}$$

These equations of the Bubnov-Galerkin method have been recommended by V. Z. Vlasov /0.4/ for the solution of the problems of the theory of shallow shells when the tangential forces are given along the entire edge contour, the function ψ satisfying the static boundary conditions (23.19) and (23.20) for the tangential forces, and the function w satisfying the geometrical boundary conditions (23.17) and (23.18).

It may easily be seen that in this case the equations (23.28) and (23.29) are indeed valid. In fact, expressing the conditions (23.19) and (23.20) in terms of ψ by means of (23.2), it is possible to prove that everywhere on the contour $\delta\psi = 0$ and $\delta\frac{\partial\psi}{\partial n} = 0$; therefore, owing to (23.11) the contour integrals in the expressions for I are also zero* and the other contour integrals in (23.27) vanish owing to (23.17) and (23.18). But if tangential displacements $\widetilde{u}_n$ and $\widetilde{u}_\tau$ are given on at least some parts of the contour, the quantities $\delta\psi$ and $\delta\frac{\partial\psi}{\partial n}$ may have arbitrary values on these parts of the contour. Consequently, the contour integrals enumerated can be zero only in the particular case when $\widetilde{u}_n$ and $\widetilde{u}_\tau$ satisfy certain differential relations, which we shall not give here owing to their complexity.

As may be seen from (23.27), the equations (23.28) and (23.29) are also applicable in those cases where ψ satisfies the static conditions (23.19) and (23.20) for the tangential forces along the entire contour and the function w either satisfies the static conditions (23.21) and (23.22) along the entire contour, or one of the conditions (23.17) and (23.21) is satisfied together with one of the conditions (23.18) and (23.22).

If tangential displacements are given on the entire contour, the function ψ must satisfy the condition of compatibility and the geometrical boundary conditions. As may be seen from (23.16), the variational equation (23.1) leads in this case to the same Bubnov-Galerkin equation as does the Lagrange principle.

If tangential displacements are given along a part of the contour and tangetial forces along the rest of it, but the particular case considered above not valid, one has to use the equation (23.1) or (23.7). In our opinion, in such a case the problem must be solved with a high degree of approximation.

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* Results of an analogous transformation for a plate may be found in /VI.6/.

§ 24. Application of the Formula for Statically Allowed Variations in the State of Stress to the Theory of Shallow Shells

We have examined in the previous sections the variational methods for the solution of problems of the theory of shells based on the principle of virtual displacements; we have also considered the mixed method, where the bending and the stress functions are varied.

In this section we shall elaborate a new method of solution for the problems of the theory of shallow shells based on possible variations of the tangential forces.

★ Let us give some relations of the theory of shallow shells which are needed in the following.

For $X^*_1 = 0$ and $X^*_2 = 0$ the first two equations of equilibrium (15.8) may be satisfied, with the error inherent in the theory of shallow shells, by the force functions ψ, according to formulas (15.11)

$$T^*_{11} = \frac{1}{A_2} \cdot \frac{\partial \Psi_2}{\partial \alpha_2} + \frac{\Psi_1}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1}, \quad T^*_{12} = -\frac{1}{A_2} \cdot \frac{\partial \Psi_1}{\partial \alpha_2} + \frac{\Psi_2}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1} \quad \overrightarrow{\underleftarrow{1,2}},$$

where

$$A_i \Psi_i = \psi_{,i} \qquad (i = 1, 2). \tag{24.1}$$

Substituting (24.1) in (15.8) and using Gauss' formula for the undeformed surface (2.27), it may be seen that (25.3) will be satisfied in the assumed degree of approximation.

Let M^0_{ik} (i, k = 1, 2) be the particular solutions of the third equation of equilibrium (15.9) without taking the tangential forces into account. Then, the equation (15.9) will be identically satisfied by the functions of moments ψ_1 and ψ_2 given by

$$M^*_{11} = M^0_{11} + \Psi_{22}, \quad M^*_{12} = M^0_{12} - \frac{1}{2}(\Psi_{12} + \Psi_{21}) \quad \overrightarrow{\underleftarrow{1,2}}, \tag{24.2}$$

where

$$\begin{aligned} \Psi_{11} &= \frac{1}{A_1} \cdot \frac{\partial \psi_1}{\partial \alpha_1} + \frac{\psi_2}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} + k^*_{11} \psi, \\ \Psi_{12} &= \frac{1}{A_1} \cdot \frac{\partial \psi_2}{\partial \alpha_1} - \frac{\psi_1}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} + k^*_{12} \psi \quad \overrightarrow{\underleftarrow{1,2}}. \end{aligned} \tag{24.3}$$

Sustituting (24.2) in (15.9) and using Codazzi's conditions for a deformed surface

$$(A_2 k^*_{12})_{,1} - (A_1 k^*_{11})_{,2} + k^*_{12} A_{2,1} + k^*_{22} A_{1,2} = 0 \quad \overrightarrow{\underleftarrow{1,2}}, \tag{24.4}$$

it may be seen that (15.9) will be identically satisfied. In order to determine the values of the forces, moments, and deformations on the contour, let us consider a right-handed trihedron $\{\bar{\tau}, \bar{m}, \bar{n}\}$:

$$\bar{\tau} = [\bar{m}, \bar{n}], \quad \bar{m} = [\bar{n}, \bar{\tau}], \quad \bar{n} = [\bar{\tau}, \bar{m}] \tag{24.5}$$

★ with the apex at a point on the contour C of the shell. Here $\bar{\tau}$ is the unit tangent to the contour C; $\bar{n}$ is the unit binormal to the contour, and $\bar{m}$ the unit normal to the middle surface. We shall, furthermore, introduce the following geometrical quantities:

$\tilde{\sigma} = -\sum_{i,j} k_{ij}\tau_i\tau_j$ --the normal curvature of the middle surface on the contour of the shell in the direction $\bar{\tau}$;

$\tilde{t}(s) = \sum_{i,j} k_{ij}\tau_i n_j$ --the geodesic torsion of the curve C;

$$\tau_i = \bar{e}_i\,\bar{\tau}, \quad n_i = \bar{e}_i\,\bar{n},$$

$\varkappa = \bar{\tau}\,\dfrac{d\bar{n}}{ds} = \operatorname{div}\bar{n}$ --the geodesic curvature of the curve.

We shall denote the tangential and the normal derivatives of the function ψ as follows:

$$\frac{d\psi}{ds} = \sum_{i=1}^{2}\frac{\tau_i}{A_i}\cdot\frac{\partial\psi}{\partial\alpha_i}, \qquad \frac{d\psi}{dn} = \sum_{i=1}^{2}\frac{n_i}{A_i}\cdot\frac{\partial\psi}{\partial\alpha_i}. \tag{24.6}$$

Let ε_n and ε_s be the relative elongations at the contour in the directions $\bar{n}$ and $\bar{\tau}$, and ε_{ns} the shear angle between $\bar{n}$ and $\bar{\tau}$; let also $\varkappa_n$, $\varkappa_s$, and $\varkappa_{ns}$ be the parameters of curvature in the system of the coordinate axes along $\bar{n}$ and $\bar{\tau}$.

Then, according to the transformation formulas for the components of deformation we have

$$\varepsilon_n = \sum_{i,j}\varepsilon_{ij}n_in_j, \quad \varepsilon_s = \sum_{i,j}\varepsilon_{ij}\tau_i\tau_j, \quad \varepsilon_{ns} = \sum_{i,j}\varepsilon_{ij}n_j\tau_i,$$
$$\varepsilon_{ij} = \varepsilon_n n_in_j + \varepsilon_s\tau_i\tau_j + \varepsilon_{ns}(\tau_in_j + \tau_jn_i); \tag{24.7}$$

$$\varkappa_n = \sum_{i,j}\varkappa_{ij}n_in_j, \quad \varkappa_s = \sum_{i,j}\varkappa_{ij}\tau_i\tau_j, \quad \varkappa_{ns} = \sum_{i,j}\varkappa_{ij}\tau_in_j,$$
$$\varkappa_{ij} = \varkappa_n n_in_j + \varkappa_s\tau_i\tau_j + \varkappa_{ns}(\tau_in_j + \tau_jn_i) \tag{24.8}$$

Resolving the displacement vector $\bar{v}$ of the points of the contour along the axes of the trihedron $\{\bar{\tau}, \bar{m}, \bar{n}\}$ and assuming $\bar{v} = u\bar{n} + v\bar{\tau} + w\bar{m}$, using (15.1), (15.7), (24.7) and (24.8), and the rule for differentiating the unit vectors of the trihedron /III.3/

$$\frac{d\bar{\tau}}{ds} = \tilde{\sigma}\bar{m} - \varkappa\bar{n}, \quad \frac{d\bar{m}}{ds} = \tilde{t}\bar{n} - \tilde{\sigma}\bar{\tau}, \quad \frac{d\bar{n}}{ds} = \varkappa\bar{\tau} - \tilde{t}\bar{m},$$
$$\frac{d\bar{\tau}}{dn} = -\tilde{t}\bar{m}, \quad \frac{d\bar{m}}{dn} = \tilde{t}\bar{\tau} - \tilde{\sigma}_n\bar{n}, \quad \frac{d\bar{n}}{dn} = \tilde{\sigma}_n\bar{m}, \tag{24.9}$$

where $\tilde{\sigma}_n = -(\tilde{\sigma} + k_{11} + k_{22})$ is the normal curvature in the direction $\bar{n}$, we obtain:

$$\varepsilon_n = \frac{du}{dn} + (\tilde{\sigma} + k_{11} + k_{22})\,w + \frac{1}{2}\left(\frac{dw}{dn}\right)^2,$$
$$\varepsilon_s = \frac{dv}{ds} + \varkappa u - \tilde{\sigma}w + \frac{1}{2}\left(\frac{dw}{ds}\right)^2, \tag{24.10}$$
$$2\varepsilon_{ns} = \frac{du}{ds} + \frac{dv}{dn} - \varkappa v + 2\tilde{t}w + \frac{dw}{ds}\cdot\frac{dw}{dn}.$$

$$\varkappa_n = -\frac{d^2w}{dn^2}, \quad \varkappa_s = -\frac{d^2w}{ds^2} - \varkappa\frac{dw}{dn}, \quad \varkappa_{ns} = -\frac{d}{dn}\left(\frac{dw}{ds}\right) =$$
$$= -\frac{d}{ds}\left(\frac{dw}{dn}\right) + \varkappa\frac{dw}{ds}. \tag{24.11}$$

★ Let T_n and S be the normal and tangential forces at the contour, and T_τ and S the same forces on the surface element with normal $\bar{\tau}$. Then, on replacing T_{ij} by the stress function (24.1) in the transformation formulas

$$T_n = \sum_{i,j} T_{ij} n_i n_j, \quad S = \sum_{i,j} T_{ij} \tau_i \tau_j, \quad T_\tau = \sum_{i,j} T_{i,j} \tau_i n_j \tag{24.12}$$

we find /III.11/:

$$\begin{aligned} T_n = \frac{d^2\psi}{ds^2} + \varkappa \frac{d\psi}{dn}, \quad T_\tau = \frac{d^2\psi}{dn^2}, \quad S = -\frac{d}{dn}\left(\frac{d\psi}{ds}\right) = \\ = -\frac{d}{ds}\left(\frac{d\psi}{dn}\right) + \varkappa \frac{d\psi}{ds}. \end{aligned} \tag{24.13}$$

Furthermore, let G and H be the bending and twisting moments on the contour, and G_τ and H_τ the corresponding moments on the section with a normal $\bar{\tau}$. Then, after replacing M_{ik} by the deflection in the transformation formulas

$$G = \sum_{i,j} M_{ij} n_i n_j, \quad H = -H_\tau = -\sum_{i,j} M_{ij} n_i \tau_j, \quad G_\tau = \sum_{i,j} M_{ij} \tau_i \tau_j, \tag{24.14}$$

we obtain the expressions for the moments at the contour:

$$\begin{aligned} G = -D\left(\frac{d^2w}{dn^2} + \nu \frac{d^2w}{ds^2} + \nu\varkappa \frac{dw}{dn}\right), \quad H = -H_\tau = D(1-\nu)\frac{d}{dn}\left(\frac{dw}{ds}\right), \\ G_\tau = -D\left(\frac{d^2w}{ds^2} + \varkappa \frac{dw}{dn} + \nu \frac{d^2w}{dn^2}\right). \end{aligned} \tag{24.15}$$

In view of (24.2) they can be expressed in terms of stress functions as follows:

$$\begin{aligned} G = G^0 + \frac{d\psi_s}{ds} + \varkappa\psi_n - \bar{\sigma}_n\psi \\ H = H^0 + \frac{1}{2}\left(\frac{d\psi_n}{ds} + \frac{d\psi_s}{dn} - \varkappa\psi_s\right) - \bar{t}_n\psi. \\ G_\tau = G_\tau^0 + \frac{d\psi_n}{dn} - (\bar{\sigma}_n + k_{11}^* + k_{22}^*)\psi. \end{aligned} \tag{24.16}$$

where

$$\begin{aligned} G^0 = \sum_{i,j} M_{ij}^0 n_i n_j, \quad H^0 = -\sum_{i,j} M_{ij}^0 n_i \tau_j, \\ G_\tau = \sum_{i,j} M_{ij} \tau_i \tau_j, \quad \psi_s = \sum_i \psi_i \tau_i, \quad \psi_n = \sum_i \psi_i n_i. \end{aligned} \tag{24.17}$$

Let us consider the variational formula (13.7) for shallow shells. Using the simplified relations of the theory of shallow shells given in Chapter IV, we obtain from (13.7) the variational formula of the principle of allowed variations of the state of stress

$$\begin{aligned} \iint_{(\sigma)} (\bar{v}\,\delta\bar{X} - \omega_1 \delta L_1 - \omega_2 \delta L_2)\, d\sigma + \int_C \left(\bar{v}\,\delta\bar{\Phi}_s - \frac{dw}{dn}\,\delta\,\hat{G}\right) ds + \\ + w\delta\hat{H}|_C = \delta \iint_{(\sigma)} \left(W + \sum_{i,k} \frac{1}{2} T_{ik}\omega_i\omega_k\right) d\sigma, \quad \omega_i = \frac{1}{A_i} \cdot \frac{\partial w}{\partial \alpha_i}. \end{aligned} \tag{24.18}$$

Here W is the additional work of deformation and δW its variation, which is

$$\delta W = \sum_{i,k} (\hat{\varepsilon}_{ik}\delta T_{ik} + \hat{\varkappa}_{ik}\delta M_{ik}) = \sum_{i,k} (\varepsilon_{ik}\delta T_{ik} + \varkappa_{ik}\delta M_{ik}). \tag{24.19}$$

* $\overset{\wedge}{\varepsilon}_{ik}$ and $\overset{\wedge}{\varkappa}_{ik}$ are the components of deformation of the surface, expressed in terms of forces and moments, according to the elasticity relations:

$$\overset{\wedge}{\varepsilon}_{11} = K'(T_{11} - \nu T_{22});\quad \overset{\wedge}{\varepsilon}_{22} = K'(T_{22} - \nu T_{11});\quad \overset{\wedge}{\varepsilon}_{12} = K'(1+\nu)\,T_{12};$$
$$\overset{\wedge}{\varkappa}_{11} = D'(M_{11} - \nu M_{22});\quad \overset{\wedge}{\varkappa}_{22} = D'(M_{22} - \nu M_{11});\quad \overset{\wedge}{\varkappa}_{12} = D'(1+\nu)\,M_{12}, \tag{24.19a}$$

ε_{ik} and $\varkappa_{ik}$ being the same quantities but expressed in terms of displacements according to (15.7).

The variational equation (24.18) will be valid if the variations of the forces and moments satisfy the equations of equilibrium (15.8) and (15.9), and if the variation of the state of stress is performed without variation of the contour forces and moments:

$$\delta\overline{\Phi} = 0,\quad \delta G = 0,\quad \delta\widetilde{H} = 0 \tag{24.20}$$

or if the boundary conditions for hinging or clamping

$$\bar{v} = 0,\quad G = \widetilde{G},\quad \bar{v} = 0,\ \frac{dw}{dn} = 0 \tag{24.20a}$$

are satisfied at the contour.

If the angles of rotation ω_i are not varied, then, according to (24.18) the states being approximated are the statically allowed ones under continuous deformations. But if ω_i are varied, the approximated states do not possess this property (§ 13). We shall examine only the first case, where the displacements do not vary.

On integrating by parts, we can obtain from the variational formula (24.18) the equations for forces and moments of the Bubnov-Galerkin method for the integration of the conditions of continuity; it also enables us to formulate the boundary conditions for the theory of shallow shells in terms of forces and moments.

Substituting for $\delta\overline{X}$ and $\delta\overline{L}_i$ from the equations of equilibrium (7.1) and (7.5) in the variational equation (24.18) and integrating by parts the terms with derivatives of forces and moments by the formula

$$\iint\limits_{(\sigma)} \sum_{i=1}^{2} f\frac{d\varphi}{d\alpha_i}\,d\sigma = \int\limits_{C} \sum_{i=1}^{2} f\varphi A_i n_i\,ds - \iint\limits_{(\sigma)} \sum_{i} \varphi\frac{\partial f A_1 A_2}{\partial\alpha_i}\,d\alpha_1 d\alpha_2, \tag{24.21}$$

we find a new variational equation

$$\int\limits_{C}\left[\bar{v}\delta(\overline{\Phi}_s - \widetilde{\Phi}) - \frac{dw}{dn}\delta(\widetilde{G} - G)\right] ds + w\delta(\widetilde{H} - H)\Big|_C =$$
$$= \iint\limits_{(\sigma)} \sum_{ik}[(\overset{\wedge}{\varepsilon}_{ik} - \varepsilon_{ik})\delta T_{ik} + (\overset{\wedge}{\varkappa}_{ik} - \varkappa_{ik})\delta M_{ik}]\,d\sigma. \tag{24.22}$$

Since in (24.22) we approximated the statically allowed states, the contour term and the non-integrated term vanish. Consequently,

$$\iint\limits_{(\sigma)} \sum_{i,k}[(\overset{\wedge}{\varepsilon}_{ik} - \varepsilon_{ik})\delta T_{ik} + (\overset{\wedge}{\varkappa}_{ik} - \varkappa_{ik})\delta M_{ik}]\,d\sigma = 0. \tag{24.23}$$

Whence, in view of the arbitrariness of δT_{ik} and δM_{ik}, one can derive the elasticity

★ relations. These also lead to the conditions of continuity for forces and moments and also to the natural boundary conditions for the functions of forces and moments.

In order to obtain the equations of the Bubnov-Galerkin method, we substitute for δT_{ik} and δM_{ik} from (24.1), (24.2), into (24.23). Since the particular solutions do not vary, we obtain from (24.23)

$$\iint\limits_{(\sigma)} \Big[(\hat{\varepsilon}_{11} - \varepsilon_{11}) \left(\frac{\partial \delta\Psi_2}{A_2 \partial \alpha_2} + \frac{\delta\Psi_1}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1} \right) +$$

$$+ (\hat{\varepsilon}_{12} - \varepsilon_{12}) \left(- \frac{\partial \delta\Psi_1}{A_2 \partial \alpha_2} + \frac{\delta\Psi_2}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1} \right) + \underline{\overrightarrow{1,2}} \Big] + \Big[(\hat{\varkappa}_{11} - \varkappa_{11}) \delta\Psi_{22} +$$

$$+ (\hat{\varkappa}_{22} - \varkappa_{22}) \delta\Psi_{11} - (\hat{\varkappa}_{12} - \varkappa_{12}) \delta(\Psi_{12} + \Psi_{21}) \Big] d\sigma = 0.$$

Since ω_i does not vary and $\delta\psi$ and $\delta\psi_i$ are absolutely arbitrary, we obtain two variational equations:

$$\iint\limits_{(\sigma)} \Big\{ \Big[(\hat{\varepsilon}_{11} - \varepsilon_{11}) \left(\frac{\partial \delta\Psi_2}{A_2 \partial \alpha_2} + \frac{\delta\Psi_1}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1} \right) + \dots \Big] -$$

$$- [(\hat{\varkappa}_{11} - \varkappa_{11}) k_{22}^{*} + (\hat{\varkappa}_{22} - \varkappa_{22}) k_{11}^{*} - 2 (\hat{\varkappa}_{12} - \varkappa_{12}) k_{12}^{*}] \delta\psi \Big\} d\sigma = 0; \tag{24.24}$$

$$\iint\limits_{(\sigma)} \Big[(\hat{\varkappa}_{11} - \varkappa_{11}) \left(\frac{1}{A_2} \cdot \frac{\partial \delta\psi_2}{\partial \alpha_2} + \frac{\delta\psi_1}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1} \right)$$

$$- (\hat{\varkappa}_{12} - \varkappa_{12}) \left(\frac{1}{A_1} \cdot \frac{\partial \delta\psi_2}{\partial \alpha_1} - \frac{\delta\psi_1}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} \right) + \underline{\overrightarrow{1,2}} \Big] d\sigma = 0, \tag{24.25}$$

the former containing the variation $\delta\psi$ and the latter the variation $\delta\psi_i$. Let us consider the first of these variational equations. The first square brackets contain the derivatives $\partial\delta\psi/\partial\alpha_i$ and $\partial^2\delta\psi/\partial\alpha_i\partial\alpha_k$. After integrating the expression (24.21) twice by parts we can get rid of them. After integrating by parts the terms containing $\varkappa_{ik}$ and ε_{ik} in equation (24.24) we obtain the condition of compatibility for the actual state (Gauss' condition). Consequently, the corresponding expression obtained by integration becomes equal to zero. In equation (24.25) the terms containing $\varkappa_{ik}$ and their derivatives vanish by virtue of the conditions of continuity, which follows from Codazzi's conditions.

As a result of these calculations, which we shall not give here owing to their laboriousness, we finally obtain istead of (24.24):

$$\iint\limits_{(\sigma)} L \delta\psi \, d\alpha_1 d\alpha_2 = I_1, \tag{24.26}$$

where L is the left-hand side of the equation of compatibility (15.16):

$$L = K' \Delta\Delta\phi - (\hat{\varkappa}_{12}^2 - \hat{\varkappa}_{11}\hat{\varkappa}_{22} - \hat{\varkappa}_{11}k_{22} - \hat{\varkappa}_{22}k_{11} + 2k_{12}\hat{\varkappa}_{12}), \tag{24.26a}$$

and I_1 represents the contour integral

$$I_1 = \int\limits_C \Big[\left(\varkappa A_n - \varkappa A_s - \frac{dA_s}{dn} + 2 \frac{dA_{ns}}{ds} \right) \delta\psi + A_s \frac{d\delta\psi}{dn} \Big] ds - \psi A_{ns} \Big|_C . \tag{24.27}$$

★ Here

$$A_n = \varepsilon_n - K'(T_n - \nu T_\tau), \quad A_s = \varepsilon_s - K'(T_\tau - \nu T_n), \qquad (24.28)$$
$$A_{ns} = \varepsilon_{ns} - K'(1+\nu)S.$$

The quantities A_n, A_s are considered, for the present. as being different from zero, because they will approach zero during the solution of the problem.

The variational equation (24.21) enables us to establish the boundary condition for the force function ψ and to validate the application of the Bubnov-Galerkin method to the integration of the equation (15.16).

1. Let the tangential forces T_n and S be given on the contour. Since these forces are given, we obtain from (24.13) the conditions

$$\delta T_n = \frac{d^2\delta\psi}{ds^2} + \varkappa\frac{d\delta\psi}{dn} = 0, \quad \delta S = -\frac{d}{ds}\left(\frac{d\delta\psi}{dn}\right) + \varkappa\frac{d\delta\psi}{ds} = 0,$$

which are satisfied for $\delta\psi = \frac{d\delta\psi}{dn} = 0$. Hence, in this case the contour integral I_1 in (24.27) vanishes so that the Bubnov-Galerkin method may be used for integrating the equation of compatibility (15.16). This case has been considered by V.Z. Vlasov.

2. Let the tangential displacements $\tilde{u}$ and $\tilde{v}$ be given on the contour of the shell, and let $\delta\psi$ and $\frac{d\delta\psi}{dn}$ be arbitrary functions. This case has been considered by N.A. Alumyae /VI.6/.

In view of the arbitrariness of $\delta\psi$ and $\frac{d\delta\psi}{dn}$, the integral I_1 vanishes if the following conditions are satisfied on the contour:

$$\varkappa A_n - \frac{dA_s}{dn} + 2\frac{dA_{ns}}{ds} = 0, \quad A_s = 0, \quad \psi A_{ns}\Big|_C = 0,$$

or, taking into account (24.28)

$$\varkappa\varepsilon_n - \frac{d\varepsilon_s}{dn} + 2\frac{d\varepsilon_{ns}}{ds} = \varkappa F_n - \frac{dF_s}{dn} + 2\frac{dF_{ns}}{ds}, \qquad (24.29)$$
$$\varepsilon_s = F_s, \quad \psi A_{ns}|_C = 0,$$

where

$$F_n = K'(T_n - \nu T_\tau), \quad F_s = K'(T_\tau - \nu T_n), \quad F_{ns} = K'(1+\nu)S. \qquad (24.30)$$

We assume that T_n, T_τ, S are expressed in terms of ψ according to (24.13). Substituting for ε_n, ε_s and ε_{ns} from (24.10) in (24.29) and using the Guass-Codazzi conditions in the form given in /III.11/,

$$\frac{d\hat{\sigma}_n}{ds} + \frac{d\hat{t}}{dn} + 2\varkappa\hat{t} = 0, \quad \frac{d\hat{t}}{ds} + \frac{d\hat{\sigma}}{dn} + \varkappa(\hat{\sigma} - \hat{\sigma}_n) = 0, \qquad (24.31)$$
$$\varkappa^2 + \frac{d\varkappa}{dn} = \hat{t}^2 - \hat{\sigma}\hat{\sigma}_n, \quad \hat{\sigma}_n = -\hat{\sigma} - k_{11} - k_{22},$$

we find the conditions which must be satisfied by the given displacements

★

$$\frac{d^2u}{ds^2} - u\frac{d\varkappa}{dn} - v\frac{d\varkappa}{ds} + \frac{d^2w}{ds^2}\cdot\frac{dw}{dn} + \varkappa\left[-\tilde{\sigma}w + \frac{1}{2}\left(\frac{dw}{dn}\right)^2 + \left(\frac{dw}{ds}\right)^2\right] +$$
$$+ w\frac{d\tilde{\tau}}{ds} + 2\tilde{\tau}\frac{dw}{ds} + \tilde{\sigma}\frac{dw}{dn} = \varkappa F_n - \frac{dF_s}{dn} + 2\frac{dF_{ns}}{ds} = f. \qquad (24.32)$$
$$\frac{dv}{ds} + \varkappa u - \tilde{\sigma}w + \frac{1}{2}\left(\frac{dw}{ds}\right)^2 = F_s, \quad \delta\psi A_{ns}\Big|_C = 0.$$

Here we have eliminated the unknown mixed derivatives of the given displacements by using the formulas

$$\frac{d}{ds}\left(\frac{d\varphi}{dn}\right) - \frac{d}{dn}\left(\frac{d\varphi}{ds}\right) = \varkappa\frac{d\varphi}{ds}. \qquad (24.33)$$

where φ is an arbitrary function. The right-hand side of (24.32), denoted by f, may be written as

$$f = K'\left[T_n\varkappa(1-\nu) - \frac{dT_\tau}{dn} + (2+\nu)\frac{dS}{dn}\right],$$
$$F_s = K'(T_\tau - \nu T_n), \qquad (24.34)$$

when using the condition of equilibrium on the contour /III.11/

$$\frac{dT_n}{d_n} + \frac{dS}{ds} + (T_n - T_\tau)\varkappa = 0.$$

The formulas (24.32) may be considerably simplified for particular cases. For instance, if the contour is a geodesic middle surface then $\varkappa = 0$ (rectangular cylindrical strip). They may also be simplified when the bending w is subordinated to the boundary conditions of hinging or clamping. For these cases we obtain, respectively, the conditions

$$\tilde{\sigma}\frac{dw}{dn} = K'\left[(2+\nu)\frac{dT_n}{ds} - \frac{dT_\tau}{ds}\right], \quad T_\tau = \nu T_n; \quad \delta\psi A_{ns}\Big|_C = 0, \qquad (24.35)$$

$$(2+\nu)\frac{dT_n}{ds} - \frac{dT_\tau}{dn} = 0, \quad T_\tau = \nu T_n, \quad \delta\psi A_{ns}\Big|_C = 0. \qquad (24.36)$$

3. Let the normal force T_n and the binormal displacement $\tilde{v}$ be given on the contour of the shell:

$$T_n = \frac{d^2\psi}{ds^2} + \varkappa\frac{d\psi}{dn}, \quad v = \tilde{v} \qquad (24.37)$$

Substituting for $\frac{d\delta\psi}{dn} = -\frac{1}{\varkappa}\cdot\frac{d^2\delta\psi}{ds^2}$ in (24.27) we obtain

$$\int\limits_C\left[\left(\varkappa A_n - \varkappa A_s - \frac{dA_s}{dn} + 2\frac{dA_{ns}}{ds}\right)\delta\psi - \frac{A_s}{\varkappa}\cdot\frac{d^2\delta\psi}{ds^2}\right]ds = \delta\psi A_{ns}\Big|_C.$$

or integrating the last term by parts, since $\delta\psi$ is arbitrary, we obtain:

$$(A_n - A_s)\varkappa - \frac{dA_s}{dn} + 2\frac{dA_{ns}}{ds} - \frac{d^2}{ds^2}\left(\frac{A_s}{\varkappa}\right) = 0,$$
$$\left\{\delta\psi A_{ns} + \frac{A_s}{\varkappa}\cdot\frac{d\delta\psi}{ds} - \delta\psi\frac{d}{ds}\left(\frac{A_s}{\varkappa}\right)\right\}\Big|_C = 0.$$

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★ Whence, substituting for A_n, A_s, and A_{ns} from (24.28) we obtain

$$\frac{\varkappa}{2}\left[\left(\frac{dw}{dn}\right)^2+\left(\frac{dw}{ds}\right)^2\right]+\frac{d^2w}{ds^2}\cdot\frac{dw}{dn}+\tilde{\sigma}\frac{dw}{dn}+2\tilde{t}\frac{dw}{ds}+w\frac{d\hat{t}}{ds}-\frac{d\varkappa v}{ds}-$$
$$-\frac{d^2}{ds^2}\left[\frac{1}{\varkappa}\cdot\frac{dv}{ds}-\frac{\tilde{\sigma}}{\varkappa}w+\frac{1}{2\varkappa}\left(\frac{dw}{ds}\right)^2\right]=$$
$$=(F_n-F_s)\varkappa-\frac{dF_s}{dn}-2\frac{dF_{ns}}{ds}-\frac{d^2}{ds^2}\left(\frac{F_s}{\varkappa}\right); \tag{24.38}$$

Here we have used the Gauss-Codazzi conditions in the form (24.31). Thus, for the integration of the equation of compatibility according to the Bubnov-Galerkin method, with the boundary conditions (24.37), it is necessary that the conditions (24.38) be satisfied and that the term outside the integral should vanish. If ψ is single-valued, this term vanishes identically. The condition (24.38) may be replaced by a simpler one for contours on which $\varkappa = 0$.

In this case, we obtain from (24.37) $\frac{d^2\delta\psi}{ds^2}=0$. Consequently, we can assume that on the contour, $\delta\psi = C$. Then the condition $I_1 = 0$ becomes

$$\int\limits_C A_s\frac{d\delta\psi}{dn}ds=0.$$

Since in the general case $\frac{d\delta\psi}{dn}\neq 0$ it results from this that $A_s = 0$. Then we obtain for ψ the boundary conditions

$$\delta\psi=0,\quad \frac{dv}{ds}-\tilde{\sigma}w+\frac{1}{2}\left(\frac{dw}{ds}\right)^2=F_s=K'(T_s-\nu T_n),\quad \delta\psi A_{ns}\Big|_C=0. \tag{24.39}$$

4. The binormal component of the contour load S and the displacement along the binormal $\tilde{u}$ are on the contour of the shell

$$S=-\frac{d}{ds}\left(\frac{d\psi}{dn}\right)+\varkappa\frac{d\psi}{ds};\ u=\tilde{u}. \tag{24.40}$$

In this case the function ψ satisfies the conditions /III.11/

$$\frac{d^2u}{ds^2}-u\frac{d\varkappa}{dn}+\varkappa\left[\frac{1}{2}\left(\frac{dw}{dn}\right)^2+\left(\frac{dw}{ds}\right)^2-\tilde{\sigma}w\right]+w\frac{d\tilde{t}}{ds}+2\tilde{t}\frac{dw}{ds}+$$
$$+\frac{d^2w}{ds^2}\cdot\frac{dw}{dn}+\tilde{\sigma}\frac{dw}{dn}+\frac{d\varkappa}{ds}\int^s\left[\varkappa u-\tilde{\sigma}w+\frac{1}{2}\left(\frac{dw}{ds}\right)^2\right]ds=$$
$$=\varkappa F_n-\frac{dF_s}{dn}+2\frac{dF_{ns}}{ds}+\frac{d\varkappa}{ds}\int^s F_s ds, \tag{24.41}$$

$$\left\{\frac{d\delta\psi}{dn}\cdot\int^s A_s ds-\varkappa\delta\psi\int^s A_s ds-\delta\psi A_{ns}\right\}\Big|_C=0 \tag{24.41a}$$

The latter expresses the condition that the function ψ should be single-valued. The boundary condition for $\varkappa = 0$ may be obtained directly from (24.41). If the edge of the shell is hinged or clamped we obtain, instead of (24.41), respectively

$$\tilde{\sigma}\frac{dw}{dn}=2\frac{dF_{ns}}{ds}-\frac{dF_s}{dn} \tag{24.41b}$$

$$2\frac{dF_{ns}}{ds}=\frac{dF_s}{dn}. \tag{24.41c}$$

★ The condition $I_1 = 0$ can also be exactly satisfied. The boundary conditions for ψ, found in this section, may be used for solving the problems not only by the Bubnov-Galerkin method, but also by different ones.

From the variational equation (24.25) we can obtain two of the Codazzi conditions for the moments. Integrating by parts the terms containing derivatives of ψ_i we can obtain

$$I_2 = \iint_{(\Omega)} \left\{ \left[(A_2 \hat{\varkappa}_{21})_{,1} - (A_1 \hat{\varkappa}_{12})_{,2} - \hat{\varkappa}_{11} A_{2,1} - \hat{\varkappa}_{12} A_{1,2} \right] \delta\psi_1 + \overleftrightarrow{1,2} \right\} d\alpha_1 d\alpha_2, \tag{24.42}$$

where

$$I_3 = \int_{C} \left[(\hat{\varkappa}_s - \varkappa_s)\, \delta\psi_n - (\hat{\varkappa}_{ns} - \varkappa_{ns})\, \delta\psi_s \right] ds \tag{24.43}$$

and $\delta\psi_n$ and $\delta\psi_s$ are the projections of the vector $\delta\bar{\psi}$ on the normal and the binormal to the contour of the shell, and $\varkappa_s$ and $\varkappa_{ns}$ are the bending deformations on the contour (24.11); $\hat{\varkappa}_s$ and $\hat{\varkappa}_{ns}$ are the same quantities expressed in terms of moments:

$$\hat{\varkappa}_s = D'(G_s - \nu G), \quad \hat{\varkappa}_{ns} = D'(1+\nu) H, \quad \hat{\varkappa}_n = D'(G - \nu G_s). \tag{24.44}$$

On replacing $\hat{\varkappa}_{ik}$ by (24.19a) in the variational equation (24.42), it becomes

$$I_2 = D' \iint_{(\Omega)} (L_1 \delta\psi_1 + L_2 \delta\psi_2)\, d\alpha_1 d\alpha_2, \tag{24.45}$$

where

$$L_i = \frac{1}{A_i} \cdot \frac{\partial M}{\partial \alpha_i} - (1+\nu) N_i, \quad M = M_{11} + M_{22}, \tag{24.46}$$

and N may be determined from (24.3).

Since $\delta\psi_i$ is arbitrary, the Codazzi conditions for the moments $L_1 = L_2 = 0$ and the elasticity relation on the contour $\hat{\varkappa}_s = \varkappa_s$ and $\hat{\varkappa}_{ns} = \varkappa_{ns}$, result from (24.45). These elasticity conditions may be expressed as

$$D'(G_s - \nu G) = -\frac{d^2 w}{ds^2} + \varkappa \frac{dw}{dn}; \quad D'(1+\nu) H = \varkappa \frac{dw}{dn} - \frac{d}{ds}\left(\frac{dw}{dn}\right), \tag{24.47}$$

where G_s, G, and H may be determined from (24.14). We note that the Codazzi condition $L_i = 0$ may be satisfied by a single function of moments φ, assuming

$$M_{11} = D(\mu_{11} + \nu\mu_{22}), \quad M_{12} = D(1-\nu)\mu_{12} \quad \overleftrightarrow{1,2}, \tag{24.48}$$

where

$$\mu_{ik} = -\frac{1}{A_i} \cdot \frac{\partial \Omega_k}{\partial \alpha_i} + \frac{(-1)^{i+k+1}}{A_i A_{3-i}} \cdot \frac{\partial A_i}{\partial \alpha_{3-i}} \Omega_{3-k},$$
$$\Omega_i = \frac{1}{A_i} \cdot \frac{\partial \varphi}{\partial \alpha_i}. \tag{24.48a}$$

Then, we shall have to deal only with the variational equation (24.26).

★ Let us now formulate the principal types of boundary conditions of the theory of shallow shells, for forces and moments.

1°. On the contour of the shell or on parts of it, let there be given: the vector of the external contour load Φ, and the bending moment $\widetilde{G}$. The static boundary conditions will be expressed in the following form:

$$\Phi_n = T_n, \quad \Phi_\tau = S, \quad \Phi_{m'} = N_1 n_1 + N_2 n_2 - \frac{dH}{ds} + T_n \frac{dw}{dn} + S \frac{dw}{ds}, \qquad (24.49)$$
$$\widetilde{G} = G.$$

From these we can eliminate the derivatives of w if the contour or a part of it is a geodesic of the middle surface ($\varkappa = 0$). In this case, we obtain from (24.11)

$$\varkappa_s = -\frac{d^2 w}{ds^2}, \quad \varkappa_{ns} = -\frac{d}{ds}\left(\frac{dw}{dn}\right). \qquad (24.50)$$

Whence we find

$$\frac{dw}{ds} = -\int^s \hat{\varkappa}_s ds + c_1, \quad \frac{dw}{dn} = -\int^s \hat{\varkappa}_{ns} ds + c_2,$$

where $\hat{\varkappa}_s$ and $\hat{\varkappa}_{ns}$ are assumed to be expressed in terms of the moments according to (24.44).

For an arbitrary contour the third condition of (24.49) may be satisfied by the variations

$$\delta\Phi_{m'} = n_1 \delta N_1 + n_2 \delta N_2 - \frac{d\delta H}{ds} = 0,$$

because w does not vary, and $\delta T_n = \delta S = 0$.

2°. The edges of the shell are freely supported: w = 0, $\widetilde{G} = 0$; from (24.11) with w = 0 we obtain the relation

$$\hat{\varkappa}_s = \varkappa\left(\int_0^s \hat{\varkappa}_{ns} ds + \text{const}\right).$$

Thus, all the boundary conditions are expressed in terms of forces and moments:

$$\Phi_n = T_n, \quad \Phi_\tau = S, \quad \widetilde{G} = 0, \quad \hat{\varkappa}_s - \varkappa\int_0^s \hat{\varkappa}_{ns} ds = \varkappa c, \qquad (24.51)$$

where c is a constant.

3°. The edges of the shell are hinged:

$$u = v = w = 0, \quad G = 0. \qquad (24.52)$$

The conditions u = v = 0 are equivalent to the boundary conditions (24.35):

$$\sigma \frac{dw}{dn} = K'\left[(2+\nu)\frac{dS}{ds} - \frac{dT_\tau}{dn}\right], \quad T_\tau = \nu T_n. \qquad (24.52a)$$

From (24.11) we find two different expressions for dw/dn:

$$\frac{dw}{dn} = -\int_0^s \hat{\varkappa}_{ns} ds + C_1, \quad \frac{dw}{dn} = -\int_0^s \frac{\hat{\varkappa}_s}{\varkappa} ds + C_2;$$

★ Equating these we have

$$\int_0^s \left(\frac{\hat{\varkappa}_s}{\tau} - \hat{\tau}_{ns}\right) ds = C = \text{const.} \qquad \textbf{(24.52b)}$$

Consequently, the conditions for hinging **(24.52)** are equivalent to the conditions **(24.52a)** and **(24.52b)**. They become simpler if the contour of the shell is a geodesic line of the middle surface ($\varkappa = 0$). In this case, we obtain from (24.47) $G_s - \nu G = 0$, i.e., $G_s = 0$. The conditions **(24.52)** are then equivalent to the conditions

$$T_\tau = \nu T_n, \quad (2+\nu)\frac{dS}{ds} - \frac{dT_\tau}{dn} = 0, \quad G = 0, \quad G_s = 0. \qquad \textbf{(24.52c)}$$

4°. Let us consider the case where the edges of the shell are rigidly clamped:

$$u = v = w = \frac{dw}{dn} = 0 \qquad \textbf{(24.53)}$$

Under these conditions, we obtain from (24.41)

$$(2+\nu)\frac{dS}{ds} - \frac{dT_\tau}{dn} = 0; \quad T_\tau = \nu T_n. \qquad \textbf{(24.53a)}$$

From (24.11) it results that $\varkappa_s = \varkappa_{ns} = 0$, or according to (24.44),

$$G_s - \nu G = 0, \quad H = 0. \qquad \textbf{(24.53b)}$$

Consequently, the boundary conditions (24.53) are equivalent to the boundary conditions (24.53a) and (24.53b) for forces.

Thus, the fundamental boundary conditions of the theory of shallow shells have been formulated in terms of forces and moments.

In summing up, let us note that from the formula (24.18) for the statically allowed variations of the state of stress, one can deduce the generalized equations of the Bubnov-Galerkin method (the first of these is simply (24.26)).

$$\iint_{(\Omega)} L\delta\psi\, d\alpha_1 d\alpha_2 = I_1 \quad D' \iint_{(\Omega)} (L_1\delta\psi_1 + L_2\delta\psi_2)\, d\alpha_1 d\alpha_2 = I_2.$$

From which, if the conditions of continuity of deformation L = 0 and L_i = 0 are satisfied, or if they are satisfied in the variational form

$$\iint_{(\Omega)} L\delta\psi\, d\alpha_1 d\alpha_2 = 0; \qquad \textbf{(24. 54)}$$

$$\iint_{(\Omega)} L_i\delta\psi_i\, d\alpha_1 d\alpha_2 = 0, \qquad \textbf{(24.54a)}$$

we shall obtain the equations

$$I_1 = 0, \qquad \textbf{(24.55)}$$
$$I_2 = 0, \qquad \textbf{(24.55a)}$$

which enables us to formulate the boundary conditions.

According to the above, we reach the following conclusion: in order to be able to integrate separately one of the fundamental equations of the theory of shallow shells, namely, the equation of compatibility L = 0 in

★ the Bubnov-Galerkin method, i.e., the equation (24.54), it is necessary and sufficient that the contour integral I_1 should vanish. We may also arrive at the following conclusion: Since, for shallow shells, the Codazzi conditions $L_1 = 0$ may always be satisfied, either by means of a bending function w_1 or by a function of moments ψ, the variational equations (24.52) are identically satisfied. As a result, the condition $I_2 = 0$ must be satisfied. For that, the following elasticity relations for moments must be satisfied:

$$D(\varkappa_{11} + \nu\varkappa_{22}) = M_{11}^0 + \frac{1}{A_2} \cdot \frac{\partial \psi_2}{\partial \alpha_2} + \frac{\psi_1}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1} + (k_{22} + \varkappa_{22})\psi \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1, 2}};$$
$$D(1-\nu)\varkappa_{12} = M_{12}^0 - \frac{1}{2}\left(\frac{1}{A_1} \cdot \frac{\partial \psi_2}{\partial \alpha_1} + \frac{1}{A_2} \cdot \frac{\partial \psi_1}{\partial \alpha_2} - \right.$$
$$\left. - \frac{\psi_1}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} - \frac{\psi_2}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1}\right) - (k_{12} + \varkappa_{12})\psi. \tag{24.56}$$

or

$$M_{11} = M_{11}^0 + \frac{1}{A_2} \cdot \frac{\partial \psi_2}{\partial \alpha_2} + \frac{\psi_1}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1} + k_{22}\psi + D'(M_{22} - \nu M_{11})\psi \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1, 2}}:$$
$$M_{12} = M_{12}^0 - \frac{1}{2}\left(\frac{1}{A_1} \cdot \frac{\partial \psi_2}{\partial \alpha_1} + \frac{1}{A_2} \cdot \frac{\partial \psi_1}{\partial \alpha_2} - \right.$$
$$\left. - \frac{\psi_1}{A_1 A_2} \cdot \frac{\partial A_1}{\partial \alpha_2} - \frac{\psi_2}{A_1 A_2} \cdot \frac{\partial A_2}{\partial \alpha_1}\right) - k_{12}\psi - D'(1+\nu) M_{12} \cdot \psi. \tag{24.57}$$

Thus, when the elasticity relations for moments and Codazzi's conditions are satisfied, the statically allowed variations of the state of stress of the shell occurs without variation of the stress functions ψ_1 and ψ_2.

The variational method given above is in fact a very general method of integration of the fundamental equations of the theory of shallow shells. This method may be applied in two different ways, depending on the form in which the elasticity relations are written.

1. We satisfy, by a series, the elasticity relations (24.56) and, consequently, the third equilibrium equation (15.9), assuming

$$w = \sum f_{mn} w_{mn}(\alpha_1, \alpha_2), \quad \psi = \sum C_{mn} \psi_{mn}(\alpha_1, \alpha_2); \tag{24.58}$$

$$\psi_1 = \sum A_{mn} \psi_{1mn}(\alpha_1, \alpha_2), \quad \psi_2 = \sum B_{mn} \psi_{2mn}(\alpha_1, \alpha_2). \tag{24.59}$$

where $W_{mn}(\alpha_1, \alpha_2)$ and $\psi_{mn}(\alpha_1, \alpha_2)$ are given functions satisfying the boundary conditions. The functions $\psi_{1mn}(\alpha_1, \alpha_2)$ and $\psi_{2mn}(\alpha_1, \alpha_2)$ are not subject to any boundary conditions and may easily be chosen according to the structure of the relations (24.56). After eliminating the coefficients A_{mn} and B_{mn} from (24.56) we obtain the relations between f_{mn} and C_{mn}.

Other relations for f_{mn} and C_{mn} may be obtained after integrating by the Bubnov-Galerkin method the conditions of compatibility

$$\iint_{(\cdot)} \left[K'\Delta\Delta\psi - \left(\varkappa_{12}^2 - \varkappa_{11}\varkappa_{22} - \varkappa_{11}k_{22} - \varkappa_{22}k_{11} + 2k_{12}\varkappa_{12}\right)\right]\delta\psi \, d\alpha_1 d\alpha_2 = 0. \tag{24.60}$$

2. Let us take the series

$$M_{ik} = \sum_{m,n} \bar{f}_{mn} \bar{M}_{mnik}(\alpha_1, \alpha_2), \quad \psi = \sum_{m,n} C_{mn} \psi_{mn}(\alpha_1, \alpha_2). \tag{24.61}$$

★ and also the series (24.59); substituting for A_{mn} and B_{mn} in the elasticity relations (24.57) and eliminating them afterwards from the expressions obtained, we find the relations between $\bar{f}_{mn}$ and C_{mn}. Other relations between them may be obtained by the Bubnov-Galerkin method from the equation

$$\iint\limits_{(\Omega)} \Big\{ K' \Delta\Delta\psi - D'^{2}\left[\nu M^2 + (1+\nu)\left(M_{12}^2 - M_{11}M_{22}\right)\right] + \\ + D'\left[M_{11}(k_{22} - \nu k_{11}) + M_{22}(k_{11} - \nu k_{22}) - 2(1+\nu)\, k_{12}M_{12}\right]\Big\}\, \delta\psi \, d\alpha_1 d\alpha_2 = 0. \quad (24.62)$$ ★

This method is illustrated in Chapter XIV by the solution of problems of large bending of rectangular cylindrical strips. Let us make a few remarks on the substance of the variational method proposed in this Section. As it may be seen from the above, the fundamental characteristics of this method are:

1. All the three equilibrium equations (15.8) and (15.9) are exactly satisfied according to Castigliano's principle.

2. The condition of compatibility of deformation (15.16) is satisfied according to the Bubnov-Galerkin method.

3. Only the force function ψ is varied, while the bending remains invariable.

From this it follows that the method proposed here is different from that proposed by P. F. Papkovich, in which:

1. The condition of (15.16) is accurately satisfied.

2. The third equation of equilibrium (15.9) is satisfied according to the Bubnov-Galerkin or the Ritz method.

3. Only the bending function is varied, while the force function remains unvariable.

The mixed method set forth in Section 23 is an intermediate one, bacause according to it both the force function and the bending function are varied, while the equilibrium equation (15.9) and the condition of compatibility (15.16) are satisfied according to the Bubnov-Galerkin method. The amount of calculation work involved in these three methods is nearly the same in solving geometrically non-linear problems. But when solving physically non-linear problems, the method proposed by us may considerably reduce the amount of calculations. The consistent application of the variational method to the solution of problems of forces and moments enables us to broaden the field of solved linear as well as non-linear problems which are of practical importance.

§ 25. Fundamental Relations for Shallow Shells of Revolution and for Cylindrical Shells

We have set forth in the preceding Sections the general non-linear theory of elastic thin shells in arbitrary orthogonal coordinates, including also several uncommon particular cases; therefore the theory developed is quite complicated. The application of the formulas deduced there for the solution of particular problems becomes even more difficult owing to the large number of concepts and notations which must be looked for in the corresponding Sections of the book. In order to make it easier for the reader who is mainly interested in the most important applications of this theory, we have stated in this Section the fundamental relations of the non-linear theory of shallow shells without giving their derivation. We shall recall here the necessary concepts and notations, so that the reader can acquire a cursory knowledge of the first part of the book which is necessary for an understanding of the following part.

Let t be the constant thickness of the shell before deformation and σ its middle surface. The position of a point on this surface may be specified either by Cartesian coordinates x, y, z, or by curvilinear coordinates α_1 and α_2, considering the point as the intersection of one of the curves of the family α_1 with one of the curves of the family α_2. These families of curves form a net of coordinate curves on the surface σ (see the beginning of Section 2). The formulas (2.1) give the relation between the Cartesian and the curvilinear coordinates of a point on the surface σ. Let us consider for instance a spherical surface defined by equations

$$x = R \sin\theta \cos\varphi, \; y = R \sin\theta \sin\varphi, \; z = R \cos\theta \tag{25.1}$$

Evidently

$$x^2 + y^2 + z^2 = R^2,$$

i.e., the expression (25.1) is the equation of the surface of a sphere whose radius is R and whose center is situated at the origin of the Cartesian coordinate system.

If xoy is the equatorial plane, then $\theta = \theta_0 =$ const, or $z = R\cos\theta_0 =$ const is a parallel, the position of which is determined by the angular deviation θ_0 from the pole; $\varphi = \varphi_0 =$ const defines a meridian, situated at an angle φ_0 to the initial meridian. The position of a point on the sphere is determined by the intersection of the circumferences $\theta = \theta_0$ and $\varphi = \varphi_0$. Thus, θ and φ are the curvilinear coordinates (the so-called geographical coordinates) of the point on the sphere.

Let ds be the distance between two infinitesimally near points on the sphere (line element). Then

$$\begin{aligned} ds^2 = dx^2 + dy^2 + dz^2 = (R\cos\theta\cos\varphi\, d\theta - R\sin\theta\sin\varphi\, d\varphi)^2 + \\ + (R\cos\theta\sin\varphi\, d\theta + R\sin\theta\cos\varphi\, d\varphi)^2 + R^2\sin^2\theta\, d\theta^2 = \\ = R^2\, d\theta^2 + R^2\sin^2\theta\, d\varphi^2. \end{aligned} \tag{25.2}$$

Here $Rd\theta$ is a line element of the meridian, $R\sin\theta$ is the radius of the parallel, and $R\sin\theta d\varphi$ is a line element of the parallel.

For the general case of the arbitrary orthogonal coordinates

$$ds^2 = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2,$$

where A_1 and A_2 are arbitrary functions of α_1 and α_2. We shall assume in the following that the coordinate lines α_1 and α_2 are the lines of curvature of the surface σ, i.e., lines along which the curvatures of normal sections of the surface are maximum and minimum. We shall denote these curvatures by k_1 and k_2. We shall thus finally assume that

$$k_1 = k_{11} = 1/R_1, \quad k_2 = k_{22} = 1/R_2, \quad k_{12} = 0. \tag{25.3}$$

Besides, we shall confine ourselves to the particular case where σ is a surface of revolution or a cylindrical surface. We shall denote the corresponding Gaussian coordinates by $\alpha_1 = \alpha$ and $\alpha_2 = \beta$. The coordinates α and β may be chosen so that

$$ds^2 = d\alpha^2 + B^2 d\beta^2, \tag{25.4}$$

where B is either constant or depends on α.

Consequently,

$$A_1 = 1, \quad A_2 = B(\alpha). \tag{25.5}$$

The geometrical parameters of the surface σ must satisfy the Gauss-Codazzi conditions (2.27):

$$\frac{\partial k_1}{\partial \beta} = 0, \quad \frac{\partial (Bk_2)}{\partial \alpha} = k_1 \frac{\partial B}{\partial \alpha}, \quad \frac{\partial^2 B}{\partial \alpha^2} = -k_1 k_2 B. \tag{25.6}$$

Let $\vec{e}_1$, $\vec{e}_2$ and $\vec{m}$ be the unit vectors of the tangents to the lines α and β on the surface σ, $\vec{m}$ the unit vector of the outer normal to σ, where the trihedron $\{\vec{e}_1, \vec{e}_2, \vec{m}\}$ is right-handed. Let also σ^0 be the true middle surface of the shell before deformation, obtained from σ by the displacement $\vec{m}w^0$ whose maximum value is of the same order of magnitude as the thickness of the shell; the surface σ is either shallow, i.e.,

$$(Bk_i)^2 << 1,$$

or the initial deviations from σ are rapidly varying, so that σ^0 divides into shallow parts.

We shall specify the position of a point on the surface σ^0 by the same coorddinates α and β, but now the lines α and β will not be, in general, the lines of curvature of the surface σ^0, because the principal curvatures of σ have increased by $\varkappa_i^0$, so that the curvature of the coordinate lines has increased by $\varkappa_{12}^0 = \varkappa_{21}^0$. These initial changes in curvature (we shall assume that the stress caused by them has been removed) will be given by the formula (20.1):

$$\varkappa_1^0 = -\frac{\partial^2 w^0}{\partial \alpha^2}, \quad \varkappa_{12}^0 = -\frac{\partial}{\partial \alpha}\left(\frac{1}{B} \cdot \frac{\partial w^0}{\partial \beta}\right), \tag{25.7}$$
$$B\varkappa_2^0 = -\frac{1}{B} \cdot \frac{\partial^2 w^0}{\partial \beta^2} - \frac{\partial w^0}{\partial \alpha} \cdot \frac{\partial B}{\partial \alpha}.$$

Let us assume that under the action of the load, the middle surface of the shell transforms into the surface σ^I, composed of shallow parts; the position of a point on this surface will be specified, as before, by α and β. The projections of the displacement corresponding to this deformation, along $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$, will be denoted by u^I, v^I, w^I. Let ε_i^I be the relative elongation caused by the load of an element of σ which before deformation had the direction $\bar{e}_i$ and $2\varepsilon_{12}^I$ the change in the angle between $\bar{e}_1$ and $\bar{e}_2$, also caused by the load. Then, according to (3.5) and (20.3):

$$\begin{gathered}
\varepsilon_1^I = w^I k_1 + \frac{1}{2}\left(\frac{\partial w^I}{\partial\alpha}\right)^2 + \frac{\partial w^0}{\partial\alpha}\cdot\frac{\partial w^I}{\partial\alpha} + \frac{\partial u^I}{\partial\alpha}, \\
\varepsilon_2^I = w^I k_2 + \frac{1}{2}\left(\frac{\partial w^I}{B\partial\beta}\right)^2 + \frac{1}{B^2}\frac{\partial w^0}{\partial\beta}\cdot\frac{\partial w^I}{\partial\beta} + \frac{1}{B}\left(\frac{\partial v^I}{\partial\beta} + u^I\frac{\partial B}{\partial\alpha}\right), \\
2B\varepsilon_{12}^I = \left(\frac{\partial w^I}{\partial\alpha} + \frac{\partial w^0}{\partial\alpha}\right)\frac{\partial w^I}{\partial\beta} + \frac{\partial w^I}{\partial\alpha}\cdot\frac{\partial w^0}{\partial\beta} + \\
+ B\frac{\partial v^I}{\partial\alpha} + \frac{\partial u^I}{\partial\beta} - v^I\frac{\partial B}{\partial\alpha}.
\end{gathered} \tag{25.8}$$

The changes in the curvature $\varkappa_i^I$ and the torsion $\varkappa_{12}^I$ will be given by formulas similar to (25.7):

$$\begin{gathered}
\varkappa_1^I = -\frac{\partial^2 w^I}{\partial\alpha^2}, \quad \varkappa_{12}^I = -\frac{\partial}{\partial\alpha}\left(\frac{1}{B}\cdot\frac{\partial w^I}{\partial\beta}\right), \\
B\varkappa_2^I = -\frac{1}{B}\cdot\frac{\partial^2 w^I}{\partial\beta^2} - \frac{\partial w^I}{\partial\alpha}\cdot\frac{\partial B}{\partial\alpha}.
\end{gathered} \tag{25.9}$$

The corresponding tangential forces T_i^I, $T_{12}^I = T_{21}^I$, the bending moments M_i^I, and the twisting moments $M_{12}^I = M_{21}^I$, whose positive directions are shown in Figure 6, may be determined according to the formulas (20.5):

$$\begin{gathered}
T_1^I = K(\varepsilon_1^I + \nu\varepsilon_2^I), \quad T_{12}^I = K(1-\nu)\varepsilon_{12}^I, \quad K = Et/(1-\nu^2), \\
M_1^I = D(\varkappa_1^I + \nu\varkappa_2^I), \quad M_{12}^I = D(1-\nu)\varkappa_{12}^I, \\
D = Et^3/12(1-\nu^2). \quad \overrightarrow{\underleftarrow{1, 2}}.
\end{gathered} \tag{25.10}$$

Here and in the following the symbol $\overrightarrow{\underleftarrow{1, 2}}$ shows that the formulas which are not written may be obtained from the given formulas by permuting the indexes 1, 2 and replacing u^I by v^I, and α by β, whereas the other quantities remain unchanged.

The equations of equilibrium (20.11) and (20.12) must be satisfied on the whole surface; in the case considered they become

$$\begin{gathered}
\frac{\partial}{\partial\alpha}(BT_1^I) + \frac{\partial T_{12}^I}{\partial\beta} - T_2^I\cdot\frac{\partial B}{\partial\alpha} = 0, \\
\frac{\partial}{\partial\alpha}(BT_{12}^I) + \frac{\partial T_2^I}{\partial\beta} + T_{12}^I\frac{\partial B}{\partial\alpha} = 0,
\end{gathered} \tag{25.11}$$

$$\begin{gathered}
D\Delta\Delta w^I + T_1^I(k_1 + \varkappa_1^0 + \varkappa_1^I) + 2T_{12}^I(\varkappa_{12}^0 + \varkappa_{12}^I) + \\
+ T_2^I(k_2 + \varkappa_2^0 + \varkappa_2^I) + p = 0,
\end{gathered} \tag{25.12}$$

Here and in the following we have introduced the Laplace operator

$$\Delta = \frac{1}{B}\left\{\frac{\partial}{\partial\alpha}\left[B\frac{\partial}{\partial\alpha}(\ldots)\right] + \frac{1}{B}\frac{\partial^2}{\partial\beta^2}(\ldots)\right\}, \tag{25.13}$$

the normal pressure being $p > 0$ for external pressure. The shearing forces are determined from the formulas:

$$N_1^I = -D\frac{\partial}{\partial\alpha}(\Delta w^I), \quad BN_2^I = -D\frac{\partial}{\partial\beta}(\Delta w^I). \tag{25.14}$$

Introducing (25.7)-(25.10) in the equations (25.11) and (25.12), we obtain a system of three non-linear differential equations for u^1, v^1, w^1.

In the following we shall assume that the boundary contour C is composed of parts C_1 on which α = const and $0 \leqslant \beta \leqslant b$, and of parts C_2 on which β = const and $0 \leqslant \alpha \leqslant a$. Let the external forces and moments, along the principal directions of the surface σ, be applied to the contour C_1: the normal force p_1, the tangential displacing force τ, the shearing force $\widetilde{N}_1$, and the bending moment $\widetilde{M}_1$. Their ppositive directions coincide with the positive directions of the internal forces shown in Figure 5. The analogous forces and moments applied to the contour C_2 will be denoted by p_2, τ, $\widetilde{N}_2$, and $\widetilde{M}_2$.

According to formulas (17.29)-(17.31), the following conditions must be satisfied at the free part C

$$
\begin{gathered}
T_1^1 = p_1,\ T_{12}^1 = \tau,\ M_{12}^1 = \widetilde{M}_{12}, \qquad \overrightarrow{\underleftarrow{1,2}} \\
N_1^1 + \frac{1}{B}\cdot\frac{\partial M_{12}}{\partial \beta} + T_1^1\left(\frac{\partial w^0}{\partial \alpha} + \frac{\partial w^1}{\partial \alpha}\right) + \frac{T_{12}^1}{B}\left(\frac{\partial w^0}{\partial \beta} + \frac{\partial w^1}{\partial \beta}\right) = \widetilde{N}_1, \\
N_2^1 + \frac{\partial M_{12}}{\partial \alpha} + \frac{1}{B}T_2^1\left(\frac{\partial w^0}{\partial \beta} + \frac{\partial w^1}{\partial \beta}\right) + T_{12}^1\left(\frac{\partial w^0}{\partial \alpha} + \frac{\partial w^1}{\partial \alpha}\right) = \widetilde{N}_2.
\end{gathered} \tag{25.15}
$$

At the fixed edges the purely geometrical conditions must be satisfied:

$$u^1 = 0,\ v^1 = 0,\ w^1 = 0,\ \frac{\partial w^1}{\partial \alpha} = 0 \ (\text{at } C_1) \quad \overrightarrow{\underleftarrow{1,2}}. \tag{25.16}$$

At those parts of the edge contour where the clamping is incomplete, mixed boundary conditions must be fulfilled, as for instance:

a. when the contour C is hinged:

$$u^1 = 0,\ v^1 = 0,\ w^1 = 0,\ M_1^1 = 0; \tag{25.17}$$

b. when the contour C is freely supported

$$w^1 = 0,\ u^1 = 0,\ \tau = T_{12}^1,\ M_1^1 = \widetilde{M}_1 \tag{25.18}$$

or

$$w^1 = 0,\ T_1^1 = p_1,\ T_{12}^1 = \tau,\ M_1^1 = \widetilde{M}_1 \tag{25.19}$$

The boundary value problem in this formulation is called the problem of the theory of shells for the displacement components. It is very difficult to solve it directly, in spite of the simplifications already made; therefore, one looks very often for approximate solutions of the problem by means of the variational equations of equilibrium. For one of these equations we may take the variational equation of the principle of virtual displacements

$$\delta \vartheta^1 = 0, \tag{25.20}$$

where ϑ^1 is the sum of the potential energy of deformation and the potential energy corresponding to the work done by the external load; according to formulas (21.8) and (17.37), for the cases considered this sum equals:

$$\mathcal{Э}^{1} = \left\{ \int\limits_{0}^{b} \left(\widetilde{M}_1 \frac{\partial w^1}{\partial \alpha} - p_1 u^1 - \tau v^1 - \widetilde{N}_1 w^1 \right) B d\beta \right\}_{\alpha=0}^{\alpha=a} +$$

$$+ \left\{ \int\limits_{0}^{a} \left(\widetilde{M}_2 \cdot \frac{1}{B} \cdot \frac{\partial w^1}{\partial \beta} - p_2 v^1 - \tau u^1 - \widetilde{N}_2 w^1 \right) d\alpha \right\}_{\beta=0}^{\beta=b} +$$

$$+ \iint\limits_{(\sigma)} \left\{ p w^1 + K \left[\frac{1}{2} (\varepsilon_1^1 + \varepsilon_2^1)^2 - (1 - \nu)(\varepsilon_1^1 \varepsilon_2^1 - \varepsilon_{12}^{1\,2}) \right] + \right. \tag{25.21}$$

$$\left. + D \left[\frac{1}{2} (\varkappa_1^1 + \varkappa_2^1)^2 - (1 - \nu)(\varkappa_1^1 \varkappa_2^1 - \varkappa_{12}^{1\,2}) \right] \right\} B d\alpha d\beta.$$

The generalized Bubnov-Galerkin equation (22.5) may be successfully used for solving the problem for the components of the displacement. We shall write this equation for the case under consideration, assuming, in addition, that:

$$\delta w^1 = 0 \text{ on C.} \tag{25.22}$$

Taking into account the initial deformations and also somewhat modified notations, we obtain the equation

$$\left\{ \int\limits_{0}^{b} \left[(p_1 - T_1^1) \delta u^1 + (\tau - T_{12}^1) \delta v^1 + (M_1^1 - \widetilde{M}_1) \delta \frac{\partial w^1}{\partial \alpha} \right] B d\beta \right\}_{\alpha=0}^{\alpha=a} +$$

$$+ \left\{ \int\limits_{0}^{a} \left[(p_2 - T_2^1) \delta v^1 + (\tau - T_{12}^1) \delta u^1 + (M_2^1 - \widetilde{M}_2) \delta \frac{\partial w^1}{\partial \beta} \cdot \frac{1}{B} \right] d\alpha \right\}_{\beta=0}^{\beta=b} +$$

$$+ \iint\limits_{(\sigma)} \left\{ \left[\frac{\partial}{\partial \alpha} (B T_1^1) + \frac{\partial T_{12}^1}{\partial \beta} - T_2^1 \frac{\partial B}{\partial \alpha} \right] \delta u^1 + \left[\frac{\partial}{\partial \alpha} (B T_{12}^1) + \right. \right. \tag{25.23}$$

$$+ \left. \frac{\partial T_2^1}{\partial \beta} + T_{12}^1 \frac{\partial B}{\partial \alpha} \right] \delta v^1 - B (k_1 T_1^1 + k_2 T_2^1 + p) \delta w^1 +$$

$$+ \frac{\partial}{\partial \alpha} \left[B T_1^1 \left(\frac{\partial w^0}{\partial \alpha} + \frac{\partial w^1}{\partial \alpha} \right) + T_{12}^1 \left(\frac{\partial w^0}{\partial \beta} + \frac{\partial w^1}{\partial \beta} \right) + B N_1^1 \right] \delta w^1 +$$

$$+ \frac{\partial}{\partial \beta} \left[T_{12}^1 \left(\frac{\partial w^0}{\partial \alpha} + \frac{\partial w^1}{\partial \alpha} \right) + N_2^1 + \right.$$

$$\left. \left. + T_2^1 \left(\frac{\partial w^0}{\partial \beta} + \frac{\partial w^1}{\partial \beta} \right) \frac{1}{B} \right] \delta w^1 \right\} d\alpha d\beta = 0.$$

Let us assume that for a certain critical value of the external load, in addition to the form of equilibrium σ^1, an infinitesimally near form of equilibrium σ^* is also possible. Then we shall say that the shell is at the limit of stable equilibrium. We shall assume that the shell is either shallow or that its transition from the form of equilibrium σ^1 to the form of equilibrium σ^* occurs with the formation of a large number of "waves" so that the middle surface divides into shallow parts. The projections of the additional displacements on $\bar{e}_1$, $\bar{e}_2$, $\bar{m}$ will be denoted by u, v, w. The additional elongations, the shear, and the changes in curvature which occur in this case, are determined according to formulas (20.4) and formulas like (20.1):

$$\varepsilon_1 = \frac{\partial u}{\partial \alpha} + w k_1 + \frac{\partial w}{\partial \alpha} \left(\frac{\partial w^0}{\partial \alpha} + \frac{\partial w^1}{\partial \alpha} \right),$$

$$\varepsilon_2 = \left(\frac{\partial v}{\partial \beta} + u \frac{\partial B}{\partial \alpha} \right) \frac{1}{B} + w k_2 + \frac{\partial w}{\partial \beta} \left(\frac{\partial w^0}{\partial \beta} + \frac{\partial w^1}{\partial \beta} \right) \frac{1}{B^2},$$

$$2B\varepsilon_{12} = B \frac{\partial v}{\partial \alpha} + \frac{\partial u}{\partial \beta} - v \frac{\partial B}{\partial \alpha} + \frac{\partial w}{\partial \alpha} \left(\frac{\partial w^0}{\partial \beta} + \frac{\partial w^1}{\partial \beta} \right) +$$

$$+ \frac{\partial w}{\partial \beta} \left(\frac{\partial w^0}{\partial \alpha} + \frac{\partial w^1}{\partial \alpha} \right). \tag{25.24}$$

$$\varkappa_1 = -\frac{\partial^2 w}{\partial \alpha^2}, \quad B\varkappa_2 = -\frac{\partial w}{\partial \alpha} \cdot \frac{\partial B}{\partial \alpha} - \frac{1}{B} \frac{\partial^2 w}{\partial \beta^2}, \quad \varkappa_{12} = -\frac{\partial}{\partial \alpha} \left(\frac{1}{B} \cdot \frac{\partial w}{\partial \beta} \right).$$

The corresponding additional elastic forces and moments are

$$T_1 = K(\varepsilon_1 + \nu\varepsilon_2), \quad T_{12} = K(1-\nu)\varepsilon_{12}, \qquad \overleftrightarrow{1,2}$$
$$M_1 = D(\varkappa_1 + \nu\varkappa_2), \quad M_{12} = D(1-\nu)\varkappa_{12}. \tag{25.25}$$

In the state of neutral equilibrium, besides the equations (25.11) and (25.12) equations (20.14) must also be satisfied, which for the case under consideration become:

$$\frac{\partial}{\partial\alpha}(BT_1) + \frac{\partial T_{12}}{\partial\beta} - T_2\frac{\partial B}{\partial\alpha} = 0;$$
$$\frac{\partial}{\partial\alpha}(BT_{12}) + \frac{\partial T_2}{\partial\beta} + T_{12}\frac{\partial B}{\partial\alpha} = 0; \tag{25.26}$$

$$D\Delta\Delta w + T_1(k_1 + \varkappa_1^0 + \varkappa_1^1) + 2T_{12}(\varkappa_{12}^0 + \varkappa_{12}^1) + T_2(k_2 + \varkappa_2^0 + \varkappa_2^1) +$$
$$+ T_1^1\varkappa_1 + 2T_{12}^1\varkappa_{12} + T_2^1\varkappa_2 = 0; \tag{25.27}$$

$$N_1 = -D\frac{\partial}{\partial\alpha}(\Delta w), \quad BN_2 = -D\frac{\partial}{\partial\beta}(\Delta w). \tag{25.28}$$

Since the loss of stability of the shell occurs without additional loading, the static boundary conditions of the type (20.30) must be satisfied at the free edges:

$$T_1 = 0, \quad T_{12} = 0, \quad M_1 = 0; \quad \overleftrightarrow{1,2}$$
$$BN_1 + \frac{\partial M_{12}}{\partial\beta} + BT_1^1\frac{\partial w}{\partial\alpha} + T_{12}^1\frac{\partial w}{\partial\beta} = 0 \text{ on } C_1,$$
$$N_2 + \frac{\partial M_{12}}{\partial\alpha} + T_{12}^1\frac{\partial w}{\partial\alpha} + T_2^1\frac{1}{B}\cdot\frac{\partial w}{\partial\beta} = 0 \text{ on } C_2. \tag{25.29}$$

For fixed edges the following conditions must be satisfied

$$u = 0, \quad v = 0, \quad w = 0, \quad \frac{\partial w}{\partial\alpha} = 0 \ (\text{on } C_1) \qquad \overleftrightarrow{1,2}. \tag{25.30}$$

For mixed edge conditions, various combinations of the equation (25.29) and (25.30) must be satisfied.

The equations (25.26)-(25.27) may be replaced by the variational equation

$$\delta Э = \delta\iint\limits_{(\sigma)}\Big\{T_1^1\Big(\frac{\partial w}{\partial\alpha}\Big)^2 + \frac{2}{B}T_{12}^1\frac{\partial w}{\partial\alpha}\cdot\frac{\partial w}{\partial\beta} + \frac{T_2^1}{B^2}\Big(\frac{\partial w}{\partial\beta}\Big)^2 + \tag{25.31}$$
$$+ K[\varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2 + 2(1-\nu)\varepsilon_{12}^2] +$$
$$+ D[\varkappa_1^2 + \varkappa_2^2 + 2\nu\varkappa_1\varkappa_2 + 2(1-\nu)\varkappa_{12}^2]\Big\}Bd\alpha d\beta = 0,^*$$

where by the variation the displacements u, v, and w allowed by the constraints are approximated.

The equation of equilibrium of the tangential forces (25.11) may be approximately satisfied, within the approximations of the theory of shallow shells, by assuming

$$B^2T_1^1 = \frac{\partial^2\psi^1}{\partial\beta^2} + B\frac{\partial B}{\partial\alpha}\cdot\frac{\partial\psi^1}{\partial\alpha}, \quad T_2^1 = \frac{\partial^2\psi^1}{\partial\alpha^2},$$
$$BT_{12}^1 = -\frac{\partial^2\psi^1}{\partial\alpha\,\partial\beta} + \frac{1}{B}\cdot\frac{\partial B}{\partial\alpha}\cdot\frac{\partial\psi^1}{\partial\beta}. \tag{25.32}$$

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* The symbol Э (open Russian É) stands for energy here and in the following - Translator.

where the force function ψ^I must satisfy the condition of compatibility of deformation:

$$\Delta\Delta\psi^I - Et\left[\varkappa_{12}^{I2} + 2\varkappa_{12}^I\varkappa_{12}^0 - \varkappa_1^I(k_2 + \varkappa_2^0 + \varkappa_2^I) - \varkappa_2^I(k_1 + \varkappa_1^0)\right] = 0. \tag{25.33}$$

Besides, the functions δ^I and ψ^I must satisfy the equation (25.12) and the boundary conditions. Instead of solving this boundary value problem, it is possible to solve the equation (25.23) of the Bubnov-Galerkin method, expressing the function of deflection in the form of a series

$$w^I = C_0 w_0 + C_1 w_1 + \ldots \tag{25.34}$$

(where each term satisfies the boundary conditions with respect to w^I) and equating to zero the coefficients of δC_i. The function ψ^I, determined from equation (25.33), must satisfy the geometrical boundary conditions for u^I and v^I. The static boundary conditions for the tangential forces T_i^I and T_{12}^I will be automatically satisfied during the solution of the variational problem; the higher the approximation considered, the more accurately will these conditions be satisfied.

The variational equation (23.1) of N. A. Alumyae is very convenient for many cases, because there we approximate the functions w satisfying the geometric boundary conditions for w, and the functions ψ satisfying the static boundary conditions for the tangential forces, but these functions will not necessarily satisfy equation (24.33). Here the index I was omitted for brevity; $\tilde{u}_n$ and $\tilde{u}_\tau$ are the projections of the displacement on the normal and the tangent to the contour, given at any arbitrary part of the edge contour. In particular, if tangential forces are given along the entire edge contour and the function w satisfies the condition $\delta w = 0$ on the contour and one of the conditions $M_i = \tilde{M}_i$ or $\delta\frac{\partial w}{\partial \sigma_i}$ on C_i, we shall obtain the equation

$$\iint_{(\sigma)}\{D\Delta\Delta w + T_1(k_1 + \varkappa_1^0 + \varkappa_1) + 2T_{12}(\varkappa_{12}^0 + \varkappa_{12}) + T_2(k_2 + \varkappa_2^0 + \varkappa_2) + p\}\,\delta w\, d\sigma \quad \iint_{(\sigma)}\left\{\frac{1}{Et}\Delta\Delta\psi + \varkappa_1(k_2 + \varkappa_2^0 + \varkappa_2) + \varkappa_2(k_1 + \varkappa_1^0) - \varkappa_{12}^2 - 2\varkappa_{12}\varkappa_{12}^0\right\}\delta\psi\, d\sigma = 0, \tag{25.35}$$

That is in essence the equation of the Bubnov-Galerkin method.

§ 26. A Shallow Shell Considered as a Plate with an Initial Deflection. The Fundamental Equations in Oblique Coordinates

When taking a plane as the surface of reference σ, we may consider the middle surface of a shallow shell before the application of the load as a surface obtained from a part of this plane after the normal displacement w^0 and without residual stresses; we shall assume that the stresses have been removed by annealing. Therefore, we have to assume that in the formulas of the previous section

$$k_1 = k_2 = 0. \tag{26.1}$$

We shall specify the position of a point on the middle surface σ^0 before the deformation by the rectangular coordinates x, y, w^0 (Figure 12). Then, the equation of the surface σ^0 becomes:

$$z = w^0 = F(x, y). \tag{26.2}$$

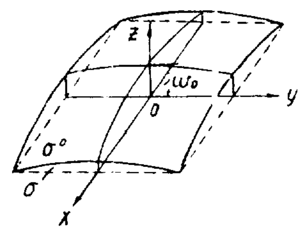

Figure 12

The planes x = const and y = const intersect σ^0 along curves which differ only slightly from the normal sections of the surface. The curvature and the torsion of these lines will be determined from (25.7), assuming B = 1, i. e.,

$$\varkappa_1^0 = -\frac{\partial \omega_1^0}{\partial x} = -\frac{\partial^2 w^0}{\partial x^2} = -\frac{\partial^2 F}{\partial x^2}, \quad \varkappa_{12}^0 = -\frac{\partial^2 w^0}{\partial x \partial y} \quad \overleftrightarrow{1, 2}. \tag{26.3}$$

The elongations and the shear caused by the load before the loss of stability, when the surface σ^0 turns into σ^I, must be determined from (25.8):

$$\begin{gathered} \varepsilon_1^1 = \frac{\partial u_1^1}{\partial x} + \frac{1}{2}\left(\frac{\partial w^1}{\partial x}\right)^2 + \frac{\partial w^0}{\partial x} \cdot \frac{\partial w^1}{\partial x}, \\ 2\varepsilon_{12}^1 = \frac{\partial u_2^1}{\partial x} + \frac{\partial u_1^1}{\partial y} + \frac{\partial w^1}{\partial x} \cdot \frac{\partial w^1}{\partial y} + \frac{\partial w^0}{\partial x} \cdot \frac{\partial w^1}{\partial y} + \frac{\partial w^0}{\partial y} \cdot \frac{\partial w^1}{\partial x}. \end{gathered} \tag{26.4}$$

In the other formulas in § 25 it is also necessary to assume $k_1 = k_2 = 0$ and B = 1. The theory of a slightly bent beam or plate given here has already been applied in particular cases in the works of I. G. Bubnov /0.2/ and /0.3/. It has been given in the general form, seemingly for the first time in 1939, by K. Marguerre /VI.11/. In the work by V. Z. Vlasov /VI.8/ this theory has been given in a rather modified form: there he assumed that the displacement of a point of the surface is specified by its projections on the tangents to the lines of intersection between the surface and the planes x = const and y = const, and on the normal to this surface. Denoting these projections by u^I, v^I, w^I, we express in terms of these the projections of the displacements on the axes $\bar{e}_1, \bar{e}_2, \bar{m}$ (i.e., on the axes x, y, w), which have been derived in this work:

$$u_1^1 \approx u^1 - \frac{\partial w^0}{\partial x} \cdot w^1 , \; u_2^1 \approx v^1 - \frac{\partial w^0}{\partial y} w^1 , \; w_1^1 \approx w^1 . \tag{26.5}$$

Introducing these expression in (26.4) we obtain the usual formulas of the theory of shallow shells*:

$$\varepsilon_1^1 = \frac{\partial u^1}{\partial x} + \frac{1}{2}\left(\frac{\partial w^1}{\partial x}\right)^2 - \frac{\partial^2 w^0}{\partial x^2} w^1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w^1}{\partial x}\right)^2 + \varkappa_1^0 w^1 \; \overrightarrow{1, 2},$$
$$2\varepsilon_{12}^1 = \frac{\partial u^1}{\partial y} + \frac{\partial v^1}{\partial x} + 2\varkappa_{12}^0 w^1 + \frac{\partial w^1}{\partial x} \cdot \frac{\partial w^1}{\partial y} . \tag{26.6}$$

Thus the formulas (26.4) and (26.6) are equivalent. When deriving them it has been assumed, as is generally done in the theory of shells, that the squares of the rotations of any elements of the surface σ^0 are small in comparison with unity because in our derivations we have used the equations

$$\cos\varphi = 1 - \frac{\varphi^2}{2} + \cdots \approx 1, \; \sin\varphi = \varphi - \frac{\varphi^3}{6} + \cdots \approx \varphi, \tag{26.7}$$

where φ is half the angle subtended at the center by the arc which determines the maximal cross-dimension of the considered part of the surface σ^0. If this arc is part of a circle of radius R, the length of the corresponding chord being a and the deflection f, the measure of the shallowness of the shell will be:

$$\frac{f}{a} \approx \frac{R\varphi^2}{2a} \approx \frac{a}{8R} . \tag{26.8}$$

It is, of course, necessary to determine the upper limit of this quantity, on the basis of the error admissible in using the approximate equations (26.7). It is often shown in literature that a shell may be considered as shallow when $a \geq 5f$ (see, for example, /VI.7/). In our opinion, such an extension of the field of application of the theory of shallow shells is risky, because for a = 5f it follows from (26.8) that $a/R \approx 8/5$ and $\varphi = 0.8$; then, by equating to unity the quantity $\cos\varphi \approx 1 - \frac{\varphi^2}{2} =$ $= 0.68$, according to (26.7), we admit a gross error.

Therefore, we must also put a restriction, even in the general case, on the magnitude of the initial deviation f of the surface σ^0 from the surface σ: we shall assume that f is maximal, i.e., of the same order of magnitude as the thickness of the shell. Evidently, this restriction is not essential from the point of view of applications of the theory, because at the present state of the manufacturing technology of thin-walled shells, the initial deflection is smaller than the thickness of the shell.

If

$$f << R, \tag{26.9}$$

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* In /VI.1/ we derived formulas (26.6) and other relations of the theory of shallow shells in the above formulation, for general coordinates and for an arbitrary surface of reference σ.

we shall call the shell "very shallow" or "a slightly bent plate". In this case, it is possible to simplify the fundamental differential equations (25.12) and (25.33) of the theory of shallow shells. With $k_1 = k_2 = 0$ and $B = 1$, they become, after introducing the force function ψ^I and the deflection function w according to (25.9) and (25.22):

$$D\Delta\Delta w^I - \frac{\partial^2\psi^I}{\partial y^2}\left(\frac{\partial^2 w^0}{\partial x^2} + \frac{\partial^2 w^I}{\partial x^2}\right) + 2\frac{\partial^2\psi^I}{\partial x\partial y}\left(\frac{\partial^2 w^0}{\partial x\partial y} + \frac{\partial^2 w^I}{\partial x\partial y}\right) - \frac{\partial^2\psi^I}{\partial x^2}\left(\frac{\partial^2 w^0}{\partial y^2} + \frac{\partial^2 w^I}{\partial y^2}\right) - p = 0; \tag{26.10}$$

$$\Delta\Delta\psi^I - Et\left[\left(\frac{\partial^2 w^I}{\partial x\partial y}\right)^2 + 2\frac{\partial^2 w^I}{\partial x\partial y}\cdot\frac{\partial^2 w^0}{\partial x\partial y} - \frac{\partial^2 w^I}{\partial x^2}\left(\frac{\partial^2 w^0}{\partial y^2} + \frac{\partial^2 w^I}{\partial y^2}\right) - \frac{\partial^2 w^I}{\partial y^2}\cdot\frac{\partial^2 w^0}{\partial x^2}\right] = 0. \tag{26.11}$$

In fact, after introducing the total deflection

$$w^n = w^0 + w^I, \tag{26.12}$$

measured from the plane XOY, we obtain the equations:

$$D\Delta\Delta w^n - \frac{\partial^2\psi^I}{\partial y^2}\cdot\frac{\partial^2 w^n}{\partial x^2} + 2\frac{\partial^2\psi^I}{\partial x\partial y}\cdot\frac{\partial^2 w^n}{\partial x\partial y} - \frac{\partial^2\psi^I}{\partial x^2}\cdot\frac{\partial^2 w^n}{\partial y^2} + p^n = 0; \tag{26.13}$$

$$\Delta\Delta\psi^I - Et\left[\left(\frac{\partial^2 w^n}{\partial x\partial y}\right)^2 - \frac{\partial^2 w^n}{\partial x^2}\cdot\frac{\partial^2 w^n}{\partial y^2} - \underline{\left(\frac{\partial^2 w^0}{\partial x\partial y}\right)^2 + \frac{\partial^2 w^0}{\partial x^2}\cdot\frac{\partial^2 w^0}{\partial y^2}}\right] = 0, \tag{26.14}$$

where

$$p^n = p - D\Delta\Delta w^0 \tag{26.15}$$

is the equivalent normal pressure. When considering a bending for which the total deflection is considerably larger than the thickness of the shell, the underlined terms in equation (26.14) may be neglected, provided that the conditions (26.9) are fulfilled. Thus, equations (26.13) and (26.14) coincide in this case with the well-known von Karman equations for the bending of a plate under the action of a certain equivalent cross-load p^n.

If a very shallow shell is traced on part of a second order surface

$$w^0 = -\varkappa_1^0\frac{x^2}{2} - \varkappa_2^0\frac{y^2}{2} - \varkappa_{12}^0 xy + cx + ey + f,$$

the curvatures $\varkappa_1^0$, $\varkappa_2^0$, $\varkappa_{12}^0$ will be constant. In this case $p^n = p$, and the equations of equilibrium of the curved plate with pronounced bending will have the form of the equations of equilibrium of a flat plate.

The generalization of the equations (26.10) and (26.11) for a shallow shell with varying thickness has been given in /VI.14/. The corresponding equations in Cartesian coordinates are of the following form:

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(\frac{1}{t}\cdot\frac{\partial^2\psi^I}{\partial x^2}\right) + 2(1+\nu)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{t}\cdot\frac{\partial^2\psi^I}{\partial x\,\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{t}\cdot\frac{\partial^2\psi^I}{\partial y^2}\right) - \\
&\quad - \nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{1}{t}\cdot\frac{\partial^2\psi^I}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{t}\cdot\frac{\partial^2\psi^I}{\partial x^2}\right)\right] = \\
&= E\left[\left(\frac{\partial^2 w^I}{\partial x\partial y}\right)^2 + 2\frac{\partial^2 w^I}{\partial x\,\partial y}\cdot\frac{\partial^2 w^0}{\partial x\,\partial y} - \frac{\partial^2 w^I}{\partial x^2}\left(\frac{\partial^2 w^0}{\partial y^2} + \frac{\partial^2 w^I}{\partial y^2}\right) - \frac{\partial^2 w^I}{\partial y^2}\cdot\frac{\partial^2 w^0}{\partial x^2}\right].\\
&\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w^I}{\partial x^2}\right) + 2(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\left(D\frac{\partial^2 w^I}{\partial x\,\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w^I}{\partial y^2}\right) + \\
&\quad + \nu\left[\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w^I}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w^I}{\partial x^2}\right)\right] = \frac{\partial^2\psi^I}{\partial y^2}\cdot\frac{\partial^2 w^n}{\partial x^2} + \\
&\quad + \frac{\partial^2\psi^I}{\partial x^2}\cdot\frac{\partial^2 w^n}{\partial y^2} - 2\frac{\partial^2\psi^I}{\partial x\partial y}\cdot\frac{\partial^2 w^n}{\partial x\partial y} - p.
\end{aligned} \tag{26.16}$$

When solving certain problems, as for instance when studying slanted plates (in the form of a parallelogram) or the problem of torsional buckling of a cylindrical shell, it is more convenient to refer the surface σ to oblique coordinates x', y', related to rectangular coordinates by the formulas:

$$y' = y + x \operatorname{tg} \varphi, \quad x' = x/\cos \varphi, \tag{26.17}$$

It has been assumed here that the angle between the axes x' and y' equals $\varphi + \pi/2$ and that the x axis is at an angle φ with the x' axis. The derivatives with respect to x and y may be expressed in terms of the derivatives with respect to x' and y', by the formulas:

$$\frac{\partial}{\partial x} = \frac{1}{\cos\varphi}\left(\frac{\partial}{\partial x'} + \sin\varphi \frac{\partial}{\partial y'}\right), \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}. \tag{26.18}$$

After carrying out these transformations in the equations (26.10) and (26.11), we obtain:

$$\Delta_k \Delta_k \psi - \frac{Et}{\cos^2\varphi}\left\{\left(\frac{\partial^2 w}{\partial x' \partial y'}\right)^2 + 2\frac{\partial^2 w}{\partial x' \partial y'} \cdot \frac{\partial^2 w^0}{\partial x' \partial y'} - \frac{\partial^2 w}{\partial x'^2}\left(\frac{\partial^2 w}{\partial y'^2} + \frac{\partial^2 w^0}{\partial y'^2}\right) - \right.$$
$$\left. - \frac{\partial^2 w}{\partial y'^2} \cdot \frac{\partial^2 w^0}{\partial x'^2}\right\} = 0, \tag{26.19}$$

$$D\Delta_k \Delta_k w - \frac{1}{\cos^2\varphi}\left\{\frac{\partial^2 \psi}{\partial y'^2}\left(\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w^0}{\partial x'^2}\right) + \frac{\partial^2 \psi}{\partial x'^2}\left(\frac{\partial^2 w}{\partial y'^2} + \frac{\partial^2 w^0}{\partial y'^2}\right)\right.$$
$$\left. - 2\frac{\partial^2 \psi}{\partial x' \partial y'}\left(\frac{\partial^2 w}{\partial x' \partial y'} + \frac{\partial^2 w^0}{\partial x' \partial y'}\right)\right\} + p = 0. \tag{26.20}$$

Here,

$$\Delta_k = \frac{1}{\cos^2\varphi}\left(\frac{\partial^2}{\partial x'^2} + 2\sin\varphi \frac{\partial^2}{\partial x' \partial y'} + \frac{\partial^2}{\partial y'^2}\right) \tag{26.21}$$

is the Laplace operator in oblique coordinates.

The expression for the forces T_1', T_{12}', T_{21}', T_2', referred to the oblique coordinate axes may be obtained by considering the equilibrium of an element with the slides dx, dy, dx'. From Figure 13,

$$T_{21}' \cos\varphi \, dx' = T_{21} \, dx + T_1 \, dy,$$
$$(T_2' - T_{21}' \sin\varphi) \, dx' = T_{12} dy + T_2 \, dx,$$
$$T_{21}' \cos\varphi = T_{21} \cos\varphi + T_1 \sin\varphi,$$
$$T_2' - T_{21}' \sin\varphi = T_{12} \sin\varphi + T_2 \cos\varphi.$$

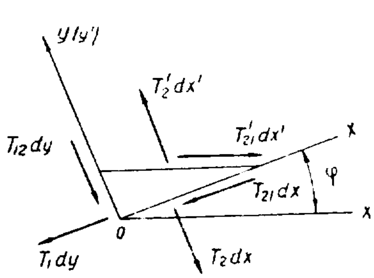

Figure 13 (26.22)

Besides,

$$T_1 = T_1' \cos\varphi, \quad T_{12} = T_{12}' - T_1' \sin\varphi. \tag{26.23}$$

Whence, taking into account that

$$T_1 = \frac{\partial^2\psi}{\partial y^2}, \quad T_{12} = T_{21} = -\frac{\partial^2\psi}{\partial x \partial y}, \quad T_2 = \frac{\partial^2\psi}{\partial x^2},$$

and also the expressions (26.18), we find

$$T_1' = \frac{1}{\cos\varphi} \cdot \frac{\partial^2\psi}{\partial y'^2}, \quad T_{12}' = T_{21}' = -\frac{1}{\cos\varphi} \cdot \frac{\partial^2\psi}{\partial x' \cdot \partial y'}, \quad T_2' = \frac{1}{\cos\varphi} \cdot \frac{\partial^2\psi}{\partial x'^2}. \tag{26.24}$$

By analogy, we find the expression for the moments. Evidently, according to (6.8) and (6.9) we obtain the same expressions, replacing in (26.22) and (26.23) the quantities T_1, T_{12}, T_{21}, T_2, T_1', T_{12}', T_{21} and T_2' by $M_{12}, M_1, -M_2, M_{21}, -M_{12}', M_1'$, $-M_2'$, M_{21}', respectively. Thus

$$\begin{gathered} M_2' = M_2 + M_{12}\,\mathrm{tg}\,\varphi, \quad M_{21} + M_2 \sin\varphi = M_1 \sin\varphi + M_{21} \cos\varphi, \\ M_{12} = M_{12}' \cos\varphi, \quad M_1 = M_1' + M_{12} \sin\varphi. \end{gathered} \tag{26.25}$$

Hence, by recalling that

$$M_1 = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \quad M_{12} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}, \ldots,$$

and once more (26.18) using we find that

$$\begin{gathered} M_1' = -\frac{D}{\cos^2\varphi}\left(\frac{\partial^2 w}{\partial x'^2} + \sin\varphi \frac{\partial^2 w}{\partial x' \partial y'} + \nu\left(\frac{\partial^2 w}{\partial y'^2} + \sin\varphi \frac{\partial^2 w}{\partial x' \partial y'}\right)\right), \\ M_2' = -\frac{D}{\cos^2\varphi}\left\{\frac{\partial^2 w}{\partial y'^2} + \sin\varphi \frac{\partial^2 w}{\partial x' \partial y'} + \nu\left(\frac{\partial^2 w}{\partial x'^2} + \sin\varphi \frac{\partial^2 w}{\partial x' \partial y'}\right)\right\}, \\ M_{12}' = -\frac{D(1-\nu)}{\cos^2\varphi}\left\{\frac{\partial^2 w}{\partial x' \partial y'} + \sin\varphi \frac{\partial^2 w}{\partial y'^2}\right\}, \\ M_{21} = -\frac{D(1-\nu)}{\cos^2\varphi}\left(\frac{\partial^2 w}{\partial x' \partial y'} + \sin\varphi \frac{\partial^2 w}{\partial x'^2}\right). \end{gathered} \tag{26.26}$$

Chapter VII

SOME PROBLEMS OF THE THEORY OF STABILITY AND LARGE DEFLECTION OF RECTANGULAR PLATES

In Chapters VII-VIII, some problems in the theory of flexure and stability of plates are dealt with, to illustrate the application of the general non-linear theory of shells to the solution of particular problems in the simpler cases. To readers interested in other problems of this kind, we recommend the monographs of P. F. Papkovich /0.17/, V. I. Feodos'ev /0.25/, and A. S. Vol'mir /0.6/, in which extensive bibliographies are also given. We also draw attention to the monograph of A. R. Rzhanitsyn /0.24/, devoted mainly to the investigation of the stability of systems of bars and plates without initial curvature.

§ 27. A Theorem of P. F. Papkovich on the Convexity of the Region of Stability of a Shell under the Simultaneous Action of Several Stresses

We shall consider the stability of shells under the simultaneous action of several stresses, assuming that these stresses $p_1\varphi_1(\alpha,\beta),\ldots,p_n\varphi_n(\alpha,\beta)$ are proportional to the numerical parameters $p_1,\ldots,p_n$ and the functions $\varphi_1,\ldots,\varphi_n$ remain fixed. As an example, we can consider a cylindrical shell under uniform axial compression and uniform pressure on its surface. By investigating a formula of type (25.31), characterizing the variation $\Delta Э$ of the sum of the deformation energy of the shell and the potential energy of the stresses acting upon it, it is easy to see that it contains the initial stresses of the shell T_1^1, T_2^1, T_{12}^1 only in the first degree. And as these are linearly dependent on the parameters of the stresses $p_1,\ldots,p_n$ then, with fixed displacements of the shell, the quantity $\Delta Э$ is a linear function of the parameters.

$$\Delta Э = Э_0 + p_1 Э_1 + \ldots + p_n Э_n,$$

where $Э_0, Э_1,\ldots,Э_n$ are coefficients dependent on the displacement of the shell.

The equilibrium of the shell will be stable if under any displacement the quantity $\Delta Э$ is positive.

If one considers the quantities $p_1,\ldots,p_n$ as coordinates of the points of some n-dimensional space, then the totality of the values of the parameters $p_1,\ldots,p_n$, for which the equilibrium is stable and the totality of the points of the n-dimensional space corresponding to them can be called the region of stability. P. F. Papkovich /0.17/ has shown that the region of stability of an elastic system is always convex.

A convex region is a region having the property that every ray issuing from any point of the region intersects its boundary not more than once. The totality of points $\bar{r} + c\bar{r}'$, where c is an arbitrary positive number, forms the ray issuing from the point $\bar{r}(p_1,\ldots,p_n)$, parallel to the vector $\bar{r}'(p_1',\ldots,p_n')$.

Points on the ray will have the coordinates $(p_1+cp_1'),\ldots,(p_n+cp_n')$. In order to demonstrate the convexity of the region of stability, we shall assume the contrary, that the region is not convex, and therefore under a monotonic increase of the parameter c and a displacement of the vertex of the vector $\bar{r} + c\bar{r}'$ along some ray issuing from the point $\bar{r}$ of the region of stability, the moving point intersects the boundary of the region of stability twice.

Here it is obvious that after the first intersection of the boundary for some value of the parameter $c_1 > 0$ the shell will be unstable, and after the second intersection of the boundary of stability with $c = c_2 > c_1$ the shell again acquires stability. As for $c = c_1$ the state of the shell is unstable, then for this value of the parameter there will exist a deflection of the shell w^I, for which the change in energy of the system is nonpositive, i. e.,

$$\Delta \vartheta \leqslant 0 \text{ with } c = c_1 > 0,$$

where

$$\Delta \vartheta = \vartheta_0 + (p_1 + cp_1')\vartheta_1 + \ldots + (p_n + cp_n')\vartheta_n. \tag{27.1}$$

On the other hand, with $c = 0$ and with $c = c_2 > c_1$ the shell is stable and therefore at $w = w^I$ we have

$$\Delta \vartheta > 0 \text{ when } c = 0 \text{ and when } c = c_2.$$

If some investigates the variation $\Delta \vartheta$ of the energy of our system for a fixed deflection w^I, then, according to (27.1) it will turn out to be a linear function of the parameter c and therefore will be either monotonically increasing or monotonically decreasing, or constant. But this contradicts the previous three inequalities, according to which with increase of c from $c = 0$ to $c = c_1$, $\vartheta, \Delta\vartheta$ decreases and with the further increase of c to $c = c_2$, $\Delta\vartheta$ again becoming positive, increases. Thus, our assumption of non-convexity of the region of stability leads to a contradiction.

We shall draw some conclusions from the theorem on the convexity of the region of stability.

We shall first consider the stability of a shell under the effect of two stresses acting simultaneously and characterized by the parameters p_1 and p_2. The boundary of the region of stability will in that case be a curve cutting off the segments p_{1k} and p_{2k} on the coordinate axes whose sizes are equal to the critical values of the parameters p_1 and p_2 under the separate influence of each of the stresses.

In the region of stability we shall consider two points with the coordinates $(p_{1k} - \eta_1, 0)$ and $(0, p_{2k} - \eta_2)$ where η_1 and η_2 are sufficiently small and whose signs coincide respectively with the signs of the numbers p_{1k} and p_{2k}. In view of the convexity of the region of stability the straight line connecting the points indicated will also belong to it.

$$\frac{p_1}{p_{1k} - \eta_1} + \frac{p_2}{p_{2k} - \eta_2} = 1.$$

Hence it follows that if the stress parameters satisfy the conditions

$$\frac{p_1}{p_{1k}} + \frac{p_2}{p_{2k}} < 1, \quad \frac{p_1}{p_{1k}} \geqslant 0, \quad \frac{p_2}{p_{2k}} \geqslant 0, \tag{27.2}$$

the equilibrium of the shell will be stable.

The results obtained can be generalized to the case when three stresses, characterized by the parameters p_1, p_2, and p_3, act simultaneously upon the shell. Obviously, the points with the coordinates $(p_{1k} - \eta_1, 0, 0)$, $(0, p_{2k} - \eta_2, 0)$ and $(0, 0, p_{3k} - \eta_3)$, where η_1, η_2, and η_3 are any small numbers having the same signs as p_{1k}, p_{2k} and p_{3k} respectively belong to the region of stability. Hence it follows that the segment connecting the first two points will belong to the region of stability, and so will any segment connecting the third point with any point of the segment which connects the first two points. Consequently, the region of stability will include every point of the area of the triangle, having for vertices the three above-mentioned points lying on the coordinate axes. The points within the triangle satisfy the relations

$$\frac{p_1}{p_{1k}-\eta_1}+\frac{p_2}{p_{2k}-\eta_2}+\frac{p_3}{p_{3k}-\eta_3}=1,\ \frac{p_i}{p_{ik}}\geqslant 0,\ i=1,2,3. \tag{27.3}$$

and consequently, every point of the parameter space which satisfies the relations

$$\frac{p_1}{p_{1k}}+\frac{p_2}{p_{2k}}+\frac{p_3}{p_{3k}}<1,\ \frac{p_i}{p_{ik}}\geqslant 0,\ i=1,2,3 \tag{27.4}$$

lies in the region of stability, for then one can always choose the numbers η_1, η_2, and η_3 so that the relations (27.3) are satisfied.

§ 28. Stability of a Long Plate under Simultaneous Longitudinal, Transverse, and Shearing Stresses

We shall investigate the stability of a plate in the form of a long strip, freely supported at the adges, under the action of longitudinal, transverse, and shearing stresses. These stresses are assumed to be uniformly distributed along the edges of the plate. We shall denote the respective linear forces by p_1, p_2, and

In the literature /VII.1, 2, 3, 4/ one finds Wagner's formula for determining the sets of the critical parameters of plate stresses

$$\sigma_{12}^2 = (2 + 2\sqrt{1+\sigma_2} + \sigma_1)(6 + 2\sqrt{1+\sigma_2} + \sigma_1), \qquad \text{(a)}$$

where

$$\sigma_1 = p_1 \frac{b^2 t}{\pi^2 D}, \quad \sigma_2 = p_2 \frac{b^2 t}{\pi^2 D}, \quad \sigma_{12} = \tau \frac{b^2 t}{\pi^2 D}, \qquad (28.1)$$

b is the width of the plate, $D = Et^3/12(1-\nu^2)$.

T. V. Nevskaya has shown in her dissertatio ı /VII.5/ that this formula is erroneous. In fact, in the special case $\sigma_2 = 0$, f ›rmula (a) takes the form

$$(\sigma_1 + 6)^2 - \sigma_{12}^2 = 4. \qquad \text{(b)}$$

In that case, the boundary of stability will be the set of two branches of a hyperbola (Figure 14).

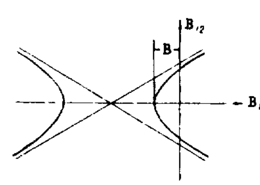

Figure 14

Obviously, the origin of coordinates is in th‹ region of stability. The ray issuing from the origin of coordinates intersects t ıe stability boundary at two points, which contradicts the theorem on the convexity of he region of stability.

In the literature /VII.1, 2, 3, 4/ one also fir ds Wagner's equation for the stability of a strip clamped at the edges, which, for $\sigma_2 = 0$, has the form

$$\sigma_{12}^2 = \left(\frac{8}{\sqrt{3}} + \frac{4}{3} + \sigma_1\right)\left(\frac{8}{\sqrt{3}} + 8 + \sigma_1\right). \qquad \text{(c)}$$

As in the preceding, it can be shown that this formula, too, cannot represent the equation of the boundary of a convex region of stab.lity.

We shall investigate the derivation of the correct equation for the boundary of the stability region for a freely supported plate given in /VII.5/. To solve the problem we shall apply the energy method, choosing the trial deflection function

$$w = f\cos\frac{\pi y}{b}\cos\frac{\pi}{\lambda}(x - my), \tag{28.2}$$

where λ is the distance between two crests, measured along the x axis, m is the tangent of the angle between the direction of the wave crests and the y axis. Along the long edges of the plate this function satisfies the condition

$$w = 0 \quad \text{for } y = \pm b/2.$$

Along the short edges the boundary condition w = 0 is not satisfied. However, if the plate is sufficiently elongated, it is natural to expect that this will not cause substantial errors.

The stability boundary will be determined from the condition that the variation of the sum of the deformation energy of the plate and the potential energy of its stresses be equal to zero. For a flat plate with $k_1 = k_2 = 0$, we obtain according to (25.31) the equation

$$\begin{gathered}\delta\mathcal{E} = \delta\int\int\Big\{D\Big[\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\Big)^2 + 2(1-\nu)\Big(\Big(\frac{\partial^2 w}{\partial x\partial y}\Big)^2 - \\ - \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\Big)\Big] + K\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + 2\nu\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \\ + \frac{(1-\nu)}{2}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2\Big] + T_1^1\Big(\frac{\partial w}{\partial x}\Big)^2 + T_2^1\Big(\frac{\partial w}{\partial y}\Big)^2 + \\ + 2T_{12}^1\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\Big\}\,dx\,dy = 0.\end{gathered} \tag{28.3}$$

The displacements u and v should be chosen so as to obtain, as a result of the computations, the smallest absolute values for the compressive stresses.

It is easy to realize that the smallest values of the compressive stresses are obtained if one sets u = v = 0. For convenience in the further computations, we shall introduce an oblique coordinate system, setting

$$x - my = z,\ y = s. \tag{28.4}$$

In that case it is evident that

$$\begin{gathered}\frac{\partial(\dots)}{\partial x} = \frac{\partial(\dots)}{\partial z},\quad \frac{\partial(\dots)}{\partial y} = \frac{\partial(\dots)}{\partial s} - m\frac{\partial(\dots)}{\partial z}, \\ w = f\cos\pi s/b\cos\pi z/\lambda.\end{gathered} \tag{28.5}$$

By making use of these relations, the evaluation of the integral in (28.3) can be simplified; here, as the function w is periodic and the plate sufficiently long, it can, with sufficient accuracy, be considered equal to the corresponding integral along one wavelength, multiplied by the number of waves in the buckled plate. Using this and (28.1), we shall set up the equation $\partial\mathcal{E}/\partial f = 0$ or

$$\frac{b^2}{\lambda^2}\left\{\left[1+\left(m-\frac{\lambda}{b}\right)^2\right]^2+\left[1+\left(m+\frac{\lambda}{b}\right)^2\right]^2\right\}+2\sigma_1+ \\ +2\sigma_2\left[m^2+\left(\frac{\lambda}{b}\right)^2\right]-4m\sigma_{12}=0. \tag{28.6}$$

To simplify the calculations we shall introduce the notation

$$\psi=(1+m^2)(b/\lambda)^2, \quad \left(\frac{b}{\lambda}\right)^2=\frac{\psi}{1+m^2}. \tag{28.7}$$

If we introduce this quantity in the equation (28.6), we obtain

$$[1+m^2+2\psi+6m^2\psi+\psi^2(1+m^2)]+\sigma_1\psi+ \\ +\sigma_2(m^2\psi+1+m^2)-2\sigma_{12}m\psi=0. \tag{28.8}$$

Noting that in the equilibrium condition the variation of the energy of the system under the possible displacements is zero, we shall determine the values of m and ψ from the conditions

$$\partial Э/\partial m=0, \quad \partial Э/\partial\psi=0$$

or

$$2m[1+6\psi+\psi^2+\sigma_2(\psi+1)]-2\sigma_{12}\psi=0; \tag{28.9}$$
$$2+6m^2+2\psi+2m^2\psi+\sigma_1+\sigma_2m^2-2\sigma_{12}m=0. \tag{28.10}$$

Multiplying equation (28.9) by m/2 and taking (28.8) into account we obtain

$$(1+\psi)^2+\sigma_1\psi+\sigma_2-\sigma_{12}m\psi=0. \tag{28.11}$$

Similarly, multiplying equation (28.10) by ψ and subtracting it from (28.8) we shall obtain

$$(1+m^2)(1-\psi^2+\sigma_2)=0, \quad \psi=\sqrt{1+\sigma_2}. \tag{28.12}$$

Introducing this expression in (28.11) we have

$$\sigma_{12}m=2+2\sqrt{1+\sigma_2}+\sigma_1. \tag{28.13}$$

Subtracting equation (28.10) multiplied by ψ from equation (28.11) and using (28.12), we obtain the equation

$$\sigma_{12}/m=6+2\sqrt{1+\sigma_2}+\sigma_2. \tag{28.14}$$

Multiplying (28.13) by (28.14) we obtain the required equation for the boundary of the region of stability:

$$\sigma_{12}^2=(2+2\sqrt{1+\sigma_2}+\sigma_1)(6+2\sqrt{1+\sigma_2}+\sigma_2). \tag{28.15}$$

§ 29. Determination of the Reduction Coefficient of an Infinite Plane Plate Supported by a Ribbed Network under Longitudinal Compression

We shall consider a wide plate, uniformly compressed in the direction parallel to its short sides. As long as the load compressing the plate is less than the critical load under which loss of stability occurs, its dependence on the mutual approach of the edges is linear. However, when the compressing load exceeds the critical value, then the further approach of the plate edges proceeds almost without increase in loading. Thus we see that the "specific resistance" of the plate, i.e., the ratio of the compressive load P to the mutual approach of the edges, starting from the moment of the loss of stability, will decrease in inverse proportion to the approach of the edges towards each other.

We call the ratio of the load P compressing the plate to that load P' which would be necessary to attain a given mutual approach of its edges without buckling the "reduction coefficient" φ of the plate for a given mutual approach of its edges:

$$\varphi = P/P'. \tag{29.1}$$

The merit of introducing the concept of the reduction coefficient into practical building calculations belongs to I. G. Bubnov /02.3/. He investigated the combined work of a wide plate with ribs, placed along its shorts edges. If the ratio of the plate width to the length is large, then one can consider that the loading necessary to compress the plate after the loss of stability, without very large deflections, is defined by Euler's formula. The loading P, as had been shown earlier, can be considered as equal to the compressive force at the instant of loss of stability of the plate:

$$P = \sigma_{\mathfrak{z}} s. \tag{29.2}$$

Here s is the cross-sectional area of the plate, and $\sigma_{\mathfrak{z}}$ is the stress appearing in the plate at the instant of its loss of stability, defined by the Euler formula. On the other hand, it is obvious that if there existed supplementary supports and the plate did not lose its stability, then the stresses in the plate would be equal to the stresses σ_c in the ribs supporting the short sides of the plate; then the loading of the plate P' would be defined by the formula:

$$P' = \sigma_c s. \tag{29.3}$$

Taking account of (29.1)-(29.3) we obtain for the reduction coefficient the quantity:

$$\varphi = 1/n, \; n = \sigma_c/\sigma_{\mathfrak{z}} > 1. \tag{29.4}$$

I. G. Bubnov in his works /0.3/ considers also the definition of the reduction coefficient of twisted plates of large width having an initial deflection (see /VII.6/).

We shall further consider a narrow plate, i.e., a plate with a small ratio of width to length. In that case, a considerable increase in membrane stresses will occur under compression even after the loss of stability. Under uniform compression of the edges, when all the points of its transverse edge are uniformly displaced, we shall have to consider the fact that while the middle part of the plate, removed from the longitudinal ribs, hardly resists further reduction of the distance between the transverse edges after the loss of stability, the deflection of the parts close to the longitudinal ribs is made more difficult, and therefore in those parts of the plate stresses can appear which are considerably larger than the stresses in the middle of the plate. This leads to the fact that the reduction coefficient for a narrow plate will be much larger than the quantity indicated by (29.4).

A number of authors /VII.6/ occupied themselves with the problem of determining the reduction coefficient. By starting from insufficiently well-founded hypotheses, they derived a series of computational formulas, which can be considered only as first, rather crude, approximations to reality.

Among the existing solutions of this problem, the solution of P. A. Sokolov /0.17/ deserves special attention. He considered the problem of determining the reduction coefficient of a plate of very large dimensions, supported on a network of mutually equidistant longitudinal and transverse ribs. We shall denote the distances between two neighboring longitudinal and transverse ribs by b and a respectively. The equation of equilibrium and the equation of compatibility of deformations of a flat plate are obtained from (26.10) and (26.11), setting $w^0 = 0$ (in view of the absence of initial bending). If, as distinct from the above, one considers one movement as positive when its direction coincides with the positive direction of the deflection, these equations will have the form

$$\Delta\Delta\psi = Et\left[\left(\frac{\partial^2 w^{\mathrm{I}}}{\partial x \partial y}\right)^2 - \frac{\partial^2 w^{\mathrm{I}}}{\partial x^2} \cdot \frac{\partial^2 w^{\mathrm{I}}}{\partial y^2}\right], \tag{29.5}$$

$$D\Delta\Delta w^{\mathrm{I}} - \frac{\partial^2 \psi}{\partial y^2} \cdot \frac{\partial^2 w^{\mathrm{I}}}{\partial x^2} - \frac{\partial^2 \psi}{\partial x^2} \cdot \frac{\partial^2 w^{\mathrm{I}}}{\partial y^2} + 2\,\frac{\partial^2 \psi}{\partial x \partial y} \cdot \frac{\partial^2 w^{\mathrm{I}}}{\partial x \partial y} - p = 0 \tag{29.6}$$

Here w^{I} is the deflection of the plate, ψ is the stress function giving the membrane stresses

$$T_1 = \frac{\partial^2 \psi}{\partial y^2}, \quad T_2 = \frac{\partial^2 \psi}{\partial x^2}, \quad T_{12} = -\frac{\partial^2 \psi}{\partial x \partial y},$$

P is the transverse pressure on the plate, and x and y are rectangular Cartesian coordinates. We shall effect a change of variables to dimensionless quantities which considerably facilitate the calculations:

$$w = \frac{w^{\mathrm{I}}}{t}, \quad \Phi = \frac{\psi}{Et^3}, \quad \xi = 2x\pi/b, \quad \eta = 2y\pi/b. \tag{29.7}$$

Then the preceding equations will take the form

$$\Delta\Delta\Phi = \left(\frac{\partial^2 w}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 w}{\partial\xi^2} \cdot \frac{\partial^2 w}{\partial\eta^2}, \tag{29.8}$$

$$\frac{1}{12(1-\nu^2)}\,\Delta\Delta w - \frac{\partial^2\Phi}{\partial\eta^2} \cdot \frac{\partial^2 w}{\partial\xi^2} - \frac{\partial^2\Phi}{\partial\xi^2} \cdot \frac{\partial^2 w}{\partial\eta^2} + 2\,\frac{\partial^2\Phi}{\partial\xi\partial\eta} \cdot \frac{\partial^2 w}{\partial\xi\partial\eta} - p_z = 0, \tag{29.9}$$

where

$$p_z = \frac{b^4}{16Et^4\pi^4}\,p, \quad \Delta(\ldots) = \frac{\partial^2(\ldots)}{\partial\xi^2} + \frac{\partial^2(\ldots)}{\partial\eta^2}. \tag{29.10}$$

Utilizing the previous relations, it is not hard to convince oneself of the correctness of the following theorems on freely supported similar plates which we define as those having the same ratio of their sides and divided by ribs into the same number of strips.

a) Similar plates which are under transverse loads only, have the same relative deflections w, if the transverse loads acting upon them are in the ratio

$$Et^4/b^4;$$

b) Similar plates, which are under longitudinal loads applied to their edges, have the same relative deflections w, if the longitudinal loads per unit length of the plate edges are in the same ratio as the corresponding values of Et^3/b^2.

Hence it follows that for similar plates, the critical strains under which infinitely small bucklings occur, are proportional to the respective values of Et^2/b^2, and that to the same relative deflections of the plate correspond the same values of the ratio $T_1/T_{1,k}$, where by $T_{1,k}$ is denoted the critical value of strain for the start of buckling of the plate.

We shall clarify the mutual relation for similar plates between the shortening of the distances between the transverse edges, to which correspond the coordinates

$$x=(2N_1+1)\frac{a}{2}=\frac{L_1}{2}, \quad x=-\frac{L_1}{2}.$$

Here it is assumed that the length of the plate L_1 is divided by ribs into $2N_1 + 1$ parts of length a, and that the origin of coordinates is situated at the center of the plate and the axes are oriented parallel to its edges. The decrease of the distance Δ_1 between the transverse edges of the plate is given by the formula

$$\Delta_1=-\int_{-L_1/2}^{L_1/2}\frac{\partial u^1}{\partial x}\,dx.$$

According to (25.8) and (26.10), in our case

$$\varepsilon_1=\frac{\partial u^1}{\partial x}+\frac{1}{2}\left(\frac{\partial w^1}{\partial x}\right)^2=\frac{T_1-\nu T_2}{Et}=\frac{1}{Et}\left(\frac{\partial^2\psi}{\partial y^2}-\nu\frac{\partial^2\psi}{\partial x^2}\right), \tag{29.11}$$

$$\varepsilon_2=\frac{\partial v^1}{\partial y}+\frac{1}{2}\left(\frac{\partial w^1}{\partial y}\right)^2=\frac{T_2-\nu T_1}{Et}=\frac{1}{Et}\left(\frac{\partial^2\psi}{\partial x^2}-\nu\frac{\partial^2\psi}{\partial y^2}\right),$$

$$2\varepsilon_{12}=\frac{\partial v^1}{\partial x}+\frac{\partial u^1}{\partial y}+\frac{\partial w^1}{\partial x}\cdot\frac{\partial w^1}{\partial y}=-2\,\frac{(1+\nu)}{Et}\cdot\frac{\partial^2\psi}{\partial x\partial y},$$

$$\Delta_1=-\int_{-L_1/2}^{L_1/2}\left[\frac{1}{Et}\left(\frac{\partial^2\psi}{\partial y^2}-\nu\frac{\partial^2\psi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial w^1}{\partial x}\right)^2\right]dx. \tag{22.12}$$

In order to attain such an approach of the plate edges in the absence of buckling, one would have to apply the stresses

$$T_1'=-Et\,\Delta_1/L_1.$$

We shall assume that along the width of the plate L_2 there are $2N_2 + 1$ strips. Then the total load in the absence of buckling of the plate would be

$$P'=-L_2T_1'=Et\Delta_1L_2/L_1. \tag{29.13}$$

In reality the load applied to the transverse edges of the plate is

$$P = -\int_{-L_2/2}^{L_2/2} T_1 dy = -\int_{-L_2/2}^{L_2/2} \frac{\partial^2\psi}{\partial y^2} dy.$$

Making use of the dimensionless variables (29.7), we obtain the formulas

$$\varphi = \frac{P}{P'} = \frac{L_1}{L_2} \cdot \frac{\int\limits_{-\lambda_2}^{\lambda_2} \frac{\partial^2\Phi}{\partial\eta^2} d\eta}{\int\limits_{-\lambda_1}^{\lambda_1}\left[\frac{\partial^2\Phi}{\partial\eta^2} - \nu\frac{\partial^2\Phi}{\partial\xi^2} - \frac{1}{2}\left(\frac{\partial w}{\partial\xi}\right)^2\right] d\xi} \tag{29.14}$$

where

$$\lambda_1 = (2N_1 + 1)\frac{a\pi}{b}, \quad \lambda_2 = (2N_2 + 1)\pi.$$

Hence it is apparent that with the same relative deflections of similar plates the reduction coefficients are also the same. This theorem can be quite useful for experimental determination of the reduction coefficients.

Utilizing similarity considerations, one can limit oneself, in order to simplify further calculations, to considering a plate, which is divided by a rib network into strips with the dimensions $2\pi a/b$ and 2π; the modulus of elasticity and the thickness t of the plate will be assumed equal to unity.

Let the plate be under the action of loads compressing it in the longitudinal and transverse directions, where the mean shearing stress in the plate is zero. In that case it can be expected that the deflection will be symmetrical with respect to the centers of the strips; therefore we give its approximate expression in the form of an even periodic function of the coordinates:

$$w_n = \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn}\cos\delta\left(m - \frac{1}{2}\right)\xi\cdot\cos\left(n - \frac{1}{2}\right)\eta, \quad \delta = \frac{b}{a}. \tag{29.15}$$

The coefficients A_{mn} of this formula shall be determined below from energy considerations.

Obviously the expression (29.15) satisfies the conditions for the absence of deflection at the points of contact of the plate with the ribs. Substituting it in the right-hand member of the equation (29.8), after simple trigonometric transformations, we obtain

$$\Delta\Delta\Phi = \sum_{m=0}^{2M}\sum_{n=0}^{2N} D_{mn}\cos m\delta\xi\cos n\eta, \tag{29.16}$$

where D_{mn} are second degree algebraic polynomi ıls in A_{mn}:

$$\begin{aligned}
D_{10} &= -(\delta^2/32)(A_{11}^2 + 9A_{21}^2 + 25A_{31}^2 + 2A_{\,1}A_{12} + 2A_{11}A_{12}),\\
D_{02} &= -(\delta^2/4)(A_{11}A_{12} + A_{11}A_{13}), \quad D_{03} = -(9\delta^2/32)(A_{12}^2 + A_{11}A_{13}),\\
D_{11} &= -\delta^2\left(\frac{1}{4}A_{11}A_{21} + A_{21}A_{31} + \frac{1}{4}A_{11}A_{1\,} + A_{12}A_{21} + A_{12}A_{13}\right),\\
D_{22} &= -\delta^2(A_{12}A_{21} + 4A_{12}A_{31} + 4A_{21}A_{13}),\\
D_{21} &= -(\delta^2/16)(A_{11}A_{31} + 9A_{11}A_{13} + 25A_{1\,}A_{21} + 49A_{12}A_{31}),\\
D_{31} &= -(\delta^2/4)(A_{11}A_{31} + 16A_{12}A_{31}), \quad D_{00} = 0.
\end{aligned} \tag{29.17}$$

From these formulas one also obtains the expressions for the quantities D_{01}, D_{20}, ..., D_{13}, if in the formulas given one interchanges the indexes m and n in the quantities A_{mn}.

The particular integral of equation (29.16) is given by the formula

$$\Phi_1 = \sum_{m=0}^{2M} \sum_{n=0}^{2N} \frac{D_{mn}}{(m \cdot \delta^2 + n^2)^2} \cos m\delta\xi \cos n\eta. \tag{29.18}$$

The general solution Φ of equation (29.16) is the sum of the particular integral Φ_1 and the general solution Φ_0 of the homogenous equation

$$\Delta\Delta\Phi_0 = 0, \tag{29.19}$$

i.e.,

$$\Phi = \Phi_1 + \Phi_0 . \tag{29.20}$$

Here, in order to simplify the problem, it is assumed that the plate is displaced relative to the ribs in a plane tangent to the plate, i.e., it can slide along them, but cannot separate itself from them. The reduction coefficient thus obtained from the plate will turn out to be somewhat lower, which will lead to a rise of the safety factor of the structure. In view of the fact that the ribs do not transmit any tangential forces to the plate, the stress function will have no discontinuities in its second derivatives which define the membrane forces in the plate.

The type of formula (29.18), representing Φ_1 as a function with continuous derivatives, accords with our assumption. To determine the biharmonic function Φ_0 we shall temporarily make the assumption that the plate we are investigating extends to infinity in all directions, and we shall require that the function Φ_0 characterizes stresses, whose values are bounded in the infinite plane. Here, the second partial derivatives of that function with respect to ξ and η should be bounded at infinity and, therefore, the function $\Delta\Phi_0$ should also be bounded, which is harmonic, as it satisfies the Laplace equation: $\Delta(\Delta\Phi_0) = 0$. But, as is well known /VII.9/, a harmonic function bounded everywhere can be only a constant, i.e.,

$$\Delta\Phi_0 = \text{const} = C.$$

Hence follows that

$$\Phi_0 = C\frac{\xi^2}{2} + \chi,$$

where χ is a harmonic function. Consequently, the function $\partial^2\chi/\partial\xi^2$ is also harmonic, as

$$\Delta(\partial^2\chi/\partial\xi^2) = \frac{\partial^2}{\partial\xi^2}(\Delta\chi) = 0.$$

But $\partial^2\chi/\partial\xi^2 = \partial^2\Phi_0/\partial\xi^2 - C$ is a bounded function and, therefore, from the theorem just mentioned $\partial^2\chi/\partial\xi^2 = \text{const}$. Therefore, $\partial^2\Phi_0/\partial\xi^2$ is also a constant.

In an analogous manner it can be shown that

$$\partial^2\Phi_0/\partial\eta^2 = \text{const}, \quad \partial^2\Phi_0/\partial\xi\partial\eta = \text{const}.$$

Hence it follows that Φ_0 is a second degree polynomial in ξ and η in which the linear terms can be set equal to zero, as they do not affect the values of the membrane stresses. Thus,

$$\Phi_0 = p_2\xi^2/2 - \tau\xi\eta + p_1\eta^2/2, \tag{29.21}$$

where p_1, p_2, τ represent the values of the mean compression and shearing stresses. If the plate has finite, but nevertheless sufficiently large dimensions, and is supported on a considerable number of longitudinal and transverse ribs, then it can be expected that for the majority of the plate strips, with the exception of those which are close to its boundary, the stress function is sufficiently well represented by the formulas (29.18), (29.20), and (29.21). To determine the deflection of the plate we shall make use of the minimum total potential energy principle, according to which the equilibrium state of the plate is characterized by a minimum of the sum of elastic energy of the plate and of the potential energy of the loads acting upon the plate:

$$Э = Э_{\text{defl}} + Э_{м} - A = \min. \tag{29.22}$$

Here $Э_{\text{defl}}$ is the elastic energy of the deflection of the plate, $Э_{м}$ is the elastic energy of the membrane stresses, and A is the work of the external forces.

The work of the load compressing the plate along the axis x is

$$A_1 = -\int_{-\lambda_2}^{\lambda_2} p_1\Delta_1 d\eta = p_1\int_{-\lambda_1}^{\lambda_1}\int_{-\lambda_2}^{\lambda_2}\frac{\partial u^1}{\partial\xi}d\xi d\eta.$$

Analogously, the work of the load compressing the plate along the y axis is

$$A_2 = p_2\int_{-\lambda_1}^{\lambda_1}\int_{-\lambda_2}^{\lambda_2}\frac{\partial v^1}{\partial\eta}d\xi d\eta.$$

The work of the shearing loads, applied to the transverse edges of the plate is

$$\int_{-\lambda_2}^{\lambda_2}\tau v^1(\xi=\lambda_1)\,d\eta - \int_{-\lambda_2}^{\lambda_2}\tau v^1(\xi=-\lambda_1)\,d\eta = \tau\int_{-\lambda_2}^{\lambda_2}\int_{-\lambda_1}^{\lambda_1}\frac{\partial v^1}{\partial\xi}d\xi d\eta.$$

An analogous expression can be obtained for the work of the shearing stresses, applied to the longitudinal edges of the plate.

The work of all the shearing loads is equal to:

$$A_3 = \tau\int_{-\lambda_2}^{\lambda_2}\int_{-\lambda_1}^{\lambda_1}\left(\frac{\partial u^1}{\partial\eta} + \frac{\partial v^1}{\partial\xi}\right)d\xi d\eta.$$

In our case, when E = 1, t = 1, $\psi = \Phi$, by making the use of formulas (29.11), we find

$$A = A_1 + A_2 + A_3 = \int_{-\lambda_2}^{\lambda_2}\int_{-\lambda_1}^{\lambda_1}\Big\{p_1\Big[\frac{\partial^2\Phi}{\partial\eta^2} - \nu\frac{\partial^2\Phi}{\partial\xi^2} - \frac{1}{2}\Big(\frac{\partial w}{\partial\xi}\Big)^2\Big] +$$
$$+ p_2\Big[\frac{\partial^2\Phi}{\partial\xi^2} - \nu\frac{\partial^2\Phi}{\partial\eta^2} - \frac{1}{2}\Big(\frac{\partial w}{\partial\eta}\Big)^2\Big] - \tau\Big[2(1+\nu)\frac{\partial^2\Phi}{\partial\xi\partial\eta} +$$
$$+ \frac{\partial w}{\partial\xi}\cdot\frac{\partial w}{\partial\eta}\Big]\Big\}\, d\xi d\eta. \tag{29.23}$$

Further, from formulas (29.11) and (17.37), one can easily derive the expressions

$$Э_м = \frac{1}{2}\int\int\Big\{(\Delta\Phi)^2 + 2(1+\nu)\Big[\Big(\frac{\partial^2\Phi}{\partial\xi\partial\eta}\Big)^2 - \frac{\partial^2\Phi}{\partial\xi^2}\cdot\frac{\partial^2\Phi}{\partial\eta^2}\Big]\Big\}\, d\xi d\eta,$$
$$Э_{\text{defl}} = \frac{D}{2}\int\int\Big\{(\Delta w)^2 + 2(1-\nu)\Big[\Big(\frac{\partial^2 w}{\partial\xi\partial\eta}\Big)^2 - \frac{\partial^2 w}{\partial\xi^2}\cdot\frac{\partial^2 w}{\partial\eta^2}\Big]\Big\}\, d\xi d\eta, \tag{29.24}$$

where in our case $D = 1/12(1-\nu^2)$.

Using (29.20) and (29.21), and setting in what follows $\tau = 0$, the first of the formulas (29.24) can be given in the form

$$Э_м = \frac{1}{2}\int\int\Big\{(\Delta\Phi_1)^2 + 2(1+\nu)\Big[\Big(\frac{\partial^2\Phi_1}{\partial\xi\partial\eta}\Big)^2 - \frac{\partial^2\Phi_1}{\partial\xi^2}\cdot\frac{\partial^2\Phi_1}{\partial\eta^2}\Big] +$$
$$+ (p_1+p_2)^2 - 2(1+\nu)p_1p_2 + (p_1 - p_2)\Delta\Phi_1 -$$
$$- 2(1+\nu)\Big(p_1\frac{\partial^2\Phi_1}{\partial\xi^2} + p_2\frac{\partial^2\Phi_1}{\partial\eta^2}\Big)\Big\}\, d\xi d\eta.$$

Integrating by parts and using (29.18), it is not hard to show that

$$\int_{-\lambda_2}^{\lambda_2}\int_{-\lambda_1}^{\lambda_1}\Big(\frac{\partial^2\Phi_1}{\partial\xi\partial\eta}\cdot\frac{\partial^2\Phi_1}{\partial\xi\partial\eta} - \frac{\partial^2\Phi_1}{\partial\xi^2}\cdot\frac{\partial^2\Phi_1}{\partial\eta^2}\Big)\, d\xi d\eta =$$
$$= \Big\{\int_{-\lambda_1}^{\lambda_1}\Big(\frac{\partial\Phi_1}{\partial\xi}\cdot\frac{\partial^2\Phi_1}{\partial\xi\partial\eta}\Big)\, d\xi\Big\}_{\eta=-\lambda_2}^{\eta=\lambda_2} - \Big\{\int_{-\lambda_2}^{\lambda_2}\Big(\frac{\partial\Phi_1}{\partial\xi}\,\frac{\partial^2\Phi_1}{\partial\eta^2}\Big)\, d\eta\Big\}_{\xi=-\lambda_1}^{\xi=\lambda_1} = 0,$$

and the last three terms of the expression for $Э_м$ are zero. Thus,

$$Э_м = \frac{1}{2}\int\int\{(\Delta\Phi_1)^2 + (p_1+p_2)^2 - 2(1+\nu)p_1p_2\}\, d\xi d\eta. \tag{29.25}$$

Analogously, the second of the formulas (29.24) can be simplified and brought into the form:

$$Э_{\text{defl}} = \frac{D}{2}\int\int(\Delta w)^2\, d\xi d\eta. \tag{29.26}$$

Introducing (29.15) and (29.18) in the expressions (29.23), (29.25), and (29.26), and carrying out the integration, we shall obtain the following expression for the total energy of the system:

$$Э = \frac{1}{2}\lambda_1\lambda_2\Big\{\sum_{m=1}^{2M}\sum_{n=1}^{2N}\frac{D_{mn}^2}{(m^2\delta^2+n^2)^2} +$$
$$+ \sum_{m=1}^{M}\sum_{n=1}^{N}A_{mn}^2\Big[(D+p_1)\Big(m-\frac{1}{2}\Big)^2\delta^2 +$$
$$+ (D+p_2)\Big(n-\frac{1}{2}\Big)^2\Big] - 8\,[p_1^2 + p_2^2 - 2\nu p_1p_2]\Big\}. \tag{29.27}$$

From condition (29.22) follow the equations:

$$\partial Э/\partial A_{mn} = 0$$

or

$$\sum_{\alpha=1}^{2M}\sum_{\beta=1}^{2N}\frac{D_{\alpha\beta}}{(m^2\delta^2+n^2)^2}\cdot\frac{\partial D_{\alpha\beta}}{\partial A_{mn}}+$$
$$+\left\{(D+p_1)\left(m-\frac{1}{2}\right)^2\delta^2+(D+p_2)\left(n-\frac{1}{2}\right)^2\right\}A_{mn}=0, \qquad (29.28)$$
$$\begin{pmatrix} m=1,\ 3,\ \ldots,\ M \\ n=1,\ 3,\ \ldots,\ N \end{pmatrix},$$

where $D_{\alpha\beta}$ are quantities defined by (29.17). As that system of equations is non-linear, several states of equilibrium can correspond to one and the same loading. Among them, the stableness will be those for which, under arbitrary small changes δA_{mn} the inequality

$$\sum_{mn}\sum_{m'n'}\frac{\partial^2 Э}{\partial A_{mn}\partial A_{m'n'}}\,\delta A_{mn}\delta A_{m'n'}>0. \qquad (29.29)$$

is satisfied. In order that the quadratic form in the left-hand member of the above should be positive, it is sufficient, as is well known, that the principal minors of its matrix be positive. The relations thus obtained allow one to pick out from the totality of all the solutions of the system (29.28) those which correspond to the states of stable equilibrium.

Having determined A_{mn}, from (29.18), (29.21), and (29.20), we obtain Φ, and then we calculate the reduction coefficient φ from (29.14).

Figure 15 gives the curves of the dependence of the reduction coefficient on the magnitude of compression of the plate.

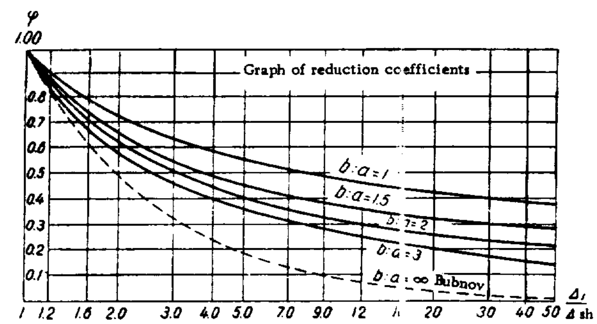

Figure 15

Here the values of φ are along the ordinates and along the abscissae the values of the compression parameters Δ_1/Δ_{sh} representing the ratio of the mutual approach of the plate edges Δ_1 to the value of the shortening Δ_{sh} of the plate at the instant of stability loss.

In conclusion we shall prove that in the case under consideration, the edges of the plate remain straight after buckling.

In fact, from the third equation (29.11) we have, after differentiation with respect to y

$$\frac{\partial^2 u^1}{\partial y^2} = -\frac{2(1+\nu)}{Et}\cdot\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{\partial}{\partial y}\left(\frac{\partial w^1}{\partial x}\cdot\frac{\partial w^1}{\partial y}\right) - \frac{\partial^2 v^1}{\partial x\partial y}.$$

Differentiating the second of the equations (29.11) with respect to x, we determine from it the value of $\partial^2 v^1/\partial x\partial y$:

$$\frac{\partial^2 v^1}{\partial x\partial y} = \frac{1}{Et}\left(\frac{\partial^3\psi}{\partial x^3} - \nu\frac{\partial^3\psi}{\partial x\partial y^2}\right) - \frac{\partial w^1}{\partial y}\cdot\frac{\partial^2 w^1}{\partial x\partial y}.$$

Introducing this expression in the preceding equation, we obtain

$$\frac{\partial^2 u^1}{\partial y^2} = -2\frac{(1+\nu)}{Et}\cdot\frac{\partial^3\psi}{\partial x\partial y^2} - \frac{1}{Et}\left(\frac{\partial^3\psi}{\partial x^3} - \nu\frac{\partial^3\psi}{\partial x\partial y^2}\right) - \frac{\partial w^1}{\partial x}\cdot\frac{\partial^2 w^1}{\partial y^2}. \qquad (29.30)$$

Introducing in this the formulas (29.15) and (29.18) we convince ourselves that

$$\frac{\partial^2 u^1}{\partial y^2} = 0 \text{ for } x = \pm(2N_1+1)a\pi/b.$$

This quantity is zero also at the values of $x = \pm\frac{\pi a}{b}$, corresponding to the edges of the central strip of the plate, which testifies to the fact that they do not twist.

§ 30. Determination of the Reduction Coefficient Under the Combined Action of Compressive and Shearing Edge Loads

We shall consider the problem of determining the finite deflections of a plate, supported by a large number of uniformly spaced longitudinal ribs and loaded at its edges by shearing stresses τ, a longitudinal stress p_1, and a transverse stress p_2. To simplify the investigation we shall initially assume that the plate can freely slide along the ribs, having no possibility of separating itself from them. This problem was solved by Kromm and Marguerre /VII.10/.

As experiments show, with the buckling of the plate due to the action of shearing loads, diagonal waves appear on its surface (Figure 16).

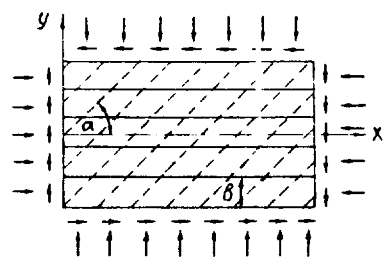

Figure 16

Taking that into account, the deflection of the plate can be approximated by the function

$$w = f\cos\frac{\pi y}{b}\cos\frac{\pi}{l}(x - my). \tag{30.1}$$

Here b is the distance between neighboring ribs, l is the length of the segments cut off by the wave crests on the axis of abscissas, m is the cotangent of the rise angle α of the waves with the abscissa axis. Introducing the expression (30.1) in the compatibility equation (29.5), we shall obtain

$$\Delta\Delta\psi = -Et\frac{f^2\pi^4}{2l^2b^2}\left\{\cos\frac{2\pi y}{b} + \cos\frac{2\pi}{l}(x - my)\right\}.$$

Analogously to the preceding we find the solution of this equation, which satisfies the boundedness condition of the membrane strains in the infinite plane:

$$\psi = -Et\frac{f^2}{32}\left\{\frac{b^2}{l^2}\cos\frac{2\pi y}{b} + \frac{l^2}{b^2(1+m^2)^2}\cos\frac{2\pi}{l}(x - my)\right\} + \\ + \frac{p_1 y^2}{2} + \frac{p_2 x^2}{2} - \tau xy \tag{30.2}$$

The stresses at the plate edges are defined by

$$T_2 = \frac{\partial^2 \psi}{\partial x^2} = \frac{Etf^2}{8b^2(1+m^2)^2}\cos(x - my) + p_2,$$
$$T_{12} = -\frac{\partial^2 \psi}{\partial x \partial y} = \frac{Etf^2 m}{8b^2(1+m^2)^2}\cos(x - my) + \tau.$$

Consequently, formula (30.2) represents the stress function for the case when plate edge is under a set of uniformly distributed loads p_1, τ, and p_2, and periodic loads, represented by trigonometric functions. But the influence of the periodic loads upon the state of strain of the plate decreases rapidly with distance from its edge, and therefore formula (30.2) will be sufficiently accurate in the application to the parts of the plate which are not too close to its edges, as also in the case when the plate edges are only under the above-mentioned uniform loads.

By using formulas of the type (29.24) and (29.23), one can determine the elongation energy of the plate and the work of the external loads; there it is convenient to first calculate the corresponding integrals along the surface of a single buckle of the plate, and then to multiply the quantity obtained by the number of half-waves in the buckled plate. Uncomplicated but somewhat lengthy calculations show that the potential energy of the plate and its loads is given by the quantity

$$\begin{gathered} \mathit{Э}_{\text{ж}} + \mathit{Э}_{\text{defl}} - A = L_1 Etb\left\{\frac{\pi^4 f^4}{256 b^4}\cdot\frac{(1+\beta^4)}{(1+m^2)^2} - \frac{1}{E^2 t^2}\left[\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \right.\right. \\ \left.\left. - \nu p_1 p_2 + (1+\nu)\tau^2\right] + \frac{t^2 f^2 \pi^4}{96(1-\nu^2)b^4}\left[(1+\beta^2)^2 + 4\frac{m^2}{1+m^2}\beta^2\right]\right\} + \\ + p_1 L_1 b\frac{\pi^2 f^2}{8b^2}\cdot\frac{\beta^2}{1+m^2} + \\ + p_2 L_1 b\frac{\pi^2 f^2}{8b^2}\left(1 + \frac{\beta^2 m^2}{1+m^2}\right) - \tau L_1 b\frac{\pi^2 f^2}{4b^2}\cdot\frac{mb^2}{1+m^2}. \end{gathered} \tag{30. 2}$$

The accuracy of this formulas will be sufficient, if the number of buckling waves is large, the plate is long, and once can roughly neglect the energy of those parts of half-waves which are situated at the plate edges.

Here we introduced the notation

$$\beta^2 = b^2(1+m^2)/l^2. \tag{30.4}$$

The state of equilibrium of the plate is characterized by a stationary value of the total energy of the plate and its loads; therefore we shall determine the values of the parameters f, β, and m, characterizing the plate shape, from the relations

$$\begin{gathered} \partial(\mathit{Э}_{\text{ж}} + \mathit{Э}_{\text{defl}} - A)/\partial f = 0, \ \partial(\mathit{Э}_{\text{ж}} + \mathit{Э}_{\text{defl}} - A)/\partial\beta = 0, \\ \partial(\mathit{Э}_{\text{ж}} + \mathit{Э}_{\text{defl}} - A)/\partial m = 0. \end{gathered} \tag{30.5}$$

As can be seen from expression (30.3), these equations will contain the quantities p_1, p_2, and τ only in the first power. Solving them, we find the relations

$$\begin{gathered} -p_1 = \frac{p^*}{4}[2 + 2\beta^2 - m^2(5 + 2\beta^2 + \beta^4)] + \\ + Et\frac{\pi^2 f^2}{16b^2}\left[\frac{2m^2 + \beta^2 + 2m^2\beta^2}{\beta^2(1+m^2)^2}\right], \end{gathered} \tag{30.6}$$

$$-p_2 = \frac{p^*}{4}(1 - \beta^4) + Et\frac{\pi^2 f^2}{16b^2}\left(\frac{1}{1+m^2}\right)^2. \tag{30. 7}$$

$$\tau = m\left[\frac{p^*}{4}(5 + 2\beta^2 + \beta^4) - Et\frac{\pi^2 f^2}{16b^2}\cdot\frac{\left(1 + \frac{1}{\beta^2}\right)}{(1+m^2)^2}\right]. \tag{30. 8}$$

Here we introduced the notation

$$p^* = \frac{Et}{1-\nu^2} \cdot \frac{\pi^2 t^2}{3b^2} \tag{30.9}$$

To simplify the calculations we shall have to confine ourselves to the consideration of some special cases.

We shall first consider the case where transverse compression of the plate is absent

$$p_2 = 0. \tag{30.10}$$

We shall start by clarifying the relationship between the longitudinal compression and the shear at the instant of stability loss. To that end, we set f = 0 in the formulas (30.6), (30.7), and (30.8). Then, using (30.10), we obtain from (30.7)

$$\beta = 1.$$

Introducing this in (30.6) and (30.8), we obtain

$$-p_1 = p^* - 2p^* m^2, \quad \tau = 2mp^*.$$

Determining the value m from the last equation and substituting it the preceding equation, we obtain the equation for the stability boundary of the plate under a simultaneous loading by shearing and longitudinal loads

$$-p_1/p^* = 1 - (\tau/\sqrt{2}\,p^*)^2. \tag{30.11}$$

By setting $\tau = 0$ in that formula, we obtain the critical value of the compression stress in the absence of shearing stresses

$$p_{1k} = -p^*.$$

In an analogous way, setting $p_1 = 0$ in formula (30.11), we determine the critical value of the shearing stress in the absence of compression

$$\tau_k = \sqrt{2}\,p^* = \frac{Et}{1-\nu^2} \cdot \frac{\sqrt{2}\,\pi^2 t^2}{3b^2}. \tag{30.12}$$

This value differs by roughly 6.5% from the exact value of the critical shearing stress of an infinitely long strip.

Introducing (30.12) in (30.11), we obtain the approximate formulas

$$p_1/p_{1k} = 1 - (\tau/\tau_k)^2.$$

We return to the general case of finite deflections of a plate. According to formulas (29.12) it is easy to compute the mutual approach Δ_1 of the transverse edges of the plate (where in that formulas one should extend the limits of integration along one half-wavelength of the buckling, and multiply the result by the number of half-wave $L_1/1$):

$$\Delta_1 = L_1 [(-p_1 + \nu p_2)/Et + \pi^2 f^2/8l^2]. \tag{30.13}$$

In an analogous way one computes the transverse contraction Δ_2 of the plate

$$\Delta_2 = L_2\left\{(-p_2+\nu p_1)/Et+\frac{\pi^2 f^2}{8l^2}\left[\left(\frac{l}{b}\right)^2+m^2\right]\right\}, \tag{30.14}$$

where L_2 is the plate width.

Assuming that the ribs undergo a contraction, equal to the mutual approach Δ_1 of the plate edges, it is easy to determine the stress appearing in the ribs:

$$-\sigma_l = E(\Delta_1/L_1) = \frac{-p_1+\nu p_2}{t}+E\pi^2 f^2/8l^2. \tag{30.15}$$

We shall now compute the shear along the plate contour. For this we note that the shear angle of the projection of a small rectangular element of the plate on the plane of the plate contour is

$$2\varepsilon_{12} = \frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}.$$

This quantity is different for different points of the plate. To characterize the shear angle for the plate as a whole we shall compute the mean value of the shear of its elements along the plate surface:

$$\gamma = 2\varepsilon_{12}^c = \left[\int\int\left(\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}\right)dxdy\right]/S,$$

where S is the plate area.

If one makes use of the relation connecting the stresses and the deformation of the plate

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial u^1}{\partial y}+\frac{\partial v^1}{\partial x}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right] = \frac{(1+\nu)\,T_{12}}{Et} = -\frac{(1+\nu)}{Et}\cdot\frac{\partial^2\psi}{\partial x\partial y},$$

then the value of the mean shear of the projection of the plate elements can be represented by the formula

$$\gamma = \int\int\left[-\frac{2(1+\nu)}{Et}\cdot\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\right]dxdy/S.$$

Introducing in this formula the expressions (30.1) and (30.2), we shall obtain

$$\gamma = \frac{2(1+\nu)}{Et}\tau+\frac{m\pi^2 f^2}{4l^2}. \tag{30.16}$$

Taking into account the fact that in our case $p_2 = 0$, we obtain from equation (30.7) a formula for the determination of f:

$$\frac{Et\pi^2 f^2}{16b^2} = \frac{p^*}{4}(\beta^4-1)(1+m^2)^2. \tag{30.17}$$

Introducing (30.17) in (30.8), we obtain an expression for m in terms of τ and β:

$$m = \tau\Big/\frac{p^*}{4}\left[6+\beta^2+\frac{1}{\beta^2}\right]. \tag{30.18}$$

The preceding relations allow one to construct, for a given value of shearing load, graphs of the dependence of the values of $p_l = t\sigma_l$ and f on the value of the mean longitudinal stress p_1 in the plate. For this one has to take a series of values of the parameter β and, by means of the formula (30.18), compute the corresponding values of the quantity m. Having substituted the values obtained in formula (30.17), it is easy to calculate the corresponding values of f, which can be used directly to compute with the help of the (30.6) and (30.15) the corresponding values of p_1 and σ_1. According to (30.16) one can determine the shear angle of the plate contour and consequently also the effective shear modulus of the plate:

$$G_э = \tau/\gamma.$$

We shall now show how one solves the problem for the case when the longitudinal edges of the plate are so clamped, that the distance between them cannot change. In that case the quantity p_2 will be determined by means of the formula (30.14), from the condition that Δ_2 is zero:

$$p_2 = \nu p_1 + Et\frac{\pi^2 f^2}{8b^2}\left(1 + \frac{m^2 b^2}{l^2}\right).$$

Introducing this expression in (30.7) and taking account of (30.4), we shall obtain the equation

$$\frac{p^*}{4}(1-\beta^4) + \nu p_1 + Et\frac{\pi^2 f^2}{16b^2}\left[2 + \frac{2m^2\beta^2}{(1+m^2)} + \frac{1}{(1+m^2)^2}\right] = 0. \qquad (30.19)$$

Thus, we have the system of equations (30.6), (30.19), and (30.8).

Also in that case for a fixed value of τ one can, by taking a series of values of β, construct curves of the dependence of the quantities p_1, p_1, and f. For this one must first eliminate the quantities p_1 and f^2 from the indicated system of equations, which is not hard to do, as they enter in the equations linearly, and then

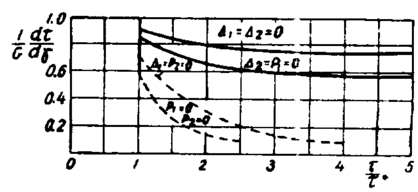

Figure 17

to solve the equation obtained for m for a series of values of β. Inasfar as the latter turns out to be complicated, it can be solved only approximately, for example, by graphical methods.

In Figure 17 are shown the dependence curves of the effective tangential shear modulus $\frac{1}{G}\cdot\frac{d\tau}{d\gamma}$ for the value τ/τ^*, where $\tau^* = \sqrt{?}\,p^*$. The upper curve relates to the case when there are ribs which hinder the mutual approach of the transverse, as well as the longitudinal edges of the plates:

$$\Delta_1 = \Delta_2 = 0.$$

Here it is assumed that the ribs do not hinder the shear in the plate contour. Below it are situated analogous curves for the cases 1) when the ribs hinder the mutual approaching of only the longitudinal edges of the plate; $\Delta_2 = 0,\ p_1 = 0$; 2) when the links hinder the mutual approach of the transverse edges only; $\Delta_1 = 0,\ p_2 = 0$; 3) for the case when both pairs of outside edges can freely approach each other; p = p = 0.

§ 31. Determination of Large Deflections of a Plate of Finite Dimensions

The solution, obtained in Section 29, of the problem of the deflections of an infinitely large plate supported on a network of rigid ribs along which the plate can slide, will be applied to the solution of some problems of large deflections of plates of finite dimensions. For this we shall turn our attention to the central strip of such a plate, whose edges are defined by the coordinates

$$\xi = \pm \pi/\delta, \quad \eta = \pm \pi.$$

As has been noted above, after the deformation, the edges of that strip remain straight. Besides, by using (29.18), it is easy to show that at the strip edges the shearing stresses are zero.

Thus, the solution found by us for the infinite plate represents the deflections of a finite plate of length $2\pi/\delta$ and width 2π, under the action of such loadings upon its edges that the following conditions are satisfied:

1) at the edges the deflections and the bending moments are zero;
2) at the edges the shearing stresses are zero;
3) the plate edges remain straight after bending.

Further, we shall consider the problem of Section 29 in a changed form, namely, we shall assume a rigid clamping of the plate to the longitudinal ribs. There, for simplicity, we consider that the plate can, just as before, slide freely along the ribs, but cannot separate from them. We shall further assume that the ribs are absolutely rigid with respect to flexure in the plane perpendicular to the plate, and their torsional rigidity is very small. The solution of this problem was given by G. G. Rostovtsev /VII.7/. At the strip boundaries, in our case, the following conditions are satisfied:

$$w = 0 \text{ for } \xi = \pm \pi/\delta \text{ and for } \eta = \pm \pi. \tag{31.1}$$

It is natural to assume that the deflection is antisymmetrical with respect to the ribs and that therefore the ribs remain straight after the deformation. From symmetry considerations one can conclude that the deforming membrane stresses near the transverse ribs are zero:

$$T_{12} = -\frac{\partial^2 \Phi}{\partial \xi \partial \eta} = 0 \text{ for } \xi = \pm \pi/\delta. \tag{31.2}$$

The condition that the curvature of the strip edges after the deformation be zero is the following:

$$\frac{\partial^2 u^1}{\partial \eta^2} = 0 \text{ for } \xi = \pm \pi/\delta. \tag{31.3}$$

Taking into consideration the relations (29.30) we shall obtain

$$(2+\nu)\frac{\partial^3 \Phi}{\partial \xi \partial \eta^2} + \frac{\partial^3 \Phi}{\partial \xi^3} + \frac{\partial w}{\partial \xi} \cdot \frac{\partial^2 w}{\partial \eta^2} = 0 \text{ for } \xi = \pm \pi/\delta. \tag{31.4}$$

In an analogous way one can obtain the linearity condition for the longitudinal edges of the strip:

$$(2+\nu)\frac{\partial^3\Phi}{\partial\xi^2\partial\eta}+\frac{\partial^3\Phi}{\partial\eta^3}+\frac{\partial w}{\partial\eta}\cdot\frac{\partial^2 w}{\partial\xi^2}=0 \text{ for } \eta=\pm\pi. \tag{31.5}$$

Thus, the solution of the problem reduces to the determination, under the boundary conditions indicated above, of the deflections of one strip clamped to two longitudinal ribs with cross-sectional area F*, equal to half the cross-sectional area of the ribs supporting the plate, since each rib of the plate simulatenously supports two neighboring strips.

We shall consider a strip of length a, width b, and thickness t, bordered by two stringers of area F*. As the longitudinal ribs are rigidly clamped to the strip edges, their extensions ε_{st} and the extensions of the strip edges adjoining them should be the same:

$$\varepsilon_{st}=\varepsilon_1=\frac{1}{Et}\left(\frac{\partial^2\psi}{\partial y^2}-\nu\frac{\partial^2\psi}{\partial x^2}\right) \text{ for } y=\pm b/2. \tag{31.6}$$

The stress in the stringer P_{st} (x) varies along its length as a result of the action of the plate upon it according to the following law:

$$P_{st}(x)=P_{st}(0)+\int_0^x \tau\Big|_{y=-b/2}dx=P_{st}(0)-\\ -\int_0^x \frac{\partial^2\psi}{\partial x\partial y}\Big|_{y=-b/2}dx. \tag{31.7}$$

Here P_{st} (0) is the stress in the longitudinal rib at the point x = 0. Utilizing this, we can write formula (31.6) in the following form:

$$\varepsilon_{st}=\frac{1}{EF^*}\left[P_{st}(0)-\int_0^x \frac{\partial^2\psi}{\partial x\partial y}\Big|_{y=-b/2}dx\right]=\\ =\frac{1}{Et}\left(\frac{\partial^2\psi}{\partial y^2}-\nu\frac{\partial^2\psi}{\partial x^2}\right)\Big|_{y=-b/2} \tag{31.8}$$

In particular, setting x = 0, we have

$$P_{st}(0)=\left(\frac{\partial^2\psi}{\partial y^2}-\nu\frac{\partial^2\psi}{\partial x^2}\right)\frac{F^*}{t}\Big|_{x=0,\ y=-b/2}. \tag{31.9}$$

Differentiating (31.8) with respect to x, we find

$$-\frac{1}{EF^*}\cdot\frac{\partial^2\psi}{\partial x\partial y}=\frac{1}{Et}\left(\frac{\partial^3\psi}{\partial x\partial y^2}-\nu\frac{\partial^3\psi}{\partial x^3}\right) \text{ for } y=-b/2. \tag{31.10}$$

Carrying out the change of variables (29.7), we easily obtain

$$P_{st}(\xi)=P_{st}(0)-\int_0^x \frac{Et^3}{b}\cdot 2\pi\frac{\partial^2\Phi}{\partial x\partial y}dx,\\ P_{st}(0)=Et^2\left(\frac{2\pi}{b}\right)^2F^*\left(\frac{\partial^2\Phi}{\partial\eta^2}-\nu\frac{\partial^2\Phi}{\partial x^2}\right),\\ -\frac{bt}{2\pi F^*}\cdot\frac{\partial^2\Phi}{\partial\xi\partial\eta}=\frac{\partial^3\Phi}{\partial\xi\partial\eta^2}-\nu\frac{\partial^3\Phi}{\partial\xi^3} \text{ for } \eta=-\pi. \tag{31.11}$$

For a plate with the dimensions $a=2\pi/\delta,\ b=2\pi$ and thickness t = 1, clamped to ribs of cross-sectional area F, the preceding formula has the form

$$-\frac{1}{F}\cdot\frac{\partial^4\Phi}{\partial\xi\partial\eta}=\frac{\partial^4\Phi}{\partial\xi\partial\eta^3}-\nu\frac{\partial^4\Phi}{\partial\xi^4}\text{ for }\eta=-\pi. \tag{31.12}$$

Comparing this equation with the preceding, it is easy to notice that the problem of determining the deflections of a strip with dimensions a and b, clamped to ribs of area F*, reduces directly to the similar problem for a plate with dimensions $2\pi/\delta$ and 2π, clamped to ribs with cross-sectional area F, defined by the relations

$$\frac{1}{F}=\frac{bt}{2\pi F^*}.$$

We turn to the consideration of a strip with the dimensions

$$a=2\pi/\delta,\quad b=2\pi,\quad t=1,\quad E=1. \tag{31.13}$$

In that case, $x=\xi$, $\eta=y$, $\Phi=\psi$.

Analogously to the above, one derives the boundary condition which has to be satisfied near the second stringer:

$$\frac{1}{F}\cdot\frac{\partial^4\Phi}{\partial x\partial y}=\frac{\partial^4\Phi}{\partial x\partial y^3}-\nu\frac{\partial^4\Phi}{\partial x^3}\text{ for }\eta=\pi. \tag{31.14}$$

As before, we shall look for an approximation to the deflection function in the form (29.15).

We shall take the expression (29.18) as the particular integral Φ_1, of the compatibility equation (29.16).

The solution of the compatibility equation satisfying the boundary conditions will be looked for in the form (29.20) where ψ_0 is the solution of the homogenous equation (29.19):

$$\Phi_0=\sum_{m=0}^{2M} f_m(y)\cos m\delta x+\frac{p_2x^2}{2}+\frac{p_1y^2}{2}. \tag{31.15}$$

Introducing this expression in the left-hand member of equation (29.19) and equating the coefficients of $\cos m\delta x$ to zero, we shall obtain the equations which must be satisfied by the functions f_m:

$$(m\delta)^4 f_m(y)-2(m\delta)^2\frac{d^2f_m}{dy^2}+\frac{d^4f_m}{dy^4}=0,\ m=1,2,\ldots. \tag{31.16}$$

In view of the symmetry of the deformations, w is an even function. It is natural to expect that the membrane stresses T_1 and T_2, and consequently also the strain functions will be even functions of the coordinates. We shall therefore consider only the even solutions of the equation (31.16):

$$f_m(y)=C_{1m}\operatorname{ch}(m\delta y)+C_{2m}\,y\operatorname{sh}(m\delta y). \tag{31.17}$$

Taking into consideration (29.20), (31.15), and (29.18), we shall obtain

$$\Phi = \sum_{m=0}^{2M}\sum_{n=0}^{2N} \frac{D_{mn}}{(m^2\delta^2+n^2)^2} \cos m\delta x \cos ny + \\ + \sum_{m=1}^{2M} \cos m\delta x \left[C_{1m} \operatorname{ch}(m\delta y) + C_{2m} y \operatorname{sh}(m\delta y)\right] + \\ + \frac{p_1 y^2}{2} + \frac{p_2 x^2}{2}. \qquad (31.18)$$

Introducing the expressions (29.15) and (31.18) in the boundary conditions (31.5), we shall obtain the equation in which it is necessary that the coefficients of $\cos m\delta x$ $(m = 1, 2, \ldots, 2M)$ be equal to zero in order that it be satisfied for all values of x.

$$- C_{1m}(m\delta)^2(1+\nu)\operatorname{sh} m\delta\pi - C_{2m}\left[(m\delta)^2\pi(1+\nu)\operatorname{ch} m\delta\pi - \right. \\ \left. - (m\delta)^2(1-\nu)\operatorname{sh} m\delta\pi\right] = 0 \quad (m = 1, \ldots, 2M). \qquad (31.19)$$

One more equation for the coefficients is obtained if one introduces (31.18) in the equation (31.14) and equates the coefficients of the functions $\sin m\delta x$ in the expressions obtained

$$\frac{1}{F}\left[\sum_{n=1}^{2N} \frac{D_{mn}\, m\delta n(-1)^{n+1}}{(m\delta^2+n^2)^2} - C_{1m}(m\delta)^2 \operatorname{sh} m\delta\pi - \right. \\ \left. - C_{2m}(m\delta \operatorname{sh} m\delta\pi + m^2\delta^2\pi \operatorname{ch} m\delta\pi)\right] = \qquad (31.20) \\ = - C_{1m} m^2\delta^2(1+\nu)\operatorname{ch} m\delta\pi + C_{2m}\left[-(m\delta)^2(1+\nu)\pi \operatorname{sh} m\delta\pi - \right. \\ \left. - (m\delta)^2 \cdot 2 \operatorname{ch} m\delta\pi\right] \quad (m = 1, 2, \ldots, 2M).$$

Solving the system of equations (31.19) and (31.20) with respect to the two unknowns C_{1m} and C_{2m}, G. G. Rostovtsev obtained explicit expressions for these quantities in terms of the deflection parameters A_{mn} and the quantities p_1 and p_2.

In order to clarify the physical meaning of the quantity p_2 entering in equation (31.18) we shall compute the mean value of the stresses T_2, acting per unit cross-section of the strip, parallel to the x axis:

$$T_{2c} = \frac{1}{2\pi/\delta}\int_{-\pi/\delta}^{\pi/\delta} T_2 dx = \frac{1}{2\pi/\delta}\int_{-\pi/\delta}^{\pi/\delta} \frac{\partial^2\Phi}{\partial x^2} dx = p_2. \qquad (31.21)$$

Thus, p_2 represents the mean value of the stress T_2. It is the same for all the cross-sections parallel to the x axis. In an analogous way one calculates the mean value of the stress T_1, acting per unit length of the cross-section parallel to the y axis:

$$T_{1c} = \frac{1}{2\pi}\int_{-\pi}^{\pi} T_1 dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2\Phi}{\partial y^2} dy = \\ = p_1 + \sum_{m=1}^{2M} \cos m\delta x \left[C_{1m} \cdot 2m\delta \operatorname{ch} m\delta\pi + \right. \\ \left. + C_{2m}(2 \operatorname{sh} m\delta\pi + 2m\delta\pi \operatorname{ch} m\delta\pi)\right].$$

Hence it follows that the mean stress in the cross-section depends on its position. This is natural, since a part of the longitudinal load is taken by the longitudinal ribs, and that fraction varies for different points of the rib. Setting $x = \pi/\delta$ in the preceding formula, we shall find an expression for the quantity p in terms of T_{1c}:

$$p_1 = T_{1c}\Big|_{x=\pi\beta} - \sum_{m=1}^{2M} (-1)^m [C_{1m}\cdot 2m\delta \operatorname{ch} m\delta\pi + \\ + C_{2m}(2\operatorname{sh} m\delta\pi + 2m\delta\pi\cdot\operatorname{ch} m\delta\pi)]. \tag{31.22}$$

The preceding formulas allow one to express all the quantities, defining the strained state of the strip, in terms of the deflection parameters A_{mn} and the quantities p_1 and p_2.

The formulas (31.11) allow one to calculate the stresses in the longitudinal ribs and the elastic energy accumulated in them:

$$Э_{\text{л}} = \int \frac{1}{2EF} P^2(x)\,dx. \tag{31.23}$$

One determines the mutual approaches Δ_1 and Δ_2 of the transverse and longitudinal edges of the strips from equations similar to (29.12), after which one determines the work of the stresses p_1 and p_2, acting on the longitudinal and transverse edges of the plate:

$$W = -p_1\Delta_1 - p_2\Delta_2. \tag{31.24}$$

In view of the fact that the calculations are elementary and the results obtained are cumbersome, we shall not give them here. The results of actual calculations by the Bubnov-Galerkin method are given in the article /VII.7/. Further, applying the energy method one can derive a system of equations for the determination of strip deflection under given loads applied to its edges. The parameter p_2 is equal to the magnitude of the transverse loads divided by the strip length, and is therefore a known quantity. The values of A_{mn} and p_1 can be determined from the minimality condition of the sum of the elastic energy of the strips, ribs, and the potential energy of the loads. This latter, up to a constant, is equal to the work of the external loads, taken with opposite sign, and therefore the parameters A_{mn} and p_1 can be determined from the system of non-linear equations:

$$\frac{\partial(Э + Э_{\text{л}} - W)}{\partial A_{mn}} = 0, \quad \frac{\partial(Э + Э_{\text{л}} - W)}{\partial p_1} = 0. \tag{31.25}$$

Here, to a stable equilibrium state correspond those solutions for which the determinants and the principal minors of the matrix of the second derivatives of the quantity $(Э + Э_{\text{л}} - W)$ with respect to A_{mn} and p_1 are positive.

In some problems of practical importance, instead of the value of the transverse load on the plate p_2, the magnitude of the mutual approach of the strip edges Δ_2 is given. Such a case is encountered, for example, when the longitudinal ribs are very rigid with respect to flexure in the plane of the plates and are clamped to the transverse ribs, preventing their mutual approach. In that case $\Delta_2 = 0$.

The solution of such a problem reduces again to the solution of the system of equations (31.25), to which is added one more equation

$$\Delta_2 = 0, \tag{31.26}$$

connecting the quantities p_1, p_2, and A_{mn}.

In article /VII.7/ are given the results of the calculation of strip deflections in the first approximation, obtained on the assumption that in the expansion (29.15) M = N = 1.

There, computations have shown that the mean stresses in the strip $T_{1c}(x)$ are almost the same along its whole length: the deviation of the quantity $T_{1c}(x)$ from its maximum value in all cases does not exceed 6%. Therefore, in the case of a strip rigidly clamped to the longitudinal ribs, it makes sense to introduce the concept of the reduction coefficient

$$\varphi = T_{1c}(x)/p_{10},$$

where p_{10} is the magnitude of that strain which would appear in the plate in the absence of buckling. Here, other conditions being the same, the reduction coefficients in plates with longitudinal edges which slip along the ribs and in plates with edges clamped to the longitudinal ribs, turn out to be almost identical. The solution of the problem of determining the deflections of a strip clamped to the longitudinal ribs was published in paper /VII.7/ for the case when the strip is clamped to the longitudinal and the transverse ribs. In order to satisfy the boundary conditions in that case one has to add to the solution of the compatibility equation (29.8), taken in the form (31.18), other terms of the form

$$\sum_{n=1}^{2N} \cos ny\,[d_{1n}\operatorname{ch} nx + d_{2n}x \operatorname{sh} nx] = \Phi_3, \tag{31.27}$$

where

$$\Delta^2\Phi_3 = 0.$$

In order to determine the constants $c_{1m}, \ldots, d_{2n}$ one has to make use of the following conditions:

1. the linearity condition of the longitudinal edges of the strips, characterized by the equations (31.19), and the analogous linearity condition for the transverse edges, obtained from (31.19) by replacing the quantities $(m\delta)$, c_{1m}, and c_{2m} by the quantities n, d_{1n}, and d_{2n};

2. the condition of clamping the plate to the longitudinal ribs of the form (31.12) and analogous conditions for the transverse ribs. Here it turns out that with the help of a finite number of parameters $c_{1m}, \ldots, d_{2n}$ it is impossible to arrive at satisfying these conditions at all points of the strip edge. Therefore, F. F. Rostovtsev proposed to determine the quantities $c_{1m}, \ldots, d_{2n}$ from the condition that the expression

$$\int_{-\pi/\delta}^{\pi/\delta} \left(\frac{\partial^3\Phi}{\partial x\partial y^2} - \nu\frac{\partial^3\Phi}{\partial x^3} + \frac{1}{F}\cdot\frac{\partial^2\Phi}{\partial x\partial y}\right)\Big|_{y=-\pi} \cos m\delta x = 0 \quad (m = 1, \ldots, 2M)$$

be zero, and from analogous equations for the transverse edges of the strip. After introducing the expression (31.18) in these formulas and integrating, one obtains linear equations for the quantities $c_{1m}, \ldots, d_{2n}$. All the preceding conditions give a system of equations whose solution allows one to express the quantities $c_{1m}, \ldots, d_{2n}$, and consequently also the stress function Φ in terms of the parameters A_{mn}, p_1, and p_2.

The further solution of the problem can be carried out by the method indicated in the preceding paragraph.

In Figure 18 are given curves of the deflections of the longitudinal axis of symmetry of a square pluralumin plate (of dimensions $40 \times 40 \times 0.05$ cm), freely supported at the edges and subject to a transverse pressure q. The plate was supported in such a way that its longitudinal edges could not approach each other: the tangential stresses at these edges were zero. The transverse edges of the plate were rigidly clamped to the transverse duralumin ribs of 2.05 cm² cross-section. The solid lines represent the theoretical data, obtained by solving the problem in the first approximation. The dash-and-dot lines represent the theoretical solutions of the problem in the second approximation, when the deflection function was sought in the form

$$\frac{w^*}{t} = w = A_{11}\cos\frac{x}{2}\cos\frac{y}{2} + A_{12}\cos\frac{3x}{2}\cos\frac{y}{2} + $$
$$+ A_{21}\cos\frac{x}{2}\cos\frac{3y}{2}.$$

The dashed lines represent experimental data.

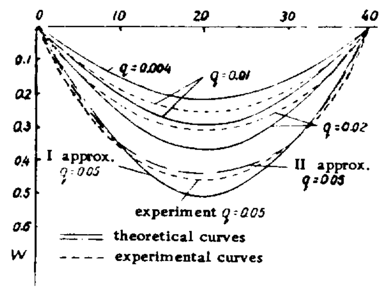

Figure 18

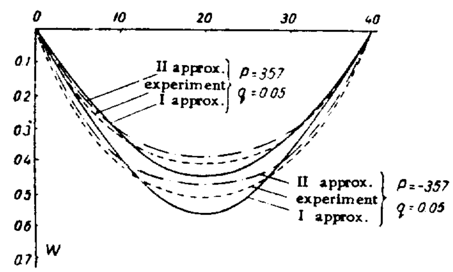

Figure 19

In Figure 19 are given the deflections of the same plate which, besides the transverse pressure q, is also under either a longitudinal compression P = −357 kg or a longitudinal tensile stress with a value of P = 357 kg*. In the paper /VII.7/ the problem of determining the deflections of a strip has also been solved, for the case where before the start of the loading the strip had an initial bending sufficiently well described by the function

$$w^0 = \sum_{m=1}^{M'}\sum_{n=1}^{N'} A_{0mn}\cos\delta\left(m-\frac{1}{2}\right)\xi\cos\left(n-\frac{1}{2}\right)\eta. \tag{31.28}$$

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* These graphs were taken from the work of G. G. Rostovtsev /VII.7/.

It should be noted that the initial bending, expressed by this function, has certain specific properties:

1. the initial bending is zero at the strip edges;
2. at the strip edges the curvatures of the initial bending are zero.

The equation of compatibility of deformations in the presence of an initial bending has, according to (26.11), the form

$$\Delta^2\Phi_1 = \left\{\left[\frac{\partial^2}{\partial\xi\partial\eta}(w + w_0)\right]^2 - \frac{\partial^2}{\partial\xi^2}(w + w_0)\cdot\frac{\partial^2}{\partial\eta^2}(w + w_0)\right\} - \\ -\left[\left(\frac{\partial^2 w_0}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 w_0}{\partial\xi^2}\cdot\frac{\partial^2 w_0}{\partial\eta^2}\right], \tag{31.29}$$

where $w = w^1/t$, $w_0 = w^0/t$.

Here the change of the stress function during the deflection of the plate has been denoted by $t\,\Phi_1$. If by $t\,\Phi_0$ one denotes the value of the stress function characterizing the initial membrane stresses, then the stress function after the deflection of the plate will be

$$\Phi = \Phi_0 + \Phi_1. \tag{31.30}$$

Since the plate has zero deflection at the edges, the equation (31.4), characterizing the linearity of the strip edges, is also entirely valid at the edges of the plate. It is easy to convince oneself that the conditions of clamping the plate to the ribs (31.11) also remain the same.

As equation (31.29) is linear in Φ, and the boundary conditions of linearity of the strip edges and the conditions of clamping of the strips to the ribs are homogeneous, the solution of the equation may be written in the form

$$\Phi_1 = \Phi_1' - \Phi_1'',$$

where Φ_1' and Φ_1'' are solutions of the equations

$$\Delta^2\Phi_1' = \left[\frac{\partial^2}{\partial\xi\partial\eta}(w + w_0)\right]^2 - \frac{\partial^2}{\partial\xi^2}(w + w_0)\frac{\partial^2}{\partial\eta^2}(w + w_0),$$
$$\Delta^2\Phi_1'' = \left(\frac{\partial^2 w_0}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 w_0}{\partial\xi^2}\cdot\frac{\partial^2 w_0}{\partial\eta^2}.$$

satisfying the above-mentioned homogeneous conditions.

The solutions of these equations are similar to the solution of the corresponding equation for an absolutely flat strip; they are obtained from it by the simple replacement of the quantity A_{mn} by the quantities ($A_{mn} + A_{0mn}$) and A_{0mn}.

The further course of solving the problem is the same as that of solving the problem of the flat plate. We shall not dwell here on the determination of the deflection of a plate, whose middle part is supported by flexible ribs, or on consideration of the case when the plate edges are not loaded. These questions are elucidated in the works /VII.7/ and /VII.8/.

Chapter VIII

SOME METHODS FOR THE SOLUTION OF PROBLEMS IN THE THEORY OF BENDING OF CIRCULAR PLATES

§ 32. Fundamental Relations of the Theory of Symmetrical Deformation of Slightly Bent Circular Plates. Application of the Method of Power Series

We shall specify the position of a point on the middle surface of a circular plate σ by polar coordinates r and θ, where the origin is the center of the plate. Let the plate in the general case have an initial bending w^0, symmetrical about the center, of the order of the plate thickness t. The length of a line element of σ before the deformation is given by

$$ds^2 = dr^2 + r^2 d\theta^2.$$

Comparing that expression with (25.4) we see that in the given case one should set the following in the formula of § 25:

$$\alpha = r, \ \beta = \theta, \ B = r. \qquad (32.1)$$

where the reference surface is a plane, i.e., $k_1 = k_2 = 0$. We shall assume that the stress applied to the plate and the boundary conditions are symmetrical with respect to the center of the plate. Then all the quantities defining the deformed and the stressed state of the plate will depend only on the coordinate r.

Our statement will hold if the stress in the plate is not very great; in the opposite case, under certain conditions an unsymmetrical buckling of the plate edge can appear /VIII. 9/.

According to (25.32) we have expressions for the radial stress T_1 and the annular stress T_2 in terms of the stress function ψ:

$$rT_1 = \frac{d\psi}{dr}, \ T_2 = \frac{d^2\psi}{dr^2}. \qquad (32.2)$$

Here, by virtue of symmetry, the shearing stress T_{12} is zero. According to (25.9) the changes in curvature are

$$\varkappa_1 = -\frac{d^2w}{dr^2}, \ \varkappa_2 = -\frac{1}{r}\cdot\frac{dw}{dr}, \ \varkappa_{12} = 0. \qquad (32.3)$$

The initial curvatures of the plate can be calculated from the preceding formulas by replacing w by w^0 in them:

$$\varkappa_1{}^0 = -\frac{d^2w^0}{dr^2}, \ \varkappa_2{}^0 = -\frac{1}{r}\cdot\frac{dw^0}{dr}, \ \varkappa_{12}^0 = 0. \qquad (32.4)$$

According to (32.1) and (25.13), the Laplacian operator has the form

$$\Delta(\ldots) = \frac{1}{r}\cdot\frac{d}{dr}\left[r\,\frac{d}{dr}(\ldots)\right]. \qquad (32.5)$$

The equation of equilibrium (25.12) can, by using (32.2)-(32.5), be written in the form

$$\frac{Et^3}{12(1-\nu^2)}\cdot\frac{1}{r}\cdot\frac{d}{dr}\left[r\frac{d}{dr}(\Delta w)\right]- \\ -\frac{1}{r}\cdot\frac{d}{dr}\left[\frac{d\psi}{dr}\cdot\frac{d}{dr}(w+w^0)\right]-\tilde{p}=0, \tag{32.6}$$

where $\tilde{p} = -p > 0$, if the transverse stress is oriented in the positive direction of the bending.

The compatibility condition (25.33) reduces to the equation

$$\frac{1}{r}\cdot\frac{d}{dr}\left[r\frac{d}{dr}(\Delta\psi)\right]+\frac{Et}{2r}\cdot\frac{d}{dr}\left[\left(\frac{dw}{dr}\right)^2+2\frac{dw}{dr}\cdot\frac{dw^0}{dr}\right]=0. \tag{32.7}$$

According to (25.10) the radial bending moment and the circumferential elongation are defined by the formulas

$$M_1=-D\left(\frac{d^2w}{dr^2}+\frac{\nu}{2}\cdot\frac{dw}{dr}\right),\quad \varepsilon_2=\frac{1}{Et}(T_2-\nu T_1)= \\ =\frac{1}{Et}\left(\frac{d^2\psi}{dr^2}-\frac{\nu}{r}\cdot\frac{d\psi}{dr}\right). \tag{32.8}$$

On the other hand, according to (25.8), $\varepsilon_2 = u/r$ where u is the radial component of displacement. Consequently,

$$u=\frac{r}{Et}\left(\frac{d^2\psi}{dr^2}-\frac{\nu}{r}\cdot\frac{d\psi}{dr}\right). \tag{32.9}$$

When the plate edge r = a is rigidly clamped, then the boundary conditions

$$w=0,\quad u=0=\frac{d^2\psi}{dr^2}-\frac{\nu}{r}\cdot\frac{d\psi}{dr}\quad \text{for } r=a. \tag{32.10}$$

should hold. When the plate edge can displace itself freely in the radial direction and an outside radial stress p_1 is applied to it, the boundary condition for the function ψ has the form

$$T_1=p_1 \quad\text{or}\quad \frac{1}{r}\cdot\frac{d\psi}{dr}=p_1 \quad \text{for } r=a. \tag{32.11}$$

Introducing the dimensionless quantities

$$w_*=w/t,\quad w_*^0=w^0/t,\quad \rho=\frac{r}{a}, \tag{32.12}$$

$$\psi_0=\psi/Et^3, \tag{32.13}$$

and also the notations

$$p_{1*}=\frac{T_1}{Et}=\frac{1}{Etr}\,\frac{d\psi}{dr},\quad q=-\frac{a}{r}\cdot\frac{d\psi}{d\rho},\quad \eta^2=\frac{t^2}{12a^2(1-\nu^2)}, \tag{32.14}$$

we write the equations (32.6), (32.7) as follows:

$$\frac{\eta^2}{\rho}\cdot\frac{d}{d\rho}\left\{\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\cdot\frac{d}{d\rho}(\rho^2 q)\right]\right\}-$$
$$-\frac{1}{\rho}\cdot\frac{d}{d\rho}[\rho^2 p_{1*}(q+q^0)]+\frac{\tilde{p}a}{Et}=0, \quad (32.15)$$
$$\frac{1}{\rho}\cdot\frac{d}{d\rho}\left\{\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\cdot\frac{d}{d\rho}(\rho^2 p_{1*})\right]\right\}+$$
$$+\frac{1}{2}\cdot\frac{1}{\rho}\cdot\frac{d}{d\rho}[\rho^2(q^2+2qq^0)]=0. \quad (32.16)$$

By studying these relations, the reader can convince himself of the fact that the similarity theorems, formulated in § 29 for a rectangular plate, hold also in the case of circular plates having the same initial relative deflections w_*^0 and the same Poisson's ratio ν. By taking this into account, one could, in what follows, limit oneself to the investigation of plates of unit radius, unit thickness, and unit modulus of elasticity of the material.

Multiplying (32.15) by ρ and integrating, and then dividing by ρ, we shall obtain

$$\eta^2\frac{d}{d\rho}\left[\frac{1}{\rho}\cdot\frac{d}{d\rho}(\rho^2 q)\right]-p_{1*}(q+q^0)\rho+$$
$$+\frac{1}{\rho}\int_0^\rho\frac{pa}{Et}\cdot\rho d\rho+\frac{c_1}{\rho}=0. \quad (32.17)$$

In order that the second derivatives of q with respect to ρ, characterizing the shearing stresses, be bounded for $\rho=0$, the constant of integration c_1 has to be set equal to zero. Making use of the identity

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\cdot\frac{d}{d\rho}(\rho^2 q)\right]=\frac{1}{\rho}\cdot 3\frac{dq}{d\rho}+\rho\frac{d^2q}{d\rho^2}=\frac{1}{\rho^2}\cdot\frac{d}{d\rho}\left(\rho^3\frac{dq}{d\rho}\right),$$

we bring the equation (32. 17) to the form

$$\frac{\eta^2}{\rho^3}\cdot\frac{d}{d\rho}\left(\rho^3\frac{dq}{d\rho}\right)-p_{1*}(q+q^0)+\frac{1}{\rho^2}\int_0^\rho\frac{pa}{Et}\rho d\rho=0. \quad (32.18)$$

By carrying out analogous transformations, from equation (32.16) we shall obtain

$$\frac{1}{\rho^3}\cdot\frac{d}{d\rho}\left(\rho^3\frac{dp_{1*}}{d\rho}\right)+\frac{1}{2}(q^2+2qq^0)=0. \quad (32.19)$$

We shall consider a plate which is under the action of a compressive edge stress ($T_1 < 0$) in the plane of the plate contour.

On the plate contour the condition (32.11) should be satisfied. Besides, the bending and the bending moment should be zero. Taking into account (32. 8) and the notations (32. 12) and (32. 14), the last of the boundary conditions may be written in the form

$$\frac{dq}{d\rho}+(1+\nu)q=0 \quad \text{for} \quad \rho=1. \quad (32.20)$$

As p_{1*} and q, by virtue of symmetry, have to be even functions of Cartesian coordinates x, y, then

C -3

$$\frac{dp_{1*}}{dr} = \frac{\partial p_{1*}}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial p_{1*}}{\partial y} \cdot \frac{\partial y}{\partial r} = 0 \quad \text{for } r = 0.$$

Consequently,

$$\frac{dp_{1*}}{dr} = \frac{dq}{dr} = 0 \quad \text{for } r = 0. \tag{32.21}$$

Friedrichs and Stoker proposed /VIII.6/ the following method of solving the problem under consideration in the case when $w^0 = 0$ and $\widetilde{p} = 0$.

Let

$$\begin{gathered} \rho = \alpha/A \quad (0 \leqslant \alpha \leqslant A), \\ p_{1*} = A^2\eta^2\varkappa, \quad q = A^2\eta k, \end{gathered} \tag{32.22}$$

where A is still an arbitrary number.

Then the equations (32.18) and (32.19) and the conditions (32.20), (32.21), imposed upon the function q, take the forms

$$\frac{1}{\alpha^3} \cdot \frac{d}{d\alpha}\left(\alpha^3 \frac{dk}{d\alpha}\right) = \varkappa k, \quad \frac{1}{\alpha^3} \cdot \frac{d}{d\alpha}\left(\alpha^3 \frac{d\varkappa}{d\alpha}\right) = -\frac{1}{2} k^2, \tag{32.23}$$

$$A\frac{dk}{d\alpha} + (1+\nu)k = 0 \quad \text{for } \alpha = A. \tag{32.24}$$

$$\frac{dk}{d\alpha} = \frac{d\varkappa}{d\alpha} = 0 \qquad \text{for } \alpha = 0. \tag{32.25}$$

We shall seek the solution of the problem in the form of power series, satisfying the conditions (32.25), setting

$$\varkappa = \sum_{s=0}^{\infty} \varkappa_s \alpha^{2s}, \quad k = \sum_{s=0}^{\infty} k_s \alpha^{2s}. \tag{32.26}$$

Here the equations (32.23) are satisfied, if the coefficients of the power series are connected by the relations

$$2s(2s+2)k_s = \sum_{m+n=s-1} \varkappa_m k_n, \tag{32.27}$$

$$2s(2s+2)\varkappa_s = -\frac{1}{2} \sum_{m+n=s-1} k_m k_n. \tag{32.28}$$

We take any values of the quantities $\varkappa_0$ and k_0 and determine the following coefficients of the series according to (32.27), (32.28). Introducing (32.26) in (32.24), we obtain the equation

$$\sum_{s=0}^{\infty} (2s+1+\nu) k_s A^{2s} = 0. \tag{32.29}$$

Solving this equation by one of the approximate methods, we determine the parameter A, which had remained free so far. Thereupon we determine the radial stress at the plate edge according to (32.26) and (32.22). Thus, taking various values of k_0, one can find solutions corresponding to different values of the parameter of edge stress.

In article /VIII.7/ it has been shown that if it is desired to obtain a solution of the problem, which, under decrease in stress, transforms continuously into the solution corresponding to the smallest of the critical stresses, then for A one has to choose the smallest positive root of the equation (32. 29).

The method described above can be applied almost without change to the calculation for a plate elastically fixed at the edges. Bodner /VIII. 7/ has carried out the corresponding calculations for a plate clamped at the edges. The results of the calculations have shown that under large compressive stresses along the edges, a region of positive membrane stresses appears inside the plate where the plate is under tension.

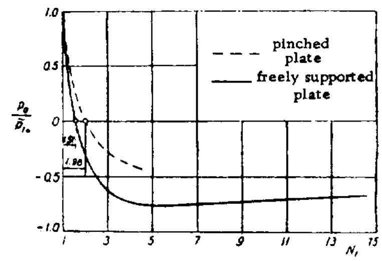

Figure 20

Figure 20 shows the dependence of the ratio of the radial stress at the center to the edge stress, and of the ratio of the compressive stress at the edge to the first critical value of that stress.

§ 33. The Method of Successive Approximations and the Method of Small Parameters

In many cases in the solution of problems of deflection of a circular plate it is convenient to use the substitutions

$$w = v \cdot t,\ w^0 = v^0 t,\ T_1 = \frac{1}{r}\frac{d\psi}{dr} = \frac{Et^3}{a^2} S,$$
$$P = pa^4(1-\nu^2)/Et^4,\ \zeta = 1 - r^2/a^2. \tag{33.1}$$

After a single integration of the equations (32.6) and (32.7), carrying out the substitutions (33.1), and assuming that p = const., we obtain the equations of equilibrium for the plate in the following form

$$\frac{d^2}{d\zeta^2}\left[(1-\zeta)\frac{dv}{d\zeta}\right] = 3(1-\nu^2) S\left(\frac{dv}{d\zeta} + \frac{dv^0}{d\zeta}\right) - \frac{3}{4}P, \tag{33.2}$$

$$\frac{d^2}{d\zeta^2}[(1-\zeta)S] = L(v) \tag{33.3}$$

where we introduced the differential operator notation

$$L(v) = -\frac{1}{2}\left[\left(\frac{dv}{d\zeta}\right)^2 + 2\frac{dv}{d\zeta}\cdot\frac{dv^0}{d\zeta}\right]. \tag{33.4}$$

Now, when the plate is clamped at the contour, the deflection should satisfy the conditions

$$v = 0,\ dv/d\zeta = 0 \quad \text{at} \quad \zeta = 0. \tag{33.5}$$

In the case when the plate edges are clamped so that they cannot displace themselves in the plane of the contour, the boundary conditions (32.10) hold for the function S which, with the notations (33.1), are brought into the form

$$2dS/d\zeta - (1-\nu)S = 0 \quad \text{at} \quad \zeta = 0. \tag{33.6}$$

When the plate edges, while fulfilling the conditions (33.5), can slide freely, the static boundary condition

$$S = S_0 \quad \text{at} \quad \zeta = 0, \tag{33.7}$$

should hold, where S_0 is the given value on the contour of the quantity S. Besides, one has to set up the boundedness condition of the quantities $dv/d\zeta$ and S at the center of the plate, i.e., for $\zeta = 1$. Integrating equation (33.3) twice with respect to ζ, we obtain an expression for S

$$S(1-\zeta) = \int_0^\zeta \int_0^\tau L(v)\,dz\,d\tau + c_1 + c_2\zeta,$$

where c_1 and c_2 are constants, determined from the boundary condition (33.6) or (33.7) and from the boundedness condition for S at $\zeta = 1$, where, according to the Dirichlet formula

$$\int_0^\zeta \int_0^\tau L(v)\,dz\,d\tau = \int_0^\zeta (\zeta - \tau) L(v)\,d\tau.$$

For the case when the plate edges can slide, we find from (33. 7) that

$$S = c_1 = S_0.$$

From the boundedness condition for S at $\zeta = 1$ it follows that

$$0 = \int_0^1 (1 - \tau) L(v)\, d\tau + c_1 + c_2.$$

Determining c_1 and c_2 from the two preceding equations and introducing them in the expression for S, we obtain, with the help of the Dirichlet formula

$$S = \frac{1}{1-\zeta}\left[\int_0^\zeta (\zeta - \tau) L(v)\, d\tau - \zeta \int_0^1 (1 - \tau) L(v)\, d\tau\right] + S. \qquad (33.8)$$

Whence, substituting for S in the equation (33. 2), we obtain an integro-differential equation which has to be satisfied by the function v. In the absence of initial deflections it has the form

$$\begin{aligned} \frac{d^2}{d\zeta^2}\left[(1-\zeta)\frac{dv}{d\zeta}\right] = 3(1-\nu^2)\frac{dv}{d\zeta}\left\{\frac{1}{2}\left[\zeta \int_0^1 (1-\tau)\left(\frac{dv}{d\zeta}\right)^2_{\zeta=\tau} d\tau - \right.\right. \\ \left.\left. - \int_0^\zeta (\zeta-\tau)\left(\frac{dv}{d\zeta}\right)^2_{\zeta=\tau} d\tau\right]\frac{1}{1-\zeta} + S_0\right\} - \frac{3}{4}P. \end{aligned} \qquad (33.9)$$

We shall investigate the solution of this equation by the method of successive approximations. As the first approximation v_1 we shall take the solution of the equation

$$\frac{d^2}{d\zeta^2}\left[(1-\zeta)\frac{dv_1}{d\zeta}\right] = -\frac{3}{4}P, \qquad (33.10)$$

satisfying the boundary conditions

$$v_1(0) = \left(\frac{dv_1}{d\zeta}\right)_{\zeta=0} = 0, \quad \left|\frac{dv_1}{d\zeta}\right|_{\zeta=1} \neq \infty, \qquad (33.11)$$

We easily find that

$$v_1 = \frac{3}{16}P \cdot \zeta^2. \qquad (33.12)$$

If one substitutes this expression for v in (33. 9), then the difference between the right- and left-hand members of that equation will be the quantity Δ_1 of the neglected compononent. Figuratively speaking, one can say that the error of the first approximation is brought about by the unbalanced state of Δ_1:

$$\begin{aligned} \Delta_1 = 3(1-\nu^2)\frac{dv_1}{d\zeta}\left\{\left[\zeta \int_0^1 (1-\tau)\left(\frac{dv_1}{d\zeta}\right)^2_{\zeta=\tau} d\tau - \right.\right. \\ \left.\left. - \int_0^\zeta (\zeta-\tau)\left(\frac{dv_1}{d\zeta}\right)^2_{\zeta=\tau} d\tau\right]\frac{1}{2(1-\zeta)} + S_0\right\}. \end{aligned} \qquad (33.13)$$

To determine the correction δ_2 to the first approximation we shall find the supplementary deflection required by the unbalanced state of Δ_1. Here, in order to avoid the necessity of solving a non-linear equation, we shall neglect the influence of membrane stresses and determine δ_2 from an equation analogous to (33. 10)

$$\frac{d^2}{d\zeta^2}\left[(1-\zeta)\frac{d\delta_2}{d\zeta}\right]=\Delta_1 \qquad (33.14)$$

with the boundary conditions

$$(d\delta_2/d\zeta)_{\zeta=0}=0, \quad \left|\frac{d\delta_2}{d\zeta}\right|_{\zeta=1}<\infty.$$

Introducing (33. 12) in (33. 13) we obtain the second approximation from equation (33. 14) by a subsequent integration

$$v_2=v_1+\delta_2=P\left\{\frac{3}{16}\zeta^2-\frac{S_0(1-\nu^2)}{32}(3\zeta^2+2\zeta^3)-\right.$$
$$\left.-\frac{9(1-\nu^2)P^3}{81920}\left(5\zeta^2+3\frac{1}{3}\zeta^3+2{,}5\zeta^4+\zeta^5+\frac{1}{3}\zeta^6\right)\right\}. \qquad (33.15)$$

To obtain the correction to the second approximation one has to calculate the change Δ_2 of the right-hand member of the equation (33. 9) which will occur if, instead of $v = v_1$ one substitutes $v = v_2$ and then solves the simple linear equation

$$\frac{d^2}{d\zeta^2}\left[(1-\zeta)\frac{d\delta_3}{d\zeta}\right]=\Delta_2. \qquad (33.16)$$

Here the third approximation will be given by the formula $v_3 = v_2 + \delta_3$, etc.

The successive approximations obtained in this way will converge, provided the magnitude of the stress upon the plate is not large. In that case, the membrane stresses are also not large, and the first correction is smaller than the maximum of the first approximation, the second correction is smaller than the maximum of the second approximation, etc. The proof of this can be found in the work /VIII. 1/.

We shall now consider the application of the method of small parameters—expansion of the deflection into a power series of a parameter of pressure.

$$v=\sum_{1}^{\infty}\gamma_k(\zeta)P^k. \qquad (33.17)$$

As the first approximation to the required solution, i. e., as the coefficient γ_1, one takes the solution of our problem according to the theory of small deflections of plates. The following coefficients of the series are determined subsequently.

We shall assume that in this way we have obtained an approximate expression

$$v=\sum_{k=1}^{n}\gamma_k(\zeta)P^k. \qquad (33.18)$$

To determine the (n + 1)th member of the series (33. 17) we shall introduce (33. 18) in the left-hand side of (33. 9) and, after raising to the power and multiplying, we shall separate on the right-hand side all members containing P to a power higher than (n + 1). Solving the equation obtained for v, we shall find a more exact approximation for v and at the same time, the value of the function $\gamma_{n+1}(\zeta)$.

Hence it is apparent that the method of the small parameter can be considered as a variation of the method of successive approximations, in the application of which the higher powers of the quantity P, considered as a small parameter, are neglected. Without dwelling on the details of the given variant – method of a small parameter, we shall only show that with $S_0 \neq 0$, in the first approximation the problem leads to the integration of a Bessel equation, and in subsequent approximations to the integration of expressions containing Bessel functions.

In what follows we set $S_0 = 0$ and go on to the exposition of a second variant of the method of a small parameter, proposed by Wei-Tsang Chien in /VIII. 14/.

Let $v_c = v(1)$ be the deflection of the center of the plate. We shall expand the deflection function and the pressure parameter P in series of powers of v_c, considered as a small parameter

$$v(\zeta) = v_1(\zeta)\, v_c + v_2(\zeta)\, v_c^3 + \ldots, \quad P = P_1 v_c + P_2 v_c^3 + \ldots. \tag{33.19}$$

Introducing these expressions in equation (33. 9) and equating the expressions on the right- and the left-hand sides containing the parameter v_c to the same power, we shall obtain the sequence of equations

$$\frac{d^2}{d\zeta^2}\left[(1-\zeta)\frac{dv_1}{d\zeta}\right] = -\frac{3}{4}P_1, \tag{33.20}$$

$$\frac{d^2}{d\zeta^2}\left[(1-\zeta)\frac{dv_2}{d\zeta}\right] = \frac{3(1-\nu^2)}{2(1-\zeta)} \cdot \frac{dv_1}{d\zeta}\left[\zeta\int_0^1 (1-\tau)\left(\frac{dv_1}{d\zeta}\right)^2_{\zeta=\tau} d\tau - \int_0^{\zeta} (\zeta-\tau)\left(\frac{dv_1}{d\zeta}\right)^2_{\zeta=\tau} d\tau\right] - \frac{3}{4}P_2, \ldots \tag{33.21}$$

The value of the function v at the center of the plate should be v_c:

$$v(1) = v_1(1)\, v_c + v_2(1)\, v_c^3 + \ldots = v_c.$$

Hence it follows that

$$v_1(1) = 1, \quad v_2(1) = 0, \quad v_3(1) = 0, \ldots. \tag{33.22}$$

The boundary conditions for the plate edge have the form

$$\frac{dv}{d\zeta} = 0, \quad \frac{dv_1}{d\zeta} = \frac{dv_2}{d\zeta} = \ldots = 0 \quad \text{for} \quad \zeta = 0. \tag{33.23}$$

By solving equation (33. 20) for the conditions $v_1(0) = (dv_1/d\zeta)_0 = 0$, $|v_1(1)| < \infty$, we obtain

$$v_1(\zeta) = \frac{3}{16}P_1\zeta^2.$$

Hence, in view of (33. 22), we obtain

$$P_1 = 16/3, \quad v_1(\zeta) = \zeta^2.$$

Introducing this expression of v_1 we obtain the solution of equation (33. 21) satisfying the conditions $v_2(0) = (dv_2/d\zeta)_0 = 0, |v_2(1)| < \infty$:

$$v_2(\zeta) = \frac{3}{16}P_2\zeta^2 - \frac{1-\nu^2}{80}\left(5\zeta^2 + 3\frac{1}{3}\zeta^3 + 2\frac{1}{2}\zeta^4 + \zeta^5 + \frac{1}{3}\zeta^6\right).$$

From (33. 22) we obtain $P_2 = 1\frac{11}{135}(1-\nu^2)$. Consequently,

$$v_2(\zeta) = \left[\frac{73}{360}\zeta^2 - \frac{1}{60}\left(5\zeta^2 + 3\frac{1}{3}\zeta^3 + 2\frac{1}{2}\zeta^4 + \zeta^5 + \frac{1}{3}\zeta^6\right)\right](1-\nu^2).$$

In /VIII. 3/ are given the calculated results for deflections and stresses in circular plates for various boundary conditions at the plate edges; in particular, one considers the question of the deflection of a plate with an elastically supported edge for various rigidities of fixing the edge, where the dependence of the deflection of the center of the plate on its stress is given by the formula*

$$P = \beta_1 w_c + \beta_2 w_c^3 + \dots \qquad (33.\ 24)$$

The values of the coefficients β_1 and β_2 for various conditions of clamping the plate edges for $\nu = 0.3$ are given in Table I.

Table I

Boundary conditions			β	β_2
$v=0$	$M_1=0$	$T_1=0$	1.3381	0.3756
$v=0$	$M_1=0$	$u=0$	1.3381	2.480
$v=0$	$dv/dr=0$	$u=0$	5.3333	2.910
$v=0$	$dv/dr=0$	$T_1=0$	5.333	0.9843

Introducing (33. 19) in (33. 8), we find an expression for S. For the third case of the boundary conditions, when at the plate edge $u = 0$, $dv/dr = 0$, it has the form at the

$$S = \frac{v_c^2}{6}\left(\frac{2}{1-\nu} + \zeta + \zeta^2 + \zeta^3\right) + \frac{v_c^4(1-\nu^2)}{7566}\left[\frac{160-104\nu}{(1-\nu^2)^2} + \right.$$
$$\left. + \frac{80-52\nu}{1-\nu}(\zeta + \zeta^2 + \zeta^3) - \frac{501-249\nu}{1-\nu}\zeta^4 - 123\zeta^5 - 39\zeta^6 - 9\zeta^7\right]. \qquad (33.\ 25)$$

Analogous formulas for elastically supported plates are given in /VIII. 3/. Calculations show that when the plate edges can move freely in the plane of the plate, regions of compressive (negative) annular stresses appear near them, which can produce a local loss of stability near its edge in an unsymmetrical form (the appearance of waves). This phenomenon has been studied by D. Yu. Panov and V. I. Feodos'ev /VIII. 9/. The results of their investigations show that the approximate solutions of the problem of deflections of a circular plate, in which the plate surface is assumed to be axially symmetric, should be applied with care if one is considering a case when the plate edges can move freely in its plane.

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* This formula for the case of a clamped plate was obtained earlier in /VIII. 1/ by a transformation of the series (33. 17) with $\zeta = 0$.

§ 34. Asymptotic Solution of the Problem of the Behavior of a Circular Plate under Large Edge Loadings

Let us return to the problem, considered in § 32, of determining the deflection of a freely-supported circular plate under the action of edge loading in the plane of the plate /VIII. 6/. The calculations of § 32 have shown that the method of power series can be applied successfully only in the case of not very large stresses on the plate, when the ratio of the value of the stress to its critical value for start of buckling does not exceed 10-15. If the above-mentioned ratio is equal to 15, then to obtain sufficient accuracy one has to evaluate nearly 30 terms of the series.

In order to obtain information on the behavior of the plate under very large compressive stresses, we shall apply the method of asymptotic solutions of differential equations.

The power series calculations have shown that with increasing edge compression on the plate, a region of almost uniform tension is produced within it where $T_1 > 0$. Under increasing edge compression this region becomes more extensive, and the edge zone, where the plate is compressed, becomes increasingly narrow. For further consideration it is convenient to introduce new substitutions

$$\beta = \sqrt{-\tilde{p}_{1*}}\,(1-\rho)/\eta, \quad p_{1*} = \tilde{p}_{1*}P_1, \quad q = -\frac{1}{\eta}\tilde{p}_{1*}Q, \tag{34. 1}$$

where $\tilde{p}_{1*}$ is the value on the contour of the quantity p_{1*}, defined—just as the quantity η—by (32. 14); p_{1k} is the critical value of $\tilde{p}_{1*}$. Here, if $q^0 = 0$, $p = 0$, equation (32. 18) takes the form

$$\frac{d^2Q}{d\beta^2} + \frac{3}{\beta - \frac{1}{\eta}\sqrt{-\tilde{p}_{1*}}} \cdot \frac{dQ}{d\beta} + P_1 Q = 0. \tag{34. 2}$$

In an analogous manner we transform equation (32. 19) into the form

$$\frac{d^2P_1}{d\beta^2} + \frac{3}{\beta - \frac{1}{\eta}\sqrt{-\tilde{p}_{1*}}} \cdot \frac{dP_1}{d\beta} = \frac{1}{2}Q^2. \tag{34. 3}$$

As according to (34. 1) $\beta = 0$ for $\rho = 1$, and $\beta = 1/\eta\cdot\sqrt{-\tilde{p}_{1*}}$ for $p = 0$, the boundary conditions (32. 11), (32. 20), and (32. 31) reduce to the equations

$$P_1(0) = 1, \quad \left(\frac{dQ}{d\beta}\right)_0 - \frac{(1+\nu)}{\sqrt{-\tilde{p}_{1*}}}\eta Q(0) = 0, \tag{34. 4}$$

$$\frac{dP_1}{d\beta}\left(\beta = \frac{1}{\eta}\sqrt{-\tilde{p}_{1*}}\right) = \frac{dQ}{d\beta}\left(\beta = \frac{1}{\eta}\sqrt{-\tilde{p}_{1*}}\right) = 0. \tag{34. 5}$$

In order to obtain simpler approximate equations, characterizing the state of the plate under large values of edge stress $\tilde{p}_{1*}$, we shall carry out the limit in the equations (34. 2)-(34. 5) by setting $-\tilde{p}_{1*} = \infty$. We shall call the equation obtained the first approximation equations of the edge effect. Denoting the respective approximate values of the functions P_1 and Q by P_1^0 and Q^0, for determining them we have the equations

$$\frac{d^2P_1^0}{d\beta^2} = \frac{Q^{02}}{2}, \quad \frac{d^2Q^0}{d\beta^2} + P_1^0 Q^0 = 0, \tag{34.6}$$

$$P_1^0(0) = 1, \quad (dQ^0/d\beta)_0 = 0, \tag{34.7}$$

$$\lim_{\beta\to\infty} \frac{dP_1^0}{d\beta} = \lim_{\beta\to\infty} \frac{dQ^0}{d\beta} = 0. \tag{34.8}$$

To solve these equations we shall introduce a new independent variable x and unknown functions y and z by means of the substitutions

$$x = \xi e^{-\omega\beta}, \quad d\beta = -dx/x\omega, \tag{34.9}$$

$$P_1^0 = -\omega^2 y, \quad Q^0 = \sqrt{2}\,\omega^2 z, \tag{34.10}$$

where ξ and ω are numbers whose values so far remain undetermined.

When $\beta = \infty$ we have x = 0, when $\beta = 0$, $x = \xi$. Consequently, the relations (34.6)-(34.8) are replaced by the equations

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + z^2 = 0, \quad x\frac{d}{dx}\left(x\frac{dz}{dx}\right) - yz = 0, \tag{34.11}$$

$$y(\xi) = -\frac{1}{\omega^2}, \quad \left(\frac{dz}{dx}\right)_{x=\xi} = 0, \tag{34.12}$$

$$\lim_{x\to 0}\left(x\frac{dy}{dx}\right) = 0, \quad \lim_{x\to 0}\left(x\frac{dz}{dx}\right) = 0. \tag{34.13}$$

We shall assume that the solution satisfying the boundary conditions can be expressed by converging power series

$$y = \sum_{s=0}^{\infty} (-1)^s y_s x^{2s}, \quad z = \sum_{s=0}^{\infty} (-1)^s z_s x^{2s+1}. \tag{34.14}$$

Substituting these series in (34.11) and equating to zero the sums of the coefficients of the same powers of x, we obtain the recurrence relations

$$(2s)^2 y_s = \sum_{m+n=s-1} z_m z_n, \tag{34.15}$$

$$(2s+1)^2 z_s = \sum_{m+n=s} z_m y_n. \tag{34.16}$$

In particular, for s = 0 we obtain from the last equation:

$$z_0 = z_0 y_0, \quad y_0 = 1. \tag{34.17}$$

According to these one can calculate the desired number of coefficients of the series (34. 14) if one takes any arbitrary value for z_0.

If the series converge, then one can differentiate them term by term, and the conditions (34.13) will be automatically satisfied.

The second of the conditions (34.12) will be satisfied if for ξ one takes a root of the left-hand member of that equation. Friedrichs and Stoker /VIII. 6/ chose the smallest of the roots, which turned out to be $\xi = 0.98618$. Then the number ω was determined from the first condition (34. 12).

When one knows ω and ξ, one can calculate all the quantities which characterize the stressed state of the plate. Computations have shown that with the increase of the edge compression of the plate, the ratio of the value of the radial membrane stresses in the interior parts of the plate to the radial stresses on the contour have as limit the negative number --0.473. Thus for large values of compression the inner part of the plate turns out to be expanded. Further calculations have shown that with a strong edge compression the largest bending stresses of the plate are approximately equal to the quantity

$$1.11\sqrt{3(1-\nu^2)}\cdot E(\tilde{T}_{11}/Et)^{2/3}(2R/t).$$

This formula has an asymptotic character, i.e., its relative error is the smaller the larger the edge compression of the plate. Of course it holds when the maximal compressive stresses do not exceed the elastic limit. One has also to take into account that for very large values of compression of the plate, the rise angles of the plate elements are no longer small, and the theory of "shallow" shells--on the basis of which our initial differential equations were derived—becomes inapplicable.

In that case, one has to make use of the corrected equations of the edge effect, which can be found, for example, in the work of Reissner /VIII. 8/. It is also not difficult to derive them from the equations (7.4) and (7.5) of this monograph.

We shall further consider the asymptotic expansion of the solution sought in powers of the small parameter $\lambda = \eta/\sqrt{-\bar{p}_{1*}}$ proposed by Friedrichs and Stoker /VIII.6/. Here, for brevity, we shall limit ourselves to two terms of that expansion. We shall assume that the functions P_1 and Q introduced above for zero values of λ have derivatives

$$P_1'(\beta) = \frac{\partial}{\partial\lambda}[P_1(\beta)]\Big|_{\lambda=0}, \quad Q' = \frac{\partial}{\partial\lambda}[Q(\beta)]\Big|_{\lambda=0}.$$

Then

$$P_1(\beta) = P_1^0(\beta) + \lambda P_1'(\beta), \quad Q(\beta) = Q^0(\beta) + \lambda Q'(\beta). \tag{34.18}$$

Substituting the quantity $1/\lambda$ for $\frac{1}{\eta}\sqrt{-\bar{p}_{1*}}$ in equations (34.3) and (34.2) and differentiating them with respect to λ," and then setting $\lambda = 0$, we shall obtain the equations

$$\frac{d^2P_1'}{d\beta^2} - Q^0Q' = 3\frac{dP_1^0}{d\beta}, \tag{34.19}$$

$$\frac{d^2Q'}{d\beta^2} + P_1^0Q' + Q^0P_1' = 3\cdot\frac{dQ^0}{d\beta}. \tag{34.20}$$

Also from the boundary conditions (34.4) we find the relations

$$P_1'(0) = 0, \quad \left(\frac{dQ'}{d\beta}\right)_0 - (1+\nu)Q^0(0) = 0. \tag{34.21}$$

Differentiating the equations (34.5) with respect to λ gives the relations

for $\beta = \frac{1}{\lambda}$ $$\frac{\partial}{\partial\lambda}\left[\frac{dP_1}{d\beta}\right] - \frac{1}{\lambda^2}\cdot\frac{d^2P_1}{d\beta^2} = 0, \quad \frac{\partial}{\partial\lambda}\left[\frac{dQ}{d\beta}\right] - \frac{1}{\lambda^2}\cdot\frac{d^2Q}{d\beta^2} = 0.$$

Numerical computations have shown that the second terms of the left-hand members of these equations have the limits zero for $\lambda \to 0$. Besides, according to (34. 18),

$$\frac{dP_1}{d\beta} = \frac{dP_1^0}{d\beta} + \lambda\frac{dP_1'}{d\beta} + \dots, \quad \frac{\partial}{\partial\lambda}\left(\frac{dP_1}{d\beta}\right) = \frac{dP_1'}{d\beta}.$$

Therefore, we shall define the functions P_1' and Q' so as to satisfy the conditions

$$\frac{dP_1'}{d\beta}(\beta=\infty) = 0, \qquad \frac{dQ'}{d\beta}(\beta=\infty) = 0. \tag{34.22}$$

We shall seek the solution of the system of non-homogeneous linear equations (34.19) and (34. 20) for P_1 and Q' in the usual way, setting

$$P_1' = \pi + \pi_1, \; Q' = k + k_1, \tag{34.23}$$

where π and k are the particular solutions of the equations (34.19) and (34.20), while the functions π_1 and k_1 satisfy the respective homogeneous equations

$$\frac{d^2\pi_1}{d\beta^2} - Q^0 k_1 = 0, \quad \frac{d^2 k_1}{d\beta^2} + P_1^0 k_1 - Q^0 \pi_1 = 0. \tag{34.24}$$

It is obvious that if $P_1^0(\beta)$ and $Q^0(\beta)$ are solutions of the system of equations (34.6), then the functions $P_1^0(\beta+\alpha)$ and $Q^0(\beta+\alpha)$, where α is some parameter, represent a one-parameter family of solutions of these equations. We shall show that the functions

$$\pi_1 = \frac{\partial}{\partial\alpha}P_1^0(\beta+\alpha), \quad k_1 = \frac{\partial}{\partial\alpha}Q^0(\beta+\alpha) \tag{34.25}$$

satisfy the equations (34. 24). In fact, introducing (34. 25) in the first of these equations and using (34. 6), we have

$$\frac{d^2}{d\beta^2}\left[\frac{\partial}{\partial\alpha}P_1^0(\beta+\alpha)\right] - Q^0\frac{\partial}{\partial\alpha}Q^0(\beta+\alpha) = \frac{\partial}{\partial\alpha}\left\{\frac{d^2P_1^0}{d\beta^2} - \frac{1}{2}Q^{02}\right\} = 0.$$

In the same way, we convince ourselves that the second of the equations (34. 24) is also satisfied. As the second one-parameter family of solutions of the equations (34.24), one can take the system of functions $\alpha^2P_1^0(\alpha\beta)$ and $\alpha^2Q^0(\alpha\beta)$ which we substitute for P_1^0 and Q^0, satisfying the equations (34.6). It is not difficult to show that the derivatives of these functions with respect to α, at $\alpha = 1$, represent one more solution of the equations (34.24):

$$\pi_1 = \beta\frac{dP_1^0}{d\beta} + 2P_1^0, \; k_1 = \beta\frac{dQ^0}{d\beta} - 2Q^0. \tag{34.26}$$

In order to find the particular integral of the non-homogeneous equations, we shall introduce the substitutions

$$x = \xi e^{-\omega\beta}, \; P_1' = 3\omega y', \; Q' = -3\sqrt{2}\,\omega z'. \tag{34.27}$$

Then the equations (34.19) and (34.20) transform into the equations

$$\begin{aligned} x\frac{d}{dx}\left(x\frac{dy'}{dx}\right) + 2zz' &= x\frac{dy}{dx}, \\ x\frac{d}{dx}\left(x\frac{dz'}{dx}\right) - yz' - zy' &= x\frac{dz}{dx}. \end{aligned} \tag{34.28}$$

The functions y and z were already defined above. They are here considered as known.

Substituting in (34.28) the power series

$$y' = \sum_{s=1}^{\infty} (-1)^s y_s' x^{2s}, \quad z' = \sum_{s=1}^{\infty} (-1)^s z_s' x^{2s+1}, \tag{34.29}$$

and also (34.15) and (34.16), and equating the coefficients of the same powers of x, it is easy to obtain the system of equations which has to be satisfied by the coefficients of the series. If one somehow chooses values for the first coefficients, then the remaining coefficients can be determined successively. In this way the particular integral of the equations (34. 19) and (34. 20) satisfying the conditions (34. 22) has been found.

In order to obtain a solution which satisfies the boundary conditions (34. 21), it is necessary to add to the particular integral the corresponding linear combination of the solutions (34. 25) and (34. 26) of the homogeneous equations.

Calculations by this method have shown that at the plate edge with r = a, the following asymptotic expansions are valid:

$$\begin{gathered} P_1 = 1, \; Q = 1.61 - 0.74\eta/\sqrt{-\tilde{p}_{1*}} + \dots, \\ \frac{dP_1}{} = -1.61 + \frac{1.94\eta}{\sqrt{-\tilde{p}_{1*}}} + \dots, \; \frac{dQ}{d\beta} = \frac{2.13\eta}{\sqrt{\tilde{p}_{1*}}} + \dots . \end{gathered} \tag{34.30}$$

The ratio of the radial stress at the center of the plate to the radial stress at the plate edge may be represented by the asymptotic formula

$$\frac{T_1(r=0)}{T_1(r=a)} = -0.47 - \frac{1.03\eta}{\sqrt{-\tilde{p}_{1*}}}. \tag{34.31}$$

Chapter IX

STABILITY OF THE MEMBRANE STATE OF EQUILIBRIUM OF CYLINDRICAL SHELLS OF MEDIUM LENGTH

§ 35. Some Relations of the Theory of Shallow Cylindrical Shells

We shall refer the middle surface σ of a shell to cylindrical coordinates $\alpha = x$ and $\beta = s$, where x is the distance measured along the generator, s is the arc distance measured along the curve of the cross-section. Then, in the formulas of § 25 one has to set $B = 1$, $k_1 = 0$, $k_2 = 1/R = k$ (R is the radius of curvature of the shell before deformation).

In particular, according to (25. 8) and (25. 9), we find the relative elongations of the middle surface and its change of curvature as it is transformed from σ^0 to σ^I:

$$\begin{gathered}
\varepsilon_1^I = \frac{\partial u^I}{\partial x} + \frac{1}{2}\left(\frac{\partial w^I}{\partial x}\right)^2 + \frac{\partial w^I}{\partial x}\cdot\frac{\partial w^0}{\partial x},\\
\varepsilon_2^I = \frac{\partial v^I}{\partial s} + kw^I + \frac{1}{2}\left(\frac{\partial w^I}{\partial s}\right)^2 + \frac{\partial w^I}{\partial s}\cdot\frac{\partial w^0}{\partial s},\\
2\varepsilon_{12}^I = \frac{\partial u^I}{\partial s} + \frac{\partial v^I}{\partial x} + \frac{\partial w^I}{\partial x}\left(\frac{\partial w^I}{\partial s} + \frac{\partial w^0}{\partial s}\right) + \frac{\partial w^0}{\partial x}\cdot\frac{\partial w^I}{\partial s},\\
\varkappa_1^I = -\frac{\partial^2 w^I}{\partial x^2},\quad \varkappa_{12}^I = -\frac{\partial^2 w^I}{\partial x \partial s} \quad \overrightarrow{\underleftarrow{1,\,2}}\,.
\end{gathered} \tag{35. 1}$$

The internal stresses and moments are defined as before by the formulas (25. 10). The equations of equilibrium (25. 11) and (25. 12) take the simple form

$$\frac{\partial T_1^I}{\partial x} + \frac{\partial T_{12}^I}{\partial s} = 0, \quad \frac{\partial T_{12}^I}{\partial x} + \frac{\partial T_2^I}{\partial s} = 0, \tag{35. 2}$$

$$\begin{gathered}
D\Delta\Delta w^I - T_1^I\left(\frac{\partial^2 w^0}{\partial x^2} + \frac{\partial^2 w^I}{\partial x^2}\right) - 2T_{12}^I\left(\frac{\partial^2 w^0}{\partial x \partial s} + \frac{\partial^2 w^I}{\partial x \partial s}\right) -\\
- T_2^I\left(\frac{\partial^2 w^0}{\partial s^2} + \frac{\partial^2 w^I}{\partial s^2} - k\right) + p = 0,
\end{gathered} \tag{35. 3}$$

where

$$\Delta(\ldots) = \frac{\partial^2(\ldots)}{\partial x^2} + \frac{\partial^2(\ldots)}{\partial s^2},$$

$p>0$ is the density of the external normal pressure.

Introducing (25. 10) and (35. 1), we express the equations (35. 2) in terms of the displacement components:

$$\frac{\partial^2 u^I}{\partial x^2} + \frac{(1+\nu)}{2}\cdot\frac{\partial^2 v^I}{\partial x \partial s} + \frac{1-\nu}{2}\cdot\frac{\partial^2 u^I}{\partial s^2} + \nu k\frac{\partial w^I}{\partial x} + f_1^I = 0, \tag{35. 4}$$

$$\frac{\partial^2 v^I}{\partial s^2} + \frac{1+\nu}{2}\cdot\frac{\partial^2 u^I}{\partial x \partial s} + \frac{1-\nu}{2}\cdot\frac{\partial^2 v^I}{\partial x^2} + \frac{\partial}{\partial s}(k w^I) + f_2^I = 0, \tag{35. 5}$$

where

$$f_1^I = \frac{\partial}{\partial x}\left\{\frac{1}{2}\left(\frac{\partial w^I}{\partial x}\right)^2 + \frac{\partial w^0}{\partial x}\cdot\frac{\partial w^I}{\partial x} + \nu\left[\frac{1}{2}\left(\frac{\partial w^I}{\partial s}\right)^2 + \frac{\partial w^0}{\partial s}\cdot\frac{\partial w^I}{\partial s}\right]\right\} + \frac{1-\nu}{2}\cdot\frac{\partial}{\partial s}\left(\frac{\partial w^I}{\partial x}\cdot\frac{\partial w^I}{\partial s} + \frac{\partial w^0}{\partial x}\cdot\frac{\partial w^I}{\partial s} + \frac{\partial w^0}{\partial s}\cdot\frac{\partial w^I}{\partial x}\right)\ \overrightarrow{\underleftarrow{1,2}}\ . \quad (35.6)$$

After combining the equations of equilibrium

$$\frac{2}{1-\nu}\cdot\frac{\partial^2}{\partial s^2}(35.4) + \frac{\partial^2}{\partial x^2}(35.4) - \frac{1+\nu}{1-\nu}\cdot\frac{\partial^2}{\partial x\partial s}(35.5) = 0,$$

we eliminate v^I and obtain the following equation, expressing the relation between u^I and w^I:

$$\Delta\Delta u^I = -k\nu\frac{\partial^3 w^I}{\partial x^3} + \frac{\partial^2}{\partial s^2}\left(k\frac{\partial w^I}{\partial x}\right) - \frac{2}{1-\nu}\cdot\frac{\partial^2 f_1^I}{\partial s^2} - \frac{\partial^2 f_1^I}{\partial x^2} + \frac{1+\nu}{1-\nu}\cdot\frac{\partial^2 f_2^I}{\partial x\partial s}. \quad (35.7)$$

Analogous to this, we find the equation expressing the relation between v^I and w^I:

$$\Delta\Delta v^I = -\frac{\partial^3}{\partial s^3}(kw^I) - (2+\nu)\frac{\partial}{\partial s}\left(k\frac{\partial^2 w^I}{\partial x^2}\right) - \frac{2}{1-\nu}\cdot\frac{\partial^2 f_2^I}{\partial x^2} - \frac{\partial^2 f_2^I}{\partial s^2} + \frac{1+\nu}{1-\nu}\cdot\frac{\partial^2 f_1^I}{\partial x\partial s}. \quad (35.8)$$

Thus, for the determination of the equilibrium state σ^I we have a system of three equations (35.7), (35.8), and (35.3), which are linear in u^I and v^I and non-linear in w^I.

In the state of neutral equilibrium, together with the equations (35.3), (35.7), and (35.8), the equations (25.26) and (25.27) also have to be satisfied:

$$\frac{\partial T_1}{\partial x} + \frac{\partial T_{12}}{\partial s} = 0, \quad \frac{\partial T_{12}}{\partial x} + \frac{\partial T_2}{\partial s} = 0, \quad (35.9)$$

$$D\Delta\Delta w - T_1\left(\frac{\partial^2 w^0}{\partial x^2} + \frac{\partial^2 w^I}{\partial x^2}\right) - 2T_{12}\left(\frac{\partial^2 w^0}{\partial x\partial s} + \frac{\partial^2 w^I}{\partial x\partial s}\right) - T_2\left(\frac{\partial^2 w^0}{\partial s^2} + \frac{\partial^2 w^I}{\partial s^2} - k\right) - T_1^I\frac{\partial^2 w}{\partial x^2} - 2T_{12}^I\frac{\partial^2 w}{\partial x\partial s} - T_2^I\frac{\partial^2 w}{\partial s^2} = 0, \quad (35.10)$$

where T_i, T_{12} are additional elongations which appear in the transformation of the surface σ^I into the surface σ^*, according to (25.24):

$$\varepsilon_1 = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\left(\frac{\partial w^0}{\partial x} + \frac{\partial w^I}{\partial x}\right), \quad \varepsilon_2 = \frac{\partial v}{\partial s} + kw + \frac{\partial w}{\partial s}\left(\frac{\partial w^0}{\partial s} + \frac{\partial w^I}{\partial s}\right),$$

$$2\varepsilon_{12} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s} + \frac{\partial w}{\partial x}\left(\frac{\partial w^0}{\partial s} + \frac{\partial w^I}{\partial s}\right) + \frac{\partial w}{\partial s}\left(\frac{\partial w^0}{\partial x} + \frac{\partial w^I}{\partial x}\right), \quad (35.11)$$

u, v, w are the projections of an infinitesimal additional displacement.

The equations (35. 9) can be replaced by the equations

$$
\begin{aligned}
\Delta\Delta u = &-k\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2}{\partial s^2}\left(k \frac{\partial w}{\partial x}\right) - \frac{2}{1-\nu} \cdot \frac{\partial^2 f_1}{\partial s^2} - \\
&- \frac{\partial^2 f_1}{\partial x^2} + \frac{1+\nu}{1-\nu} \cdot \frac{\partial^2 f_2}{\partial x \partial s}, \\
\Delta\Delta v = &- \frac{\partial^3}{\partial s^3}(kw) - (2+\nu)\frac{\partial}{\partial s}\left(k \frac{\partial^2 w}{\partial x^2}\right) - \frac{2}{1-\nu} \cdot \frac{\partial^2 f_2}{\partial x^2} - \\
&- \frac{\partial^2 f_2}{\partial s^2} + \frac{1+\nu}{1-\nu} \cdot \frac{\partial^2 f_1}{\partial x \partial s},
\end{aligned}
\qquad (35.\,12)
$$

where

$$
\begin{aligned}
f_1 = &\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}\left(\frac{\partial w^0}{\partial x} + \frac{\partial w^1}{\partial x}\right) + \nu \frac{\partial w}{\partial s}\left(\frac{\partial w^0}{\partial s} + \frac{\partial w^1}{\partial s}\right)\right] + \\
&+ \frac{1-\nu}{2} \cdot \frac{\partial}{\partial s}\left[\frac{\partial w}{\partial x}\left(\frac{\partial w^0}{\partial s} + \frac{\partial w^1}{\partial s}\right) + \frac{\partial w}{\partial s}\left(\frac{\partial w^0}{\partial x} + \frac{\partial w^1}{\partial x}\right)\right] \quad \underset{\leftarrow}{\overset{\rightarrow}{1,2}},
\end{aligned}
\qquad (35.\,13)
$$

(35. 12) and (35. 10) represent a system of three linear homogeneous equations in u, v, w.

These equations are the equations of equilibrium in the components of displacement. It is frequently more convenient to make use of the equations of a mixed method, which define the deflection and the stress function. Setting

$$
T_1^{\mathrm{I}} = \frac{\partial^2 \psi^{\mathrm{I}}}{\partial s^2}, \quad T_{12}^{\mathrm{I}} = -\frac{\partial^2 \psi^{\mathrm{I}}}{\partial x \partial s}, \quad T_2^{\mathrm{I}} = \frac{\partial^2 \psi^{\mathrm{I}}}{\partial x^2}, \qquad (35.\,14)
$$

we satisfy the equations (35. 2) identically, where the condition of the compatibility of deformations has to be satisfied:

$$
\begin{aligned}
\Delta\Delta\psi^{\mathrm{I}} - Et\Bigg\{&\left(\frac{\partial^2 w^{\mathrm{I}}}{\partial x \partial s}\right)^2 + 2\frac{\partial^2 w^{\mathrm{I}}}{\partial x \partial s} \cdot \frac{\partial^2 w^0}{\partial x \partial s} - \\
&- \frac{\partial^2 w^{\mathrm{I}}}{\partial x^2}\left(\frac{\partial^2 w^0}{\partial s^2} + \frac{\partial^2 w^{\mathrm{I}}}{\partial s^2} - k\right) - \frac{\partial^2 w^{\mathrm{I}}}{\partial s^2} \cdot \frac{\partial^2 w^0}{\partial x^2}\Bigg\} = 0.
\end{aligned}
\qquad (35.\,15)
$$

This equation together with the equation obtained from (35. 3), by the substitution (35. 14), represents a system of two non-linear equations in ψ^{I} and w^{I}. In the same way, setting

$$
T_1 = \frac{\partial^2 \psi}{\partial s^2}, \quad T_{12} = -\frac{\partial^2 \psi}{\partial x \partial s}, \quad T_2 = \frac{\partial^2 \psi}{\partial x^2}, \qquad (35.\,16)
$$

we satisfy the equations (35. 9), where the additional stress function and the additional deflection w are mutually related by the equilibrium equation (35. 10) and by the condition of compatibility of deformations (20. 2[illegible]):

$$
\begin{aligned}
\Delta\Delta\psi - Et\Bigg[&2\frac{\partial^2 w}{\partial x \partial s}\left(\frac{\partial^2 w^0}{\partial x \partial s} + \frac{\partial^2 w^{\mathrm{I}}}{\partial x \partial s}\right) - \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w^0}{\partial s^2} + \frac{\partial^2 w^{\mathrm{I}}}{\partial s^2} - k\right) - \\
&- \frac{\partial^2 w}{\partial s^2}\left(\frac{\partial^2 w^0}{\partial x^2} + \frac{\partial^2 w^{\mathrm{I}}}{\partial x^2}\right)\Bigg] = 0.
\end{aligned}
\qquad (35.\,17)
$$

If the initial deflections are negligible, then before the loss of stability a membrane state or a near-membrane state is possible. For these one can neglect the changes in curvature $\frac{\partial^2 w^{\mathrm{I}}}{\partial x^2}$, $\frac{\partial^2 w^{\mathrm{I}}}{\partial s^2}$, $\frac{\partial^2 w^{\mathrm{I}}}{\partial x \partial s}$ in comparison with k, and further the rotations $\frac{\partial w^{\mathrm{I}}}{\partial x}$ and $\frac{\partial w^{\mathrm{I}}}{\partial s}$ can be considered as quantities of the same order as the elongations.

Then the equations of equilibrium of the shell, (35. 3), (35. 4) and (35. 5), become linear, and the equations of neutral equilibrium can be considerably simplified; in the components of displacement they take the form

$$\Delta\Delta u = -k\nu \frac{\partial^3 w}{\partial x^3} + \frac{\partial^2}{\partial s^2}\left(k \frac{\partial w}{\partial x}\right),$$

$$\Delta\Delta v = -\frac{\partial^3}{\partial s^3}(kw) - (2+\nu)\frac{\partial}{\partial s}\left(k \frac{\partial^2 w}{\partial x^2}\right), \tag{35. 18}$$

$$D\Delta\Delta w + T_2 k - T_1^1 \frac{\partial^2 w}{\partial x^2} - 2T_{12}^1 \frac{\partial^2 w}{\partial x \partial s} - T_2^1 \frac{\partial^2 w}{\partial s^2} = 0, \tag{35. 19}$$

where

$$T_2 = K\left(\frac{\partial w}{\partial s} + kw + \nu \frac{\partial u}{\partial x}\right), \quad K = Et/(1-\nu^2).$$

We multiply the equation (35. 19) by R = 1/k and operate upon it by $\Delta\Delta(\ldots)$. Using (35. 18) and the relations

$$\Delta\Delta\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(\Delta\Delta u), \quad \Delta\Delta\left(\frac{\partial v}{\partial s}\right) = \frac{\partial}{\partial s}(\Delta\Delta v), \tag{35. 19b}$$

we obtain an equation for w

$$\Delta\Delta\left\{R\left(D\Delta\Delta w - T_1^1 \frac{\partial^2 w}{\partial x^2} - 2T_{12}^1 \frac{\partial^2 w}{\partial x \partial s} - T_2^1 \frac{\partial^2 w}{\partial s^2}\right)\right\} + Etk \frac{\partial^4 w}{\partial x^4} = 0. \tag{35. 20}$$

Introducing the stress functions ψ the equations (35. 18) become

$$\Delta\Delta\psi - Etk \frac{\partial^2 w}{\partial x^2} = 0, \tag{35. 21}$$

and in (35. 19) one has to set $T_2 = \frac{\partial^2 \psi}{\partial x^2}$.

It is interesting to note that if the initial deflection w^0 is a function of the argument $\lambda x + \mu s$, where λ and μ are real numbers, then for displacement components which are functions of the same argument, the non-linear terms in the equations (35. 7), (35. 8), and (35. 15), and also in (35. 3) cancel each other upon the substitution of (35. 14).

§ 36. The Stability of a Cylindrical Shell of Circular Cross-Section under Axial Compression and Normal Pressure

We shall assume that a cylindrical shell of circular cross-section is subject to the simultaneous action of an axial compression p_1, uniformly distributed along the end sections, and of a uniform external normal pressure p. We shall neglect the effect of the clamping conditions on the first form of equilibrium. Then, before the loss of stability the stresses will be:

$$T_1^1 = p_1 < 0, \; T_2^1 = -Rp, \; T_1^1 = 0 \quad (R = \text{const}).$$

We shall try to determine the additional displacement components in the form

$$\begin{gathered} u = U \sin \frac{mx}{R} \sin \frac{ns}{R}, \quad v = V \cos \frac{mx}{R} \cos \frac{ns}{R}, \\ w = W \cos \frac{mx}{R} \sin \frac{ns}{R}, \\ m = \frac{i\pi R}{L}, \; (i = 1, 2, \ldots) \end{gathered} \tag{36.1}$$

where n is the number of waves formed on the circumference with the buckling of the shell.

Then, as seen from (35.21), the stress function is of the form

$$\psi = \Psi \cos \frac{mx}{R} \sin \frac{ns}{R},$$

where

$$\Psi = -Etm^2RW/(m^2+n^2)^2. \tag{36.2}$$

Introducing (36.1) in equation (35.20) we obtain the approximate relationship between the load, the shell parameters, and the numbers of waves m and n, which is obtained from the corresponding equation of /IX.4/, even if one of the quantities m^2 or n^2 is large in comparison with unity.

$$-p_1m^2 + Rpn^2 = D(m^2+n^2)^2k^2 + Etm^4/(m^2+n^2)^2. \tag{36.3}$$

In order that all the conditions of the problem be strictly satisfied, it is necessary to solve that equation for m. Then we substitute the values found for m_1, m_2, ... in the boundary conditions and look for the smallest values of $|p_1|$ and p for which these conditions are satisfied. Here we note that as the equation (36.3) contains only even powers of m and the boundary conditions are assumed to be the same on both ends of the shell, it is only necessary to verify that they are fulfilled at one end of the shell. Denoting the roots of the equation by $\pm m_j (j = 1, 2, 3, 4)$, we obtain for w the expression

$$w = \sin \frac{ns}{R} \sum_{j=1}^{4} W_j \cos \frac{m_j x}{R}. \tag{36.4}$$

And according to (35. 18)

$$u = R\sin\frac{ns}{R}\sum_{j=1}^{4}\frac{W_j(m_j n^2 - \nu m_j^3)}{(m_j^2+n^2)^2}\sin\frac{m_j x}{R},$$

$$v = R\cos\frac{ns}{R}\sum_{j=1}^{4}\frac{W_j[n^3+(2+\nu)nm_j^2]}{(m_j^2+n^2)^2}\cos\frac{m_j x}{R}.$$

We shall assume that the edge contours $x = \pm L/2$ are absolutely rigid with respect to elongation and deflection in their plane. This means that for $x = \pm L/2$ the following conditions should hold:

$$w=0 \text{ or } \sum_{j=1}^{4} W_j\cos\mu_j = 0 \quad (\mu_j = m_j L/2R) \tag{36. 5}$$

$v = 0$ or, by (36. 5)

$$\sum_{j=1}^{4} W_j(2+\nu)nm_j^2\cos\mu_j/(m_j^2+n^2)^2 = 0. \tag{36. 6}$$

Various cases are possible for displacements perpendicular to the plane of the edge stiffening rib.

A. It can turn out that the stiffening ribs have almost no resistance to torsion and are very easily deformed. In that case the displacements u and the rotations $\frac{\partial w}{\partial x}$ occur freely and in addition to the geometrical conditions (36. 5) and (36. 6) it is necessary to satisfy the static conditions for $x = \pm L/2$

$$M_1 = -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial s^2}\right)=0,\ T_1 = K(\varepsilon_1+\nu\varepsilon_2)=0,\ \oint T_{12}ds = 0.$$

Using (36. 1) and (36. 5), one can write the first of these conditions in the form

$$\sum_{j=1}^{4} W_j\mu_j^2\cos\mu_j = 0. \tag{36. 7A}$$

By (36. 1), (36. 6), and (36. 7A) the second condition may be brought into the form

$$\sum_{j=1}^{4} W_j\nu m_j^4\cos\mu_j/(m_j^2+n^2)^2 = 0. \tag{36. 8A}$$

The third conditions is satisfied owing to the periodicity of the additional displacements relative to s.

One can satisfy all these boundary conditions by setting

$$m_1 = \ldots = m_4 = i\pi R/L \text{ (i--odd integer)}$$

B. If the edge contours are supported by stiffening ribs which prevent not only displacements in their plane, but also displacements perpendicular to these planes (with the exclusion of rigid body displacements), then for $x = \pm L/2$, apart from (36. 5) and (36. 6) the following conditions should also be satisfied

$$\frac{\partial w}{\partial x}=0 \quad \text{or} \quad \sum_{j=1}^{4} W_j m_j \sin \mu_j = 0, \qquad \frac{\partial u}{\partial s}=0 \tag{36.7B}$$

or, by (36.7B)

$$\sum_{j=1}^{4} W_j^3 m_j^3 \sin \mu_j / (m_j^2+n^2)^2 = 0, \qquad \int_c T_1 ds = 0, \quad \int_c T_{12} ds = 0. \tag{36.8B}$$

The last two conditions are satisfied identically owing to the periodicity of the solution under investigation in s.

I. We shall investigate the case p = 0* in greater detail.

In this case, according to (36.3)

$$-p_1 = k\sqrt{EtD}(\zeta + 1/\zeta), \quad \zeta = t(m^2+n^2):\left[2m^2R\sqrt{3(1-\nu^2)}\right]. \tag{36.9}$$

The minimum of p_1 is at $\zeta = 1$:

$$-p_1 = Et^2/R\sqrt{3(1-\nu^2)} \approx 0.6Et^2/R \quad (\nu = 0.3). \tag{36.10}$$

This absolute minimum of the stress is reached for values of m and n of the order of $\sqrt{R/t}$. Therefore every term of (36.6) which contains a factor of the order of $1/n$ in comparison with the corresponding term of (36.5) can be considered as negligibly small if we admit an inaccuracy of the order of $\sqrt{t/R}$ in comparison with unity. Consequently, with that degree of accuracy the boundary condition (36.6) is approximately satisfied also in the variant B. Analogous to this, equation (36.8B), all terms of which contain factors of the order of $1/n^2$ in comparison with the corresponding terms of equation (36.7B), can be considered as approximately satisfied if one neglects t/R in comparison with unity. Hence, to satisfy the remaining conditions (36.5) and (36.7B) for case B, one can take the corrugated face after the loss of stability to be of the form

$$w = \sin\frac{ns}{R}\left(W_1 \cos\frac{m_1 x}{R} + W_2 \cos\frac{m_2 x}{R}\right).$$

Then m_1 and m_2 are found from the equations

$$W_1 \cos\mu_1 + W_2 \cos\mu_2 = 0, \quad W_1 m_1 \sin\mu_1 + W_2 m_2 \sin\mu_2 = 0. \tag{36.11}$$

The compatibility condition for W_1 and W_2 is

$$\mu_1 \operatorname{tg}\mu_1 = \mu_2 \operatorname{tg}\mu_2. \tag{36.11}$$

and this can be satisfied for every such pair of values μ_1, μ_2 for which $\mu_2 - \mu_1 \leqslant \pi$. As $L/2R \sim 1$, m_1 and m_2 are large, and so the values of T_1^I at m = m_1 and m_2 differ only slightly from its absolute minimum. Therefore the critical axial stress can be determined with good accuracy from (36.10). It is applicable also in the case A, as with

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* See articles /IX. 2/ and /IX. 3/.

$$m_1 = m_2 = i\pi R/L \text{ (i--odd integer)}$$

the boundary conditions are exactly satisfied.

Note that as the boundary conditions in case B are only approximately fulfilled the form of the corrugated surface can vary considerably from the actual one, but as T_1^I changes very slowly near its minimum with change of m and n, the critical stress will be determined with an accuracy up to t/R in comparison with unity. In fact, according to (36. 9), for p = 0 we have in the neighbourhood of the minimum

$$\zeta = 1 + \zeta_1,$$

where ζ_1 is a small quantity. Let ζ_1 be a quantity of the order of $\sqrt{t/R}$. Then

$$\zeta + \frac{1}{\zeta} \approx 1 + \zeta_1 + 1 - \zeta_1 + \zeta_1^2 = 2 + \zeta_1^2,$$

i. e., an error in the quantity ζ of the order of $\sqrt{t/R}$ leads to an error of the order of t/R in comparison with unity in T_1^I.

II. Let the critical values of the axial and contour stress be connected by the relation*:

$$-p_1 = \lambda_1 pR, \tag{36.12}$$

where λ_1 denotes a given quantity (for uniform compression, e. g., $\lambda_1 = 0.5$). In this case, according to (36. 3)

$$Rp = [D(m^2 + n^2)^2 k^2 + Etm^4/(m^2 + n^2)^2] : (n^2 + \lambda_1 m^2)$$

or, setting

$$\delta = (m^2 + n^2) : m, \quad \varepsilon = t : R\sqrt{12(1 - \nu^2)}, \tag{36.13}$$

we have

$$Rp = Etm\,(\delta^2\varepsilon^2 + 1/\delta^2) : [\delta + (\lambda_1 - 1)\,m]. \tag{36.14}$$

From the minimal condition $\partial p/\partial\delta = 0$ we find:

$$\varepsilon^2\delta^4 = \left[3 + \frac{2m(\lambda_1 - 1)}{\delta}\right] : \left[1 + \frac{2m(\lambda_1 - 1)}{\delta}\right].$$

If $\lambda_1 \neq \infty$, then $\partial p/\partial m > 0$.

Consequently, $m = m_0 = \pi R/L$.

With $\lambda_1 = 1$, i. e., if the axial stress before buckling is equal to the contour stress

$$\delta = \delta_1 = \sqrt[4]{3} : \sqrt{\varepsilon}. \tag{36.15}$$

In the general case

$$\delta = \delta_1 : (1 + \beta), \tag{36.16}$$

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* The formulas given below were obtained by us in 1950. In 1953 they were generalized for the case of a conical shell by A. V. Sachenkov in his candidate's dissertation presented at the Kazan' State University. See also /IX. 8/ and /IX. 6/.

where β satisfies the equation

$$\theta' = 2(1-\lambda_1)\, m_0/\delta_1 = [1-(1+\beta)^4] : \left\{(1+\beta)\left[1-\frac{1}{3}(1+\beta)^4\right]\right\}. \qquad (36.17)$$

$$\theta' = 3.61\,(1-\lambda_1)\,\theta, \quad \theta = \sqrt{tR} : [L\sqrt{2}\,(1-\nu^2)^{1/4}].$$

Hence it follows that

$$-0.2 \leqslant \beta \leqslant 0.2, \quad \text{for} \quad 0.8 \geqslant \theta' \geqslant -2.9. \qquad (36.18)$$

Here for β one can take the smallest (in absolute value) root of the equation*:

$$\beta^2(9-5\theta') + \beta(6-\theta') + \theta' = 0, \qquad (36.19)$$

and the approximate value of the critical pressure is

$$p_k = 1.31\,Et\,\frac{m_0\theta^{4/3}}{R}\,(1.33+2\beta^2) : \left[1-\frac{1}{2}\theta'(1+\beta)\right]. \qquad (36.20)$$

the change in the value of p is slow near the minimum. Therefore the critical pressure found from (36.19) for a given β differs even at the boundaries of the region (36.18) from its value for a β satisfying the minimization condition (36.17) by less than 0.6%, even though the error in β is 6%.

For shells satisfying the condition

$$-0.1 \leqslant \beta \leqslant 0.1 \quad \text{or} \quad 0.49 \geqslant 3.61\,(1-\lambda_1)\,\theta \geqslant -0.82, \qquad (36.21)$$

one can set $\beta = 0$, in (36.17), admitting an increased error of about 1%. Thus we arrive at the simple formula

$$p_k = \frac{1.2\,Et^2\theta}{R^2(1-\nu^2)^{1/2}} : [1-1.81\,(1-\lambda_1)\,\theta]. \qquad (36.22)$$

In the case of uniform compression $\lambda_1 = 0.5$ and with the condition (36.21)

$$p_k = \frac{0.85\,Et}{L(1-\nu^2)^{3/4}}\cdot\left(\frac{t}{R}\right)^{3/2} : \left[1-\frac{0.65\sqrt{tR}}{L(1-\nu^2)^{1/4}}\right]. \qquad (36.23)$$

In the general case, using (36.17) and (36.13), one can bring (36.14) into the form

$$p_k = 0.85\,Et^{5/2}\alpha : [L(1-\nu^2)^{3/4}R^{3/2}], \qquad (36.24)$$

where

$$\alpha = 1 : \left\{(1+\beta)\left[1-\frac{1}{3}\theta'(1+\beta)\right]\right\}, \quad \theta' = \frac{2.56\,(1-\lambda_1)\sqrt{tR}}{L(1-\nu^2)^{1/4}}. \qquad (36.25)$$

For a given β the corresponding values of θ are easily determined from (36.17). A table of values of α has been prepared on the basis of formula (36.25). It includes the results for $-0.1 \leqslant \beta \leqslant 0.1$, as they are covered, as shown above, by the simple formula (36.22).

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* In the article /IX. 8/ this equation contains a typographical error, but the remaining formulas are given correctly.

Table II

θ'	1.18	1.12	1.05	0.99	0.92	0.85	0.79	0.72	0.64	0.59	0.49
α	1.97	1.90	1.82	1.75	1.69	1.62	1.55	1.49	1.43	1.36	1.32
$-\theta'$	12.5	7.55	5.19	3.81	2.90	2.25	1.76	1.38	1.08	0.82	—
α	0.12	0.19	0.25	0.32	0.39	0.45	0.51	0.57	0.64	0.70	—

In conclusion, let us note that the formulas for the determination of the critical stress which were derived in this section for the case of the combined action of external normal pressure and axial compression, are applicable only for freely-supported edges. If there is an edge clamping of type B, then the non-fulfillment of the boundary condition (36. 11) can bring about a real error in the value of the critical stress. As a matter of fact, if for the loss of stability from external normal pressure, the buckling of the shell occurs with the formation of one half-wave along its length, i. e., if the shell is of medium length, then $\pi R \sim L$, $\mu_1 \sim \pi$, $\mu_2 \sim \pi$ and consequently, the values of p for $\mu = \mu_1$ and $\mu = \mu_2$ will differ markedly from the absolute minimum of p. If the shell is very short, then μ_1 and μ_2 are considerably larger than π, but the theory developed here, based on the assumption of a membrane state before buckling, turns out to be dubious.

§ 37. The Stability of a Cylindrical Shell of Arbitrary Cross-Section under Longitudinal Compression*. Shell with Elliptic Cross-Section. Shell with Longitudinal Corrugation.

Before the loss of stability, just as in the case of a shell of circular cross-section,

$$T_1^1 = \text{const}, \quad T_2^1 = T_{12}^1 = 0. \tag{37.1}$$

Let the cross-section of the shell have an axis of symmetry. Then its curvature can be expressed as a function of arc in the following way

$$k = \sum_{r=0}^{\infty} a_r \cos ris, \quad i = \pi n_1/l, \tag{37.2}$$

where 2l is the length of the section contour and n_1 is an integer.

Obviously, with the loss of stability of the shell waves appear along the section contour and their form will depend also on the form of the contour. Therefore we shall seek the solution of the equations (35.20), (35.21) of neutral equilibrium in the form

$$\psi = \cos\mu x \sum_{p=-\infty}^{\infty} B_p \cos(j + ip)s, \tag{37.3}$$

$$w = \cos\mu x \sum_{p=-\infty}^{\infty} C_p \cos(j + ip)s.$$

where p is an integer and $j = \pi n/l$. Here, for

$$\mu = (2p+1)\frac{\pi}{L} \tag{37.4}$$

as shown in § 36, boundary conditions of type A are fulfilled at $x = \pm L/2$.

For the determination of the coefficients B_p and C_p we have an infinite number of equations. Eliminating B_p from them we obtain a system of equations in C_p:

$$\sum_{q=1}^{\infty} \lambda_{p-q}^{p} C_{p-q} + (\lambda_p^p - t_0) C_p + \sum_{q=1}^{\infty} \lambda_{p+q}^{p} C_{p+q} = 0, \tag{37.5}$$

where we set

$$\lambda_p^p = \frac{\eta^2}{\xi_p} + a_0^2 \xi_p + \sum_{\rho=1}^{\infty} \frac{a_\rho^2}{4} (\xi_{p-\rho} - \xi_{p+\rho}),$$

$$\eta^2 = \frac{t^2}{12(1-\nu^2)}, \quad t_0 = -\frac{T_1^1}{Et}$$

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* The theory relating to this problem has been derived in §§ 16-18 of the monograph /0.13/.

★

$$4\lambda_{p-q}^{p}=\sum_{\rho=0}^{q}\alpha_{q-\rho}\alpha_{\rho}\xi_{p-\rho}+\sum_{\rho=q}^{\infty}\alpha_{\rho-q}\alpha_{\rho}\xi_{p-\rho}+\sum_{\rho=0}^{\infty}\alpha_{\rho+q}\alpha_{\rho}\xi_{p+\rho} \qquad (37.6)$$

$$\xi_p=\mu^2:[\mu^2+(j+lp)^2], \qquad (37.7)$$

$$4\lambda_{p+q}^{p}=\sum_{\rho=0}^{q}\alpha_{q-\rho}\alpha_{\rho}\xi_{p+\rho}+\sum_{\rho=q}^{\infty}\alpha_{\rho-q}\alpha_{\rho}\xi_{p+\rho}+\sum_{\rho=0}^{\infty}\alpha_{\rho+q}\alpha_{\rho}\xi_{p-\rho}, \qquad (37.8)$$

$$q=1,2,\ldots;\quad -\infty<p<\infty.$$

For consistency of the system (37.5), the determinant of the coefficients of C_p should be zero:

$$\Delta=\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \lambda_{p-1}^{p-1}-t_0 & \lambda_{p}^{p-1} & \lambda_{p+1}^{p-1} & \cdots \\ \lambda_{p-1}^{p} & \lambda_{p}^{p}-t_0 & \lambda_{p+1}^{p} & \cdots \\ \lambda_{p-1}^{p+1} & \lambda_{p}^{p+1} & \lambda_{p+1}^{p+1}-t_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}=0 \qquad (37.9)$$

The unknown $-T_1 l$ whose minimum is to be determined and which is an eigenvalue of the boundary value problem under consideration, is contained only in the diagonal terms of the characteristic determinant. Besides, as is apparent from (37.6),

$$\lambda_{p}^{p-q}=\lambda_{p-q}^{p}. \qquad (37.10)$$

i.e., Δ is a symmetrical determinant. Consequently, the equation (37.9) is the so-called secular equation, all of whose roots are real.

As is well known, for the infinite determinant to converge, it is necessary and sufficient that the derivatives of the diagonal terms and the sum of the non-diagonal terms converge absolutely.

We shall multiply the p-th row of the determinant by ξ_p/η^2. Then the product of the diagonal terms has the form:

$$\prod_{p=-\infty}^{\infty}\left\{1-\frac{t_0\xi_p}{\eta^2}+\frac{\alpha_0^2\xi_p^2}{\eta^2}+\frac{\xi_p}{\eta^2}\sum_{\rho=1}^{\infty}\frac{\alpha_\rho^2}{4}(\xi_{p-\rho}+\xi_{p+\rho})\right\}=\prod_{p=-\infty}^{\infty}(1+a_{pp}).$$

$\sum a_{pp}$ converges absolutely even upon the replacement of the quantities $\xi_{p-\rho}$ and $\xi_{p+\rho}$ by the largest of them, as $\sum_{\rho=1}^{\infty}\alpha_\rho$ converges, and $\zeta_p\to 0$ as $1/p^4$. Consequently, this product converges absolutely.

We shall replace $\xi_{p-\rho}$ and $\xi_{p+\rho}$ by the largest of them, ξ_m, in (37.6). Then the sum of the non-diagonal terms of the p-th row will be less than

$$\xi_m\sum_{\rho=1}^{\infty}\sum_{r=1}^{\infty}\alpha_r\alpha_\rho.$$

and the sum of all non-diagonal terms will be less than

$$\sum_{p=-\infty}^{\infty}\xi_p\xi_m\sum_{\rho=1}^{\infty}\sum_{r=1}^{\infty}\alpha_r\alpha_\rho/\eta^2$$

and will, consequently, converge as $1/p^4$.

★ Limiting oneself to a finite number of terms in the series (37. 3), we find the approximate eigenfunction of the problem. As is well known, the sequence of the latter converges if the problem has a Green's function, finite at a singular point /0. 23/. In our case the Green's functions are displacements under the action of point forces and have finite values everywhere. The actual determination of the critical stress from (37. 9) can be carried out only approximately, taking instead of Δ a determinant of finite order. It can be shown that of all determinants of a given order, the best approximation can be given by the one in which the diagonal terms are

$$\lambda_{-1}^{-1},\ \lambda_0^0 \text{ and } \lambda_1^1$$

Without dwelling on the formal proof of this statement, we shall, in what follows, determine the approximate value of the critical stress from the equation

$$\begin{vmatrix} \lambda_{-1}^{-1}-t_0 & \lambda_0^{-1} & \lambda_1^{-1} \\ \lambda_{-1}^{0} & \lambda_0^0-t_0 & \lambda_1^0 \\ \lambda_{-1}^{1} & \lambda_0^1 & \lambda_1^1-t_0 \end{vmatrix}=0, \tag{37. 11}$$

which can also be obtained from the energy criterion for stability, expressed in the form (25. 31). In fact, according to the nature of the problem, one can expect to obtain a sufficiently good approximation by retaining a small number of terms of the Ritz series, taking the components of the additional displacement in the following form:

$$\begin{gathered} u=\sin\mu x\sum_{p=-2}^{2}A_p\cos(j+pi)s,\quad v=\cos\mu x\sum_{p=-2}^{2}B_p\sin(j+ip)s, \\ w=\cos\mu x\sum_{p=-1}^{1}C_p\cos(j+ip)s. \end{gathered} \tag{37. 12}$$

Setting $\alpha_2=\alpha_3=\ldots=0$ in (37. 2) using (25. 31) and the condition B = 1, we obtain 13 equations in 13 unknowns A_{-2}, A_{-1}, ..., C_1:

$$\frac{\partial Э}{\partial A_p}=\frac{\partial Э}{\partial B_p}=\ldots=\frac{\partial Э}{\partial C_p}=0,$$

where A_p and B_p simply expressions in C_p. Thus, in order to determine C_{-1}, C_0, C_1 we obtain three equations, the consistency conditions of which gives the characteristic equation (37. 11). The reader can convince himself of this by carrying out the calculations indicated which are simple in principle, but rather laborious.

Before going on to the determination of the approximate value of the critical stress from the equation (37. 11), we shall note that limiting ourselves to a finite number of terms of the Ritz series or, what is the same, replacing Δ by the determinant of a finite order, we find the critical stress with an error on the larger side, as this imposes supplementary constraints which hinder the buckling of the shell.

We shall consider the problem further for the two limiting cases when either $i^2 \ll j^2$, or $i^2 \gg j^2$.

A. Stability of a shell with elliptical cross-section

Let the cross-section of the middle surface of the shell be given by the equation

$$\rho=r_0(1+\lambda\cos n_1\varphi), \tag{37. 13}$$

where r_0 is the mean radius and λ and n_1 are quantities characterizing the deviation of the shape of the cross-section from a circle of radius r_0. Then

$$R = \frac{r_0\left[(1 + \lambda\cos n_1\varphi)^2 + \lambda^2 n_1^2 \sin^2 n_1\varphi\right]^{3/2}}{(1 + \lambda\cos n_1\varphi)^2 + 2\lambda^2 n_1^2 \sin^2 n_1\varphi + (1 + \lambda\cos n_1\varphi)\lambda n_1^2 \cos n_1\varphi}.$$

Such a section can be approximately replaced by the section (37. 3) investigated by us if the parameters α_i are so chosen that the curvature of the fictitious curve is equal to the curvature of the given curve. Setting $\alpha_2 = \alpha_3 = \ldots = 0$ and equating the curvature of the curve (37. 13) at the points of greatest and smallest curvature to the curvatures of the curve (37. 3) at the corresponding points, we find that

$$\alpha_0 = \frac{1 - \lambda^2 - 2\lambda^2 n_1^2}{r_0(1 - \lambda^2)^2}, \quad \alpha_1 = \frac{\lambda(n_1^2 - 1) + \lambda^3(n_1^2 + 1)}{r_0(1 - \lambda^2)^2}. \tag{37. 14}$$

In the case of a shell of elliptical cross-section

$$n_1 = 2, \quad i = 2\pi/l. \tag{37. 15}$$

★ With the loss of stability of a thin cylindrical shell with circular cross-section, waves appear along the length and the circumference of the shell, so that the quantities $m^2 = \mu^2 r_0^2$, $n^2 = j^2 r_0^2$ can be considered as large in comparison with unity. We assume that at least one of the quantities μ^2 and j^2 is large in comparison with i^2 also for an elliptical section of small eccentricity, i. e., we consider that

$$2i^2 \ll \mu^2 + j^2.$$

Then, according to (37. 8)

$$\begin{gathered} \xi_1 = \xi_0(1 - \zeta), \ \xi_{-1} = \xi_0(1 + \zeta), \ \zeta = 4ij/\mu^2 + j^2, \\ \xi_2 = \xi_0(1 - 2\zeta), \ \xi_{-2} = \xi_0(1 + 2\zeta). \end{gathered} \tag{37. 16}$$

Introducing these expressions in (37. 9) we find

$$\begin{gathered} \lambda_0^0 = \frac{\eta^2}{\xi_0} + \left(\alpha_0^2 + \frac{\alpha_1^2}{2}\right)\xi_0, \ \lambda_{-1}^{-1} = \lambda_0^0 - \vartheta\zeta, \ \lambda_1^1 = \lambda_0^0 + \vartheta\zeta, \\ \lambda_{-1}^0 = \lambda_0^{-1} = \alpha_0\alpha_1\xi_0(1 + \zeta/2), \ \lambda_0^1 = \lambda_1^0 = \alpha_0\alpha_1\xi_0(1 - \zeta/2), \\ \lambda_{-1}^1 = \lambda_1^{-1} = \alpha_1^2\xi_0/4, \ \vartheta = \frac{\eta^2}{\xi_0} - \left(\alpha_0^2 + \frac{\alpha_1^2}{2}\right)\xi_0. \end{gathered} \tag{37. 17}$$

Consequently, setting

$$\tau = \lambda_0^0 - t_0, \tag{37. 18}$$

the equation (37. 11) can be brought into the form

$$\tau^3 - \tau\left(2\alpha_0^2\alpha_1^2\xi_0^2 + \frac{\alpha_1^4\xi_0^2}{16} + \zeta^2\vartheta^2\right) - 2\alpha_0^2\alpha_1^2\xi_0^2\zeta^2\vartheta + \frac{\alpha_0^2\alpha_1^4\xi_0^3}{2} = 0. \tag{37. 19}$$

Solving this equation to the first approximation and considering that the eccentricity of the section is small, we shall neglect the terms containing ζ^2 and α_1^4. Thus, we find the approximate values of the roots

$$\tau_1 = 0, \ \tau_2 = \alpha_0\alpha_1\xi_0\sqrt{2}, \quad \tau_3 = -\alpha_0\alpha_1\xi_0\sqrt{2}.$$

★ The smallest value of the critical stress gives the root $\tau = \tau_2$, for which, minimizing t_0 with respect to ξ_0, we find

$$t_{0,\min} = 2\eta \sqrt{\alpha_0^2 + \frac{\alpha_1^2}{2} - \alpha_0\alpha_1\sqrt{2}}, \quad \vartheta = -\alpha_0\alpha_1\sqrt{2}\xi_0.$$

Introducing these values of τ and ϑ in (37.19), we convince ourselves that it was admissible to neglect--just as we did--the terms containing ζ^2, if $\zeta^2 \ll 1$, as we had assumed. The term $\tau\alpha_1^4\xi_0^2/16$ is also small, provided that $\alpha_1^2 << 8\alpha_0^2$. Of the terms neglected earlier in the equation (37.19), the largest is

$$\alpha_0^2\alpha_1^4\xi_0^2/2.$$

In the second approximation, setting

$$\tau = \tau_2(1+\theta)$$

and neglecting the squares and higher powers of θ in comparison with unity, we find that $\theta = -\alpha_1/8\alpha_0\sqrt{2}$.

Thus, in the second approximation

$$t_{0,\min} = 2\eta \sqrt{\alpha_0^2 - \alpha_0\alpha_1\sqrt{2} + \frac{5\alpha_1^2}{8}}. \tag{37.20}$$

Whence it follows that $\xi_0 \sim t/\alpha_0$, or, in view of (37.14)

$$\xi_0 \sim tr_0.$$

On the other hand,

$$\xi_0 = \mu^2 : (\mu^2 + j^2)^2;$$

therefore, if our assumption $\mu^2 + j^2 >> i^2$ is not true and $\mu^2 \sim i^2$, $j^2 \sim i^2$, then we would obtain

$$\xi_0 \sim 1/i^2 \sim l^2/4\pi^2 \sim r_0^2$$

which contradicts the condition $\xi_0 \sim tr_0$.

This means that (37.16) indeed gives $\zeta^2 << 1$, which we actually assumed when simplifying the characteristic equation. ★

For a shell of elliptical section, a form of buckling is also possible for which j = 0. This form of loss of stability can be called almost-symmetrical in analogy with the axially symmetrical form of loss of stability of a shell of circular section. Here, instead of (37.3) we seek a solution of the system of equations of equilibrium in the form

$$\Phi = \sum_{p=1}^{\infty} B_p \cos pis, \quad W = \sum_{p=0}^{\infty} C_p \cos pis. \tag{37.21}$$

We shall obtain the corresponding characteristic equation from (37.9) by equating to zero all the λ_p^q and ξ_p with negative indices. Limiting ourselves again to a determinant of the third order, we arrive at the equation

$$\Delta_3 = \begin{vmatrix} \lambda_0^0 - t_0 & \lambda_1^0 & \lambda_2^0 \\ \lambda_0^1 & \lambda_1^1 - t_0 & \lambda_2^1 \\ \lambda_0^2 & \lambda_1^2 & \lambda_2^2 - t_0 \end{vmatrix} = 0,$$

in expanding which we note that in the case under consideration, $j=0,\ \zeta=0,\ \xi_2\approx\xi_1\approx\xi_0$ (since $i^2 \ll \mu^2$). Solving the cubic equation obtained in the second approximation, we find

$$t_{0,k}=2\eta\sqrt{\alpha_0^2-s_k\alpha_1\sqrt{2}+\frac{9}{16}\alpha_1^2}. \tag{37.22}$$

The comparison of (37.22) with (37.20) shows that a shell of elliptical section under axial compression buckles in an almost symmetrical form.

B. Stability of a corrugated shell of circular section with longitudinal corrugations.

Let n_1 be the number of corrugations along the circumference of the section, $\lambda=\delta/r_0$ for amplitude of corrugation 2δ.

According to (37.14)

$$\alpha_0\approx(1-2\lambda^2n_1^2):r_0,\quad \alpha_1\approx\lambda n_1^2:r_0,\quad \alpha_1^2\gg\alpha_0^2. \tag{37.23}$$

If each circumference wave which is formed under buckling encompasses several corrugations, then

$$(i+j)^4\gg j^4,\quad (i-j)^4\gg j^4;$$

therefore according to (37.8)

$$\lambda_0^0\approx\frac{\eta^2}{\xi_0}+\alpha_0^2\xi_0+\frac{\alpha_1^2}{4}(\xi_1+\xi_{-1});\quad \lambda_1^1\approx\frac{\eta^2}{\xi_1}+\frac{\alpha_1^2}{4}\xi_0;\ \ldots$$

Neglecting the small terms we obtain from (37.11) the equation

$$t_0^3-at_0^2+bt_0-c=0,$$

where

$$a=\eta^2\left(\frac{1}{\xi_1}+\frac{1}{\xi_{-1}}\right)+\frac{\alpha_1^2}{2}\xi_0,\quad b=\frac{\eta^4}{\xi_1\xi_{-1}}+\eta^2\frac{\alpha_1^2}{4}\xi_0\left(\frac{1}{\xi_1}+\frac{1}{\xi_{-1}}\right),$$
$$c=\frac{\eta^2\xi_0}{\xi_1\xi_{-1}}\left[\frac{\eta^2}{\xi_0}+\frac{\alpha_1^2}{4}(\xi_1+\xi_{-1})\right]^2+\eta^4\frac{\alpha_0^2\xi_0}{\xi_1\xi_{-1}}.$$

The smaller positive root of this equation is

$$t_0\approx\frac{c}{b}\left(1+\frac{ac}{b^2}\right)=-\frac{T_1^1}{Et}.$$

To the first approximation

$$t_0\approx\frac{c}{b}=\frac{\eta^2}{\xi_0}+\frac{\alpha_1^2}{4}(\xi_1+\xi_{-1})+\frac{\eta^2\alpha_0^2}{\frac{\eta^2}{\xi_0}+\frac{\alpha_1^2}{4}(\xi_1+\xi_{-1})}. \tag{37.24}$$

In particular, for $\alpha_1=0$ this becomes the Lorentz-Timoshenko formula for a shell of circular section. For large values of α_1, according to (37.24) the absolute minimum of t_0 is not reached for admissable values of μ and j. Therefore one can take the following as the critical values of μ and j

$$\mu_k = \pi/L, \ j_k = 2\pi/l, \ n_k = 2, \tag{37.25}$$

which give the smallest possible value for t_0.

In the theory of stability of shallow shells (which we have used in investigating this problem) it had been assumed that $n^2 \gg 1$. Therefore our solution for n = 2 can turn out to be unsatisfactory. In the given case, however, t_0 increases monotonically together with n, and at first the increase in n is small within quite broad limits of t_0, as $i^2 \gg j^2$; therefore, for the values (37.25), the formula (37.24) can be considered as satisfactory for the determination of the critical stress. Hence, the number of waves on the circumference remains undetermined. Note that in (37.24) one can set, within an error of 1-2%,

$$\mathfrak{k}_1 + \mathfrak{k}_{-1} = 2\mu^4 (i^2 + 12 j^2)/i^6.$$

Example:

$$2r_0 = L = 40 \text{ cm}; \ 2\delta = 0.915 \text{ cm}; \ t = 3.9 \cdot 10^{-2} \text{ cm};$$
$$n_1 = 42; \ E = 7.2 \cdot 10^5 \text{ kg/cm}^2; \ \nu = 0.3.$$

We have

$$\lambda = 2.29 \cdot 10^{-2}; \ j = \frac{n}{20 \cdot 1.2}; \ i = \frac{35}{r_0}, \ \eta^2 = 1.39 \cdot 10^{-4}; \alpha_0 = -\frac{0.85}{20};$$
$$\alpha_1 = 2.02; \ \mu_k^2 = 2.467/400.$$

According to (27.24)

$$T_1^I = -44 \text{ kg/cm for } n = 2,$$
$$T_1^I = -45 \text{ kg/cm for } n = 5.$$

For a smooth shell of the same radius, $T_1^I = -32.6$ kg/cm. Thus, in the case given, the corrugation increases the critical stress by 40%.

§ 38. The Stability of a Shell of Circular Section under Torsion and under the Action of Combined Stresses

Let a cylindrical shell of radius R be in equilibrium under the action of shearing tangential stresses τ and normal stresses p_1, uniformly distributed along the edge sections $x = \pm L/2$, and also a normal external pressure p, uniformly distributed along the lateral surface of the shell. The equations of neutral equilibrium (35. 18) and (35. 10) are satisfied if the components of displacements which appear with the buckling of the shell are taken in the form

$$u = \sum_k u_k \sin \frac{m_k x + ns}{R}, \quad v = \sum_k v_k \sin \frac{m_k x + ns}{R}, \qquad (38.1)$$
$$w = \sum_k w_k \cos \frac{m_k x + ns}{R},$$

where m_k ($k = 1, 2, \ldots, 8$) are the roots of the equation

$$-p_1 m^2 + Rpn^2 - 2\tau mn = Et\left[(m^2 + n^2)^2 \varepsilon^2 + \frac{m^4}{(m^2 + n^2)^2}\right], \qquad (38.2)$$
$$\varepsilon^2 = t^2 : 12R^2(1 - \nu^2).$$

Let

$$p_1 = -2\lambda_3\tau, \; Rp = 2\lambda_2\tau, \; \mu = m : n, \qquad (38.3)$$

where λ_3 and λ_2 are given constants.

Then the characteristic equation (38. 2) is brought into the form

$$\frac{2\tau}{Et\varepsilon}(\lambda_3\mu^2 - \mu + \lambda_2) = \mu^2\left(\beta + \frac{1}{\beta}\right), \; \beta = \frac{\mu^2}{n^2\varepsilon(1 + \mu^2)^2}. \qquad (38.4)$$

The case of a long shell, when $R \sim L\sqrt{\varepsilon}$, was considered in § 19. It had been shown that in this case n = 2 and the value of μ corresponding to the critical stress has to be determined from the equation

$$\partial\tau/\partial\mu = 0. \qquad (38.5)$$

irrespective of any boundary condition being satisfied.

In the case of a shell of medium length which we do not consider here, it is necessary to satisfy at least the most important boundary condition

$$w = 0 \quad \text{for} \quad x = \pm L/2. \qquad (38.6)$$

For this we set

$$w = W_1 \cos \frac{m_1 x + ns}{R} + W_2 \cos \frac{m_2 x + ns}{R}, \qquad (38.7)$$

where m_1 and m_2 are the real roots of the equation (34. 8) which are small in absolute value: the boundary conditions (38. 6) are satisfied, provided that

$$W_1 = W_2 = W,\ m_1 - m_2 = \frac{2\pi R}{L} = 2m_0, \qquad (38.8)$$
$$\mu_1 - \mu_2 = 2\mu_0 = \frac{2m_0}{n}.$$

Note that as had been shown in /IX. 7/, two of the eight roots of the equation (38.2) with $p_1 = p = 0$, are real, and the others are complex. To satisfy the condition (38.6), we use both real roots m_1 and m_2. The locus of the points for which $w = 0$ is defined by the equation

$$\cos\frac{1}{R}\left[(m_1 + m_2)\frac{x}{2} + ns\right]\cos\frac{\pi x}{L} = 0,$$

from which, using (38.8), we obtain

$$\left(m_1 - \frac{\pi R}{L}\right)x + ns = \frac{\pi R}{2} i,$$

where i is an odd integer.

Hence it follows that with the buckling of the shell, waves are formed, inclined to the generators at an angle ϑ given by the formula

$$\operatorname{tg}\vartheta = |(m_1 - m_0)| : n = |\mu_1 + \mu_2| : 2. \qquad (38.9)$$

In the case of a long shell $\vartheta \ll 1$ and for a very short shell $R \gg L$. In the limit, when $R = \infty$, the problem of the stability of a cylindrical shell under torsion is transformed into the problem of the stability of a flat strip under the action of shearing forces, whose exact solution with the fulfillment of the condition of free support or clamping was given in the article of Southwell and Skan /IX. 9/. As is known from that solution and from earlier approximate solutions, with the buckling of the strip by tangential forces, $\vartheta = 45°$. We make the assumption, later verified by the solution obtained, that in the case of a shell of medium length, $\operatorname{tg}^4\vartheta$ is much smaller than unity and therefore if $|\mu_1| < |\mu_2|$ the quantity $\mu_1{}^4$ can be neglected in comparison with unity

$$\mu_1^4 \ll 1. \qquad (38.10)$$

We shall first consider the case when

$$\lambda_1 = \lambda_2 = 0. \qquad (38.11)$$

According to (38.4)

$$\frac{2\tau}{Et\epsilon} = -\mu_1\left(\beta_1 + \frac{1}{\beta_1}\right) = -\mu_2\left(\beta_2 + \frac{1}{\beta_2}\right), \qquad (38.12)$$

where $\beta_k = \beta$ for $\mu = \mu_k$ (k = 1; 2).

Since $|\mu_2|$ is considerably greater than $|\mu_1|$, ($|\mu_2| \approx 3|\mu_1|$ (as can be seen from the solution obtained) the preceding equality can hold only when $\beta_2 + 1/\beta_2$ is near its minimum, equal to two, where, obviously, to the minimal value of τ corresponds a value of μ_2 for which $\beta_2 \leqslant 1$. Hence it follows that β_1 is considerably less than β_2. Therefore, in the first approximation we assume that

$$\beta_1 \ll \frac{1}{\beta_1},\quad \beta_1 + \frac{1}{\beta_1} \approx \frac{1}{\beta_1},\quad \beta_2 + \frac{1}{\beta_2} \approx 2. \qquad (38.13)$$

Besides, in that approximation we shall take

$$\frac{1}{\beta_1} = \frac{n^2\varepsilon(1+\mu_1^2)^2}{\mu_1^2} \approx \frac{n^2\varepsilon}{\mu_1^2},$$

neglecting μ_1^2 in comparison with unity. Thus, we obtain the approximate equation

$$\frac{2\tau}{Et\varepsilon} \approx -\frac{n^2\varepsilon}{\mu_1} \approx -2\mu_2, \quad n^2\varepsilon = 2\mu_2(\mu_2 + 2\mu_0),$$

from which we find in the first approximation

$$\mu_2 = -\mu_0 - \sqrt{\mu_0^2 + 0.5n^2\varepsilon},$$

$$\frac{\tau}{Et\varepsilon} = \mu_0 + \sqrt{\mu_0^2 + 0.5n^2\varepsilon} = \frac{m_0 + \sqrt{m_0^2 + 0.5n^4\varepsilon}}{n}.$$

The value of n^4 at the critical load is determined from the equation $\partial\tau/\partial n = 0$. It is

$$n^4 = 6m_0^2/\varepsilon. \tag{38.14}$$

Here*, denoting the quantity τ in the first approximation by τ_1, we have:

$$\tau_1 = 1.35\sqrt{2m_0}\,Et\varepsilon^{5/4}, \quad -m_2 = 3m_0, \quad -m_1 = m_0 = \pi R/L. \tag{38.15}$$

To obtain the solution in the second approximation, we introduce (38.14) and (38.15) in those terms of equation (38.12) which were neglected in the first approximation or which were determined inexactly and use the condition (39.11). Thus we find

$$\mu_1\left(\beta_1 + \frac{1}{\beta_1}\right) \approx \frac{1}{\mu_1}\left[n^2\varepsilon(1+2\mu_1^2) + \frac{\mu_1^4}{n^2\varepsilon}\right] \approx$$
$$\approx \frac{1}{\mu_1}(n^2\varepsilon + 2m_0^2\varepsilon + 0.068\,m_0\sqrt{\varepsilon});$$
$$\mu_2\left(\beta_2 + \frac{1}{\beta_2}\right) \approx \mu_2(2.167 - 6.12\,m_0\sqrt{\varepsilon} + 70.8\,m_0^2\varepsilon).$$

Introducing these expressions in (38.12) we obtain the corrected equation for determining μ_2

$$2\mu_2(\mu_2 + 2\mu_0)\,a_1 = n^2\varepsilon + a_2,$$

where

$$a_1 = 1.083 - 3.06m_0\sqrt{\varepsilon} + 35.4m_0^2\varepsilon; \quad a_2 = 2m_0^2\varepsilon + 0.068m_0\sqrt{\varepsilon}.$$

Consequently,

$$\mu_2 = -\frac{m_0}{n}\left[1 + \sqrt{1 + \frac{1}{2a_1m_0^2}(n^4\varepsilon + a_2n^2)}\right].$$

From condition $\partial\tau/\partial n = 0$ or, what amounts to the same, from equation $\partial\mu_2/\partial n = 0$, we obtain

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* The formula (38.15) was obtained by Kh. M. Mushtari in /IX. 7/. See also /0. 13/.

$$\frac{n^4\varepsilon}{4a_1 m_0^2} = 1.5 + \frac{0.5a_2}{n^2\varepsilon}, \quad \frac{n^4\varepsilon}{4m_0^2 a_1} \approx 1.5 + \frac{0.4m_0\sqrt{\varepsilon}+0.014}{\sqrt{a_1}}.$$

Hence,

$$\mu_2 \approx -\frac{1.35\sqrt{2m_0}\sqrt[4]{\varepsilon}}{a_1^{1/4}}\left(1+\frac{0.2m_0\sqrt{\varepsilon}+0.007}{\sqrt{a_1}}\right),$$
$$\frac{2\tau_2}{Et\varepsilon} = -\mu_2\left(\beta_2+\frac{1}{\beta_2}\right),$$

where τ_2 is the quantity τ in the second approximation.

Calculations show that for the quantities m_0 which satisfy the condition

$$0.03 \leqslant m_0\sqrt{\varepsilon} = 2.39\theta \leqslant 0.12, \quad \theta = \sqrt{R} : [L\sqrt{2}(1-\nu^2)^{1/4}], \tag{38. 16}$$

τ_2 exceeds τ_1 by less than 6%. The succeeding approximations somewhat reduce this difference. Therefore, admitting the error indicated, with the shell parameter satisfying the condition (38. 16), we shall take for the critical shearing stress

$$\tau_k^0 = \tau_1 = 1.21\,Et\,(t/2R)^{3/4}\,(2R/L)^{1/2} : (1-\nu^2)^{5/8}. \tag{38. 17}$$

Turning to the investigation of the general case, when at least one of the quantities λ_3 and λ_2 is not equal to zero, we shall nevertheless assume that the loss of stabili is caused by the torsion and that the stresses T_1^I and T_2^I are either tensile or com pressive, but play a relatively small role in the buckling of the shell*. Denoting t T_{1k}^I, T_{2k}^I, T_{12}^I the critical stresses under pure axial compression, pure outward norm pressure and pure torsion respectively, from (35. 10), (36. 20), and (38. 17) we find

$$|T_{1k}^I| \approx 0.7\theta^{-1/2}\,|T_{12}^I|, \quad |T_{2k}^I| \approx 3.2\theta^{1/2}\,|T_{12}^I|.$$

Besides, according to (38. 14) and (38. 15)

$$-\mu_1 \approx \mu_0 \approx \theta^{1/2}, \quad \mu_2 \approx 3\mu_1.$$

Consequently, all the terms of the expression $\lambda_3\mu_1^2-\mu_1+\lambda_2$ are approximately the same if λ_3 and λ_2 take values which differ little from unity.

In order for the compressive forces to play only an auxiliary role in comparison with the shearing stress, the conditions $|\lambda_3\mu_1| \approx \lambda_3\theta^{1/2} \ll 1$, $\lambda_2 < \theta^{1/2}$ have to be satisfied. The first of these two conditions is most essential, as when it is not observed, the stability of the shell can be lost under axial compression with the forming of many half-waves along the length of the shell, which would contradict the assumed form of the buckling. We shall consider λ_3 and λ_2 as sufficiently small, so that to the first approximation one could use the approximate equalities (38. 13). Then it follows from (38. 4) that

$$\frac{2\tau}{Et\varepsilon} \approx \frac{n^2\varepsilon}{\lambda_3\mu_1^2-\mu_1+\lambda_2} \approx \frac{2\mu_2^2}{\lambda_3\mu_2^2-\mu_2+\lambda_2}. \tag{38. 18}$$

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* The exposition of this question is given here as a natural development of our article /IX. 7/. An analogous setting of the problem was considered by V. M. Darevskii /IX. 14/ in a work whose text, in the form of a published article, is not known to us.

Whence, after the substitution $\mu_1 = \mu_2 + 2\mu_0$ we obtain an equation of the fourth degree in μ_2, from which one can find μ_2 and thereupon τ, as functions of n. In view of the complexity of this direct method, we shall seek the solution of the problem by an indirect method, setting

$$\lambda_3 = -\gamma_3 : \mu_2, \quad \lambda_2 = -\gamma_2\mu_2, \tag{38. 19}$$

where γ_3 and γ_2 have yet to be determined.

As λ_2, λ_3, γ_3, and γ_2 are given quantities, while μ_2 is a quantity to be determined from the condition of minimum critical stress, the proposed substitution (38.19) has no meaning if the matter concerns an "exact" determination of the stress. If one sets oneself the problem of approximately determining the influence of the axial compressive stress and external pressure (characterized by parameters γ_3 and γ_2, small in comparison with unity) upon the value of the critical shearing stress τ, then in the expressions (38. 19) for μ_2 one can take the corresponding quantity $\mu_2{}^c$, calculated for pure shear with $p = p_1 = 0$, which does not vary in the case of the action of combined stresses. With small γ_3 and γ_2 the actual critical value of μ_2 will differ little from $\mu_2{}^c$; therefore, the introduction of (38. 19) in the small terms of the equations (38. 18) will produce in these equations an error of the second order of magnitude for γ_3 and γ_2 in comparison with unity. If, in spite of this, one considers that near the minimum $\tau(\mu_2)$ the error in the value of μ_2 has a relatively small effect upon τ, it may be expected that even with values of γ_3 and γ_2 of the order of 0.2-0.3, the critical stress will be found with sufficient accuracy.

Thus we bring (38. 18) into the form

$$\frac{2\tau}{Et\varepsilon} = -\frac{n^2\varepsilon}{\gamma_3\frac{\mu_1^2}{\mu_2} + \mu_1 + \gamma_2\mu_2} = -\frac{2\mu_2}{\gamma}, \quad \gamma = \gamma_3 + 1 + \gamma_2. \tag{38. 20}$$

The negative root of this equation is

$$\mu_2 = -\frac{m_0}{n\gamma}\left\{1 + 2\gamma_3 + \sqrt{(1 + 2\gamma_3)^2 + \gamma\left(\frac{n^4\varepsilon\gamma}{2m_0^2} - 4\gamma_3\right)}\right\}. \tag{38. 21}$$

From the minimum condition for τ (or $|\mu_2|$) we find

$$\frac{n^4\varepsilon\gamma^2}{2m_0^2} = 1.5\,(1 + 2\gamma_3^2) - 4\gamma_3\gamma +$$

$$+\sqrt{[1.5(1 + 2\gamma_3)^2 - 4\gamma\gamma_3]^2 - 4\gamma\gamma_3[4\gamma - (1 + 2\gamma_3)^2]} = 3\,(1 + \delta_1). \tag{38. 22}$$

Thus the critical shearing stress in the case of simultaneous action of normal stresses is

$$\tau_k = \tau_k{}^0\alpha, \tag{38. 23}$$

where

$$\alpha = [1 + 2\gamma_3 + \sqrt{(1 + 2\gamma_3)^2 + 3\,(1 + \delta_1) - 4\gamma\gamma_3}] : [3\gamma^{3/2}(1 + \delta_1)^{1/4}]. \tag{38. 24}$$

Taking various values of γ_3 and γ_2, we determine the corresponding δ_1 from (38. 22) and then we find α from (38. 24). Then, from (38. 22) and (38. 21) we find $|\mu_2|$ as the quantity proportional to $\sqrt{m_0\sqrt{\varepsilon}}$ and from (38. 19) we determine the corresponding λ_3 and λ_2. The values thus found for α, λ_3, and λ_2 are given in Table III.

Table III

α	0.92	0.86	0.807	0.76	0.72
$\lambda_2\sqrt{\theta}$	0.033	0.065	0.096	0	0.037
$\lambda_2/\sqrt{\theta}$	0	0	0	0.54	0.552
$-\mu_2/\sqrt{\theta}$	3.00	3.05	3.10	2.70	2.76

Knowing the geometrical parameter θ of the shell and the relations (38.3), from the tabular data one can easily construct curves bounding the region of stability of the shell under the action of combined stresses. Here it is necessary to remember that in the determination of the critical load the state of the shell before the loss of stability was taken to be a membrane state and that only one boundary condition (38.6) is fulfilled. This latter is the most essential boundary condition, if one is considering the stability of a cylindrical shell supported by frames which are rigid against bending in their plane but are only weakly resistant to torsion, where the skin can slide along the frames, and the segments between the frames are shells of medium length. If these conditions are not satisfied, then the simple solution given above cannot be considered as applicable.

Some attempts are known from literature of obtaining a thoroughly well-founded solution of the stability problem under pure torsion, convenient also for shorter shells. The first solution of this kind was proposed by L. Donnell, simultaneously with our solution* given above. Taking the displacement components (38.1), Donnell also obtains the characteristic equation of the eighth degree in m. Then he sets

$$m^4/(m^2+n^2)^2 \simeq m^4/n^4$$

and thus replaces equation (38.2) by an approximate equation of the fourth degree in m, which is equivalent to neglecting m^2 in comparison with n^2.

By further admitting such an error in the boundary conditions, Donnell manages to satisfy the boundary conditions of free support or clamping and after a laborious graphical computation, derives approximate formulas for the determination of the critical shearing stress:

$$A=\frac{(1-\nu^2)\,|T_{12}^1|\,L^2}{Et^3}=4.6+\sqrt{7.8+\frac{0.21}{\theta^3}} \qquad (38.25)$$

(when the edges are clamped, if $L^2t<62.4\sqrt{1-\nu^2}R^3$),

$$A=2.8+\sqrt{2.6+0.15/\theta^3} \qquad (38.26)$$

(when the edges are supported, if $L^2t<44\sqrt{1-\nu^2}\cdot R^3$).

Hence, in the case of shells of medium length, when $\theta\ll 1$, we obtain the formulas

$$\frac{\tau_k^0}{t}\simeq\frac{1.83E}{(1-\nu^2)^{5/8}}\left(\frac{t}{2R}\right)^{5/4}\left(\frac{R}{L}\right)^{1/2} \text{ (with clamped edges)}$$

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* See articles /IX.10/ and /IV.6/, and also the monograph of S. P. Timoshenko /0.16/.

$$\frac{\tau_k^0}{t} \approx \frac{1.67E}{(1-\nu^2)^{5/8}}\left(\frac{t}{2R}\right)^{1/4}\left(\frac{R}{L}\right)^{1/2} \text{ (with supported edges)}$$

The latter differs from our formula for the first approximation (38.17) only by a numerical coefficient which is almost 2% smaller than ours.

The validity of these formulas for shells of medium length was also demonstrated in another way in the work of N. A. Alumyae /IX. 13/. In the case of shorter shells the formulas (38.25) and (38.26) cannot be replaced by the simplified formulas (38.25a) and (38.26a). In that case, however, Donnell's fundamental formulas also turn out to be dubious, as they are obtained by neglecting m^2 in comparison with n^2. Apparently, taking this fact into consideration, S. B. Batdorf and M. Stein found it necessary to study anew the stability of a cylindrical shell under torsion*. In the case of supported edges they took the deflection function in the form of a series

$$w = \sum_{q=1}^{\infty}\left(a_q \sin\frac{ns}{R} + b_q \cos\frac{ns}{R}\right)\sin\frac{q\pi x}{L}, \tag{38.27}$$

each of whose terms satisfies the conditions $w = \frac{\partial^2 w}{\partial x^2} = 0$ for x = 0 and x = L, where each term of the series for v determined from equation (35.18) satisfies the boundary conditions

$$v = 0, \; T_1 = 0, \; \oint T_{12}\,ds = 0.$$

For the case of clamping it was assumed that

$$w = \sum_{0}^{\infty}\left\{\left(a_q \sin\frac{ns}{R} + b_q \cos\frac{ns}{R}\right)\left[\cos\frac{q\pi x}{L} - \cos(q+2)\frac{\pi x}{L}\right]\right\}. \tag{38.28}$$

Each term of this series satisfies the conditions

$$w = 0, \; \frac{\partial w}{\partial x} = 0, \; u = 0, \; \oint T_1\,ds = 0, \; \oint T_{12}\,ds = 0 \quad \text{for } x = 0 \text{ and } x = L. \tag{38.28}$$

Introducing these expressions in (35.21), we find the corresponding stress function ψ, and thereupon we substitute $T_2 = \partial^2\psi/\partial x^2$ in (35.19). We integrate the expression thus obtained with respect to w by the Bubnov-Galerkin method; we multiply the left-hand term of the equation by δw and after the substitution of the expression (38.27) or (38.28) we integrate over the shell surface. Then, equating to zero the coefficients of δa_q and δb_q, we find the infinite system of homogeneous equations in a_q and b_q, the consistency condition of which gives the characteristic equation, determining the relation between the critical stress and the number of waves n on the circumference. In the work /IX. 12/ one considers the loss of stability under the action of torsion and axial compression. If the main role is played by torsion, one can limit oneself to the second approximation and determine the critical shearing stress from the formulas

$$\left\{\frac{12\tau_k L^2(1-\nu^2)}{E t^2 \pi^2}\right\}^2 = \frac{Q_1 Q_2 Q_3}{1.44Q_1 + 0.444Q_3} \tag{38.29}$$

(for supported edges)

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* See /IX. 11/ and /IX. 12/.

$$\left\{\frac{12\tau_k L^2(1-\nu^2)}{Et^2\pi^2}\right\}^2 = \frac{(Q_1+Q_3)[2Q_0(Q_2+Q_4)+Q_2Q_4]}{1.47Q_2+4.37Q_4+22.6Q_0} \qquad \textbf{(38.30)}$$

(for clamped edges), in which

$$Q_r = \frac{\pi}{8\mu}\left[(r^2+\mu^2)^2 + \frac{3r^4}{\pi^4(r^2+\mu^2)^2\theta^4} + p_1\frac{L^2}{D\pi^2}r^2\right],$$

$$\mu = \frac{Ln}{\pi R},\quad r = 0,\ 1,\ldots,$$

the value of μ, corresponding to the minimal stress, is determined by appropriate choice.

To the first approximation

$$\mu = 1/\mu_0 \simeq \theta^{-1/2}.$$

Besides, it is assumed that the shell is not very short, so that the condition

$$\theta \leqslant 0.1. \qquad (38.31)$$

is satisfied. For comparison we shall give the critical stresses of pure shear, calculated according to different formulas for a shell, whose geometrical parameters lie near the upper limits of the region of variation of θ, as determined by the conditions (38.16). Denoting the values of the quantity A, calculated according to Donnell's formulas (38. 25), (38. 26), according to our formula (38. 17), and according to Batdorf's formulas (38. 29) and (38. 30) by A_D^3, A_D^0, A_M, A_B^0, and A_B^3 respectively, we find

θ	A_D^3	A_D^0	A_M	A_B^0	A_B^3
0.0512	45.0	40.1	36.9	36.7	39.8
0.10	20.0	17	13.5	14.35	16.2

Here it must be noted that the approximate values of critical stress found by Batdorf by the Bubnov-Galerkin method—which is essentially equivalent to the Rayleigh-Ritz method--are overestimates, since as one increases the number of terms calculated in the series (38. 27) or (38. 28), one increases the number of possible displacements of the shell which lead to the loss of stability of its equilibrium state. Consequently, from the table appended it can be seen that already at the boundary of the region (38. 16), Donnell's formulas give an overestimate of the critical stress by at least 12%. At the boundary of the region (38. 31) this error reaches 20%; therefore, taking all the aforesaid into consideration, we propose to determine the critical linear stress of the membrane equilibrium state under torsion according to our formulas (38. 17) or (38. 23), provided that θ satisfies the condition (38. 31). To evaluate the approximate formula (38. 23), proposed by us for the combined action of stresses with dominant torsion, we shall compare the value of α according to Table III for $\theta = 0.0512$, $\lambda_3 = 0.42$, $\lambda_2 = 0$ with the results of calculations according to Batdorf's formula (38. 29) applied to that case. It turns out that the value of τ_k, found according to the latter formula, is by only 2% smaller than the value found according to formula (38. 23), although there $\gamma_3 \approx 0.3$.

Chapter X

THE STABILITY AND LARGE DEFLECTIONS OF CLOSED CYLINDRICAL SHELLS OF CIRCULAR CROSS-SECTION WITH INITIAL IRREGULARITIES

§ 39. The Concept of Upper and Lower Critical Stresses

In the preceding chapter we have considered the stability of shells having an ideally cylindrical form, where we assumed that the stressed state of the shell before the loss of stability may be assumed to be a membrane state. In particular, we neglected the influence of clamping the shell edges upon the equilibrium state σ^I before the loss of stability of that state. In this way we managed to linearize the equations of neutral equilibrium of the shell not only with respect to the components of additional displacements, but also with respect to the components of displacements before the loss of stability. We shall call the critical stresses found in this way the "upper" critical stresses. Obviously they limit the value of the critical stress from above, as in reality the shells can have initial deviations from the shape considered, which facilitate buckling, as well as initial deflections from the load even before the loss of stability. Besides, we did not consider the possible dynamic character of the stress, which also promotes the loss of stability of the shell equilibrium. Because of this, the upper critical stresses σ_K^u, found theoretically, turn out to be much greater than the experimental values of critical stress σ_K^E. The ratios of $\sigma_K^E:\sigma_K^u$ under axial compression are shown in **Figure 21.**

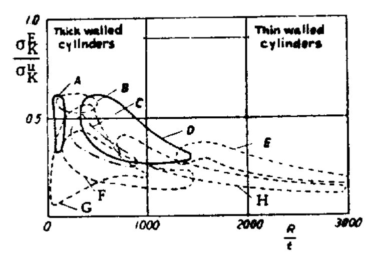

Figure 21

A--celluloid, B--steel, C—steel and bronze, D--duraluminum, E--steel, F--steel, G--bronze, H--steel.

Here the closed solid and dotted lines are the limits of the region of location of the experimental points, taken from the works of various authors* where we excluded the experiments in which the stability of the shells was known to be lost under plastic deformations.

Analogous results were arrived at by A. S. Vol'mir /X. 8/ and L. R. Ispravnikov /X. 12/.

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* See work /X. 14. / and the literature cited therein.

For torsion we see the experimental coefficient, according to Donnell's data, to be 0.60-0.75; with all-round compression it is, according to Ebner's data, 0.70-0.75.

In all cases the loss of stability in the region of elastic deformations occurs discontinuously (by jumps), where part of the energy is transformed into the energy of sound waves, heard in the form of a sharp "snap".

One should also note the fact that the form of buckling observed with loss of stability differs from the sinusoidal waveform on the entire middle surface of the shell predicted by the linearized theory. According to that theory, the amplitudes of the buckles directed from the center of curvature of the shell surface are equal to the amplitudes of the buckles directed toward the center of curvature, while experiments show that the shell "prefers" to buckle inwards. Besides, a noticeable buckling is most often observed only on a portion of the middle surface (Figure 22).

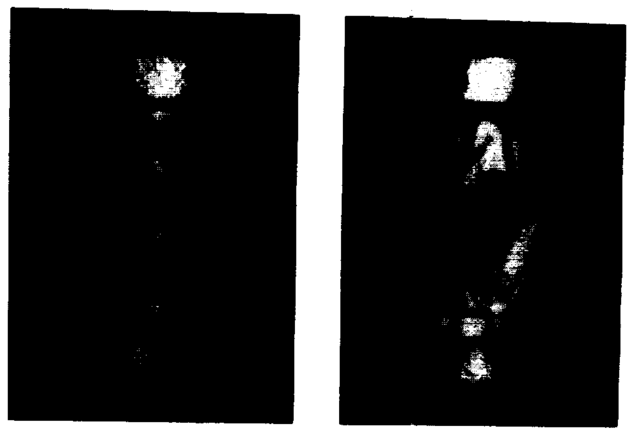

Figure 22

a--buckling under axial compression, b--under torsion

In order to explain the discrepancy between the results of calculations according to the linear theory and those of experiment, it was proposed to make the theory more exact by considering the phenomena of stability loss of the shell and its subsequent deformation with the help of the non-linear theory of shells. Here investigations are possible in three directions. The most important and difficult of these is the investigation of large deflections and the stability loss of a shall having initial irregularities in shape or initial stresses. Particularly important is the determination of the most unfavorable forms of irregularities which lower the stability of shells. Equally interesting, in our opinion, is the determination of critical stresses while taking account of the unquestionable fact that the actual state of the shell before the loss of stability should not be considered as a membrane state even in the case when, before applying the load, the shell has the ideally correct form. Finally, a question of no minor importance is the investigation of the state of a shell of ideal shape after the loss of stability with the aim of determining the minimal stress which the shell is capable of supporting after the loss of stability. If such a minimum exists, then after decreasing the stress to that minimum the buckled equilibrium shape becomes unstable and a so-called collapse occurs, i. e., a discontinuous passage to the initial membrane form of equilibrium. The stress at which this collapse phenomenon occurs will be called the lower critical stress, and the corresponding stress we shall denote by σ_K^H.

For an actual shell

$$\sigma_K^H < \sigma_K^E < \sigma_K^u.$$

We devote the following sections to the investigations of the problems indicated, in the measure in which this is rendered possible by the present state of our knowledge about so complex a question.

§ 40. Lower Critical Stress under Longitudinal Compression. A Necessary Modification of the Ritz-Timoshenko Method

The problem of determining the lower critical stress in the case of longitudinal compression of a circular cylindrical shell was posed by von Karman and Tsien /X.1/. However, in view of the fact that these authors, starting from the principle of virtual displacements and solving the problem according to the Ritz-Timoshenko method, varied the total potential energy of the system only with respect to the deflection amplitude without considering the variation in length and width of the buckles formed, the lower critical axial load, found by them, turned out to be a tensile force. A more well-founded theoretical determination of the lower critical stress was given in the articles /X.2/, /X.3/, and /X.4/.

Passing to the exposition of that solution, we assume that besides the longitudinal compressive load, an interior normal pressure $p<0$ also acts upon the shell. We shall make use of the notation of § 35. We shall determine the state of a shell σ^I, characterized by a finite deflection w^I, where for brevity we shall omit the superscript I. Let the shell have no initial irregularities. By a and b we shall denote the lengths of half-waves formed by buckling in the axial and circumferential directions respectively.

$$m=\pi R/a,\ n=\pi R/b,\ \mu=m/n. \tag{40.1}$$

The potential energy of elongation and bending, the work of the axial force, applied to the ends of the shell, and the work of the internal pressure acting on the surface of a whole wave, are defined just as in §§ 29-30.

They are equal, respectively, to the quantities:

$$\begin{gathered}\mathcal{E}_n=\frac{2}{Et}\int_0^a\int_0^b\Big\{\Big(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial s^2}\Big)^2-\\-2(1+\nu)\Big[\frac{\partial^2\psi}{\partial x^2}\cdot\frac{\partial^2\psi}{\partial s^2}-\Big(\frac{\partial^2\psi}{\partial x\partial s}\Big)^2\Big]\Big\}dxds,\\ \mathcal{E}_{\text{bend}}=2D\int_0^a\int_0^b\Big\{\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial s^2}\Big)^2-\\-2(1-\nu)\Big[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial s^2}-\Big(\frac{\partial^2 w}{\partial x\partial s}\Big)^2\Big]\Big\}dxds,\\ W_1=4\int_0^b(T_1)_{x=a}ds\int_0^a\frac{\partial u}{\partial x}dx,\quad W_2=-4\int_0^a\int_0^b wpdxds.\end{gathered}\tag{40.2}$$

The condition of compatibility of deformations (35.15) and the equation of equilibrium (35.3) take the forms

$$\Delta\Delta\phi=Et\Big[\Big(\frac{\partial^2 w}{\partial x\partial s}\Big)^2-\frac{\partial^2 w}{\partial x^2}\Big(\frac{\partial^2 w}{\partial s^2}-k\Big)\Big],\ k=1/R, \tag{40.3}$$

$$D\Delta\Delta w=\frac{\partial^2\psi}{\partial s^2}\cdot\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial s}\cdot\frac{\partial^2 w}{\partial x\partial s}+\frac{\partial^2\psi}{\partial x^2}\cdot\frac{\partial^2 w}{\partial s^2}-k\frac{\partial^2\psi}{\partial x^2}-p. \tag{40.4}$$

Substituting in the left-hand member of (40. 3) for w the solution of the linearized equations of equilibrium as a first approximation,

$$w_1 = g_{01} + g_{11} \cos mkx \cos nks,$$

we find the stress function ψ in the first approximation and then, introducing w_1 and ψ_{11} in the right-hand member of (40. 4) we determine w in the second approximation. Thus, we find an expression of the form

$$w = g_0 + g_1 \cos mkx \cos nks + g_2 \cos 2mkx + g_3 \cos 2nks + \\ + g_4 \cos mkx \cos 3nks + g_5 \cos 3mkx \cos nks. \quad (40.5)$$

In article /X.1/, starting from experimental considerations, the expression (40.5) was taken with

$$g_4 = g_5 = 0. \quad (40.6)$$

In what follows we shall take, for simplicity, as in article /X.2/,

$$g_2 = g_3. \quad (40.7)$$

Whereby, introducing (40.5) in equation (40.3), we find the particular integral

$$\psi = -Et\cdot\frac{\mu^2 R^2}{n^2}\Big[\frac{A}{16\mu^4}\cos\frac{2mx}{R} + \frac{B}{16}\cos\frac{2ns}{R} + \\ + \frac{C}{(1+\mu^2)^2}\cos\frac{mx}{R}\cos\frac{ns}{R} + \frac{F}{(1+9\mu^2)^2}\cos\frac{3mx}{R}\cos\frac{ns}{R} + \\ + \frac{G}{(9+\mu^2)^2}\cos\frac{mx}{R}\cos\frac{3ns}{R} + \frac{H}{16(1+\mu^2)^2}\cos\frac{2mx}{R}\cos\frac{2ns}{R}\Big] + \\ + \frac{p_2 x^2}{2} + \frac{p_1 s^2}{2}, \quad (40.8)$$

where

$$A = \frac{1}{2} n^2 g_1^2 + 4g_2, \quad B = \frac{1}{2} g_1^2 n^2, \quad C = 4g_1 g_2 n^2 + g_1, \\ F = G = 2g_1 g_2 n^2, \quad H = 16 g_2^2 n^2. \quad (40.9)$$

To (40.8) one could have added an arbitrary biharmonic function; however there is no necessity for it, as of all the boundary conditions we shall satisfy only the condition that the mean longitudinal stress be equal to the external stress. Let the latter be equal in absolute value to T_0. Then, according to formula (40.8)

$$T_1 = -T_0 = \frac{\partial^2\psi}{\partial s^2} = p_1. \quad (40.10)$$

As with the buckling of a cylindrical shell under longitudinal compression many waves appear in the axial as well as the circumferential direction, and the influence of the boundary conditions is attenuated already within the limits of one half-wave, neglecting the latter hardly affects the total potential energy of the shell and the critical stress, which allows one to solve the problem in the simplified setting indicated. We further find

$$Et\frac{\partial v}{\partial s} = Et\Big[\varepsilon_2 - kw - \frac{1}{2}\Big(\frac{\partial w}{\partial s}\Big)^2\Big] = p_2 + \nu T_0 - \\ - Etn^2\Big(\frac{1}{8} g_1^2 + g_2^2\Big) - Etg_0 + \text{periodic terms}$$

As in the case of an entire shell v should be a periodic function of s, we have

$$p_2 + \nu T_0 - Etn^2\left(\frac{1}{8} g_1^2 + g_2^2\right) - Etg_0 = 0. \quad (40.11)$$

According to (40. 2), (40. 5), and (40. 10) we find the total energy of the shell acting on the surface of one wave

$$Э = Э_м + Э_{\text{bend}} - W_1 - W_2. \quad (40.12)$$

This quantity should be a minimum, therefore $\frac{\partial Э}{\partial g_0} = 0$, or

$$-tg_0 + \nu T_0/E - pR/E - tn^2\left(\frac{1}{8} g_1^2 + g_2^2\right) = 0.$$

Comparison of this with (40. 11) shows that

$$p_2 = -pR, \quad g_0 = -n^2\left(\frac{1}{8} g_1^2 + g_2^2\right) + (-pR + \nu T_0)/Et. \quad (40.13)$$

Eliminating g_0 we find

$$\begin{gathered} Э_1 = \frac{2Э}{Etab} = -\frac{4T_0^2}{E^2t^2} - \left(\frac{2pR}{Et}\right)^2 + \frac{8pR\nu T_0}{E^2t^2} - \\ - \frac{n^2}{Et}(pR + \mu T_0)(g_1^2 + 8g_2^2) + [A_1g_1^4 + B_1g_1^2g_2^2 + C_1g_2^4 + \\ + t^2k^2(D_1g_1^2 + G_1g_2^2)]n^4 + F_1n^2g_1^2g_2 + 2g_2^2 + H_1g_1^2, \end{gathered} \quad (40.14)$$

where we have introduced new notations

$$\begin{gathered} A_1 = (1 + \mu^4) : 32, \quad C_1 = 16H_1 = 16\mu^4 : (1 + \mu^2)^2, \quad F_1 = (1 + C_1) : 2, \\ B_1 = C_1 + \frac{4\mu^4}{(1 + 9\mu^2)^2} + \frac{4\mu^4}{(9 + \mu^2)^2}, \quad D_1 = \frac{(1 + \mu^2)^2}{12(1 - \nu^2)} \\ G_1 = \frac{16(1 + \mu^4)}{6(1 - \nu^2)}. \end{gathered}$$

We further set up the equations

$$\frac{\partial Э_1}{\partial g_1} = \frac{\partial Э_1}{\partial g_2} = \frac{\partial Э_1}{\partial n} = \frac{\partial Э}{\partial \mu} = 0 \quad (40.15)$$

and form their different combinations just as was done in /X. 2/ for the case p = 0; we introduce the dimensionless quantities

$$\begin{gathered} \varphi = \frac{\mu^2}{(1 + \mu^2)^2 n^2} \cdot \frac{R}{t}, \quad \gamma = -g_2n^2, \quad \theta = -\frac{n^2g_1^2}{8g_2}, \quad T_0^* = \frac{T_0R}{Et^2}, \\ p^* = \frac{pR^2}{Et^2}, \quad A_2 = \frac{2\mu^4}{(1 + \mu^2)^2}, \quad B_2 = 16 + 4\left(\frac{1 + \mu^2}{1 + 9\mu^2}\right)^2 + 4\left(\frac{1 + \mu^2}{9 + \mu^2}\right)^2, \\ C_2 = 1 + \mu^4, \quad D_2 = 4C_2/(1 + \mu^2)^2 \quad E_2 = 2C_2/D_2. \end{gathered} \quad (40.16)$$

Thus we obtain the following equations relating the critical stress parameters and the buckle shapes:

$$\begin{gathered} [A_2/12(1 - \nu^2)\varphi^2]^2(D_2 - 1) + [A_2/12(1 - \nu^2)\varphi^2]\left\{\frac{1}{2} - A_2D_2 + \right. \\ \left. + (2D_2 + 1)\left(\frac{1}{2} + 4A_2\right)\gamma - [D_2(A_2B_2 - C_2) + A_2(B_2 - 4)]\gamma^2\right\} - \\ - \left\{A_2\left(\frac{1}{2} - A_2\right) + 3A_2\left(\frac{1}{2} + 4A_2\right)\gamma - \left[\frac{1}{2}(A_2B_2 - C_2) + \right.\right. \\ \left.\left. + \frac{1}{2}(1 + 8A_2)^2 + A_2^2(B_2 - 4)\right]\gamma^2 + A_2\left(\frac{1}{2} + 4A_2\right)(B_2 - 4)\gamma^3\right\} = 0. \end{gathered} \quad (40.17)$$

$$\theta = 1 + A_2 \{[(D_2 - 1)/12(1-\nu^2)\varphi^2] + (4 - B_2)\gamma^2 + \\ + (8 + B_2 - E_2)\gamma - 5\} : \{0.5 + 4A_2 - (A_2B_2 - C_2)\gamma\}. \tag{40.18}$$

$$-\frac{p^*(1+\mu^2)}{2\mu^2\varphi}\left(1+\frac{\gamma}{\theta}\right) + \frac{1}{12(1-\nu^2)\varphi^2}\left[1 + \frac{4\gamma}{(1+\mu^2)\theta}\right] = 1 + 2\gamma^3\frac{\mu^2}{\theta} + \\ + \mu^2\gamma^2\left[16 + 36\left(\frac{1+\mu^2}{9+\mu^2}\right)^3 + 4\left(\frac{1+\mu^2}{9+\mu^2}\right)^3\right] = \\ = \left[\frac{(1+\mu^2)^3(1-\theta)}{4\mu^4} - 4(1-\mu^2)\right]\gamma. \tag{40.19}$$

$$T_0^* + \frac{p^*}{\mu^2} = \varphi\left\{\frac{1}{12(1-\nu^2)\varphi^2} + 1 - \left(8 + \frac{1}{A} - \frac{C_2\theta}{A_2}\right)\gamma + B_2\gamma^2\right\}. \tag{40.20}$$

For a given p* the problem consists in finding the smallest of all the values of T_0^* for which the system of equations (40. 17)-(40. 20) in the parameters μ, φ, γ, and θ has real roots, satisfying the conditions

$$\varphi > 0, \ \gamma/\theta > 0.$$

This problem may be solved by the following semi-inverse method: 1) we take some value of μ^2; 2) from (40. 16) we calculate the corresponding values of $A_2, \ldots, E_2$; 3) we take a series of values of γ; 4) from (40. 17) we determine for a given γ two possible values of $\varphi > 0$; 5) from (40. 18) we find the corresponding θ's; 6) from the totality of values of γ, φ, and θ we choose those for which the equation (40. 19) is satisfied; 7) substituting them in (40. 20) we find T_0^*; 8) repeating the calculation for various μ^2, we find the smallest T_0^* for a given p*.

With p* = 0, these very laborious computations were carried out in /X. 2/. It turned out that

$$\min(T_0^*) \approx 0.195 \quad \text{for} \quad \mu \approx 0.4. \tag{40.21}$$

As is well known, the parameter of upper critical stress is

$$T_0^{**} \approx 0.606 \quad (\text{for } \nu = 0.3).$$

Thus, the lower critical stress at which the return "collapse" occurs for an ideal cylindrical shell is less than one third of the critical stress for the start of the "snap".

In contrast to the above-stated process of solution of the problem given, von Karman and Tsien took (in /X. 1), as has been indicated above), as the stationarity conditions of the total energy in the state of equilibrium, the equations

$$\frac{\partial Э_1}{\partial g_0} = \frac{\partial Э_1}{\partial g_1} = \frac{\partial Э_1}{\partial g_2} = 0,$$

varying only the looked-for deflection amplitudes of the assumed wavelengths. Then, utilizing the relations obtained, they found an expression for the stress parameter T_0^* in terms of the parameters m and n characterizing the buckle shape. Here, one had in view the minimization of T_0^* by varying these parameters, without caring whether the obtained values of $T^*_{0\,\min}$, m, and n correspond to the stationary value of the total energy. Thus, the problem of determining the minimum T_0^* while observing the minimization condition for the total energy was replaced by the problem of finding the absolute minimum of T_0^*. In the case considered it turned out that with changing μ the quantity T_0^* decreases monotonically, subsequently assuming negative values as well. Therefore, in the article indicated it had been proposed, based on experimental data, to take μ = 1, as a result of which the value T_0^* = 0.196 was obtained. It almost coincides with the value (40. 21). However, this solution of von Karman and Tsien may not be considered as theoretically well-founded, since it had been found in a semi-empirical way.

The necessity of changing the form of the usual minimization procedure in the Ritz-Timoshenko method, as shown in this section, was first pointed out in the article of Friedrichs /XIII.4/ without, as it seems to us, sufficient explanation. Some considerations of this question were given in the article /XIII.7/. They will be set forth in Chapter XIII. Here we note in addition that the necessity of minimizing the total energy with respect to the parameters of buckling frequency cannot only be dispensed with, but also becomes entirely meaningless if the buckle shape is given by an infinite trigonometric series. However, when we seek the solution in the form of a trigonometric polynomial with a small number of terms, the problem of choosing the most suitable wavelengths of the solution sought becomes very real, and the usual minimization procedure can lead to quite an incorrect result. To avoid misunderstanding, it is necessary to note that by applying the Ritz-Timoshenko method according to Lagrange's principle, we admit such displacements which can occur without violating the geometrical relations. In the case under consideration, this means that δw, determined from (40.5) with the condition of varying the quantities $g_0, \ldots, g_5$, m, and n, should be a virtual displacement. In particular, for a complete cylinder the condition of periodicity of the displacement components and their variation with respect to the variable s should be fulfilled. It is obvious that the latter condition is not satisfied if one considers the buckling of the entire surface of the shell. But if one assumes that finite deflections extend only over a part of the surface, then m and n will no longer be discretely varying quantities, characterized by the integral numbers of the half-waves which are formed, but will be continuously varying quantities characterizing the sizes of the buckles in the part of the shell under consideration. In this case the question of the periodicity of these or other quantities no longer arises. Unfortunately, however, by approximating the deflection of a part of the shell by means of a periodic function and not considering the attenuation of buckling on the remaining part of the shell, we make the assumption that the total energy corresponding to this zone of the edge effect is negligibly small. Consequently, the solutions obtained in this way must be considered as satisfactory at the present state of the theory only insofar as they still take into consideration the experimentally observed local character of the shell buckling.

In article /X.4/ Kempner utilizes for the approximation to the solution the expression (40.5) with the condition (40.6), assuming that $g_2 \neq g_3$, and finds the value

$$\min(T_0^*) = 0.182 \text{ (for } \mu = 0.362), \tag{40.22}$$

which is only 7 % smaller than the value (40.20).

L. Kirste in the article /X.7/ comes to a result almost the same as (40.22), considering the shell after buckling as a polyhedron freely supported by its ribs on a cylindrical surface. This assumption is lent some support by the fact that a cylindrical surface can apparently be easily deformed with considerable deflections, if its middle surface remains close to a developable surface. In the case of the transformation of a cylinder into a polyhedron, this condition is satisfied (with the exception of angles). Thus we break up the shell into longitudinal strips of length a and width b, which are considered as compressed beams on an elastic foundation, which resists normal displacements, as well as torsion of the beams. The critical compressive load for such a beam according to a formula of article /X.6/ is

$$p_k = \frac{\pi^2 EI}{a^2} + \varkappa \frac{a^2}{\pi^2} + M, \tag{40.23}$$

where I is the moment of inertia of the transverse section of the strip,

$\varkappa$ is the coefficient of normal reaction of the elastic foundation, M is the reaction moment.

If the shell is divided into shallow parts, each strip may be considered as a slightly-bent circular profile. Consequently,

$$I = \frac{2}{45} R^4 t a^5 + \frac{bt^3}{12} = bt\left(\frac{b^4}{720R^2} + \frac{t^2}{12}\right),$$

where t is the shell thickness, a = b/R.

The elastic foundation favors the circumstance that the longitudinal edges remain, in view of the action of the neighboring strips, on the cylindrical surface. Therefore, the strips twist in a transverse direction, where under a sinusoidal deformation a transverse unit strip of length b is under the reaction force

$$Q = \int_0^b D \frac{\partial^4 w}{\partial s^4} ds = \int_0^b D \frac{\pi^4}{b^4} w_0 \sin\frac{\pi s}{b} ds = 2D\frac{\pi^3}{b^3} w_0,$$

where w_0 is the deflection vector. Consequently,

$$\varkappa = 2D\pi^3/b^3, \quad D = Et^3/12(1-\nu^2).$$

Analogously, for the reaction moment we find the expression M = 2Dc/b, where c is some dimensionless coefficient. Substituting these quantities in (40.23) we have

$$T_0 = p_k : b = \frac{E\pi^2 t}{6(1-\nu^2)}\left(\frac{b^4}{120 R^2 a^2} + \frac{t^2}{2a^2} + \frac{t^2 a^2}{\pi b^4} + \frac{ct^2}{\pi b^2}\right). \tag{40.24}$$

Minimizing this quantity with respect to a and b, we find

$$\mu^2 = \frac{b^2}{a^2} = -\frac{c}{\pi}, \quad \frac{b^8}{a^4 R^2 t^2} = 120\left(\frac{1}{\pi} - \frac{c^2}{2\pi^2}\right).$$

$$T_{0,\min} = E\frac{t^2}{R} \cdot \frac{\pi^2}{3(1-\nu^2)} \sqrt{\frac{1}{120}\left(\frac{1}{\pi} - \frac{c^2}{2\pi^2}\right)}$$

For a thin shell $\mu^4 = c^2/\pi^2$ is much less than unity. For example, for the lower critical load, according to (40.22) $c^2 = 0.017\pi^2$; therefore

$$T_{0,\min} \approx 0.187\, Et^2/R. \tag{40.25}$$

This solution, based on a number of assumptions whose validity cannot be strictly proved, nevertheless merits attention, as in it one makes the attempt of constructing an elementary theory of the snap phenomenon, starting from the probable physical picture of that phenomenon.

In conclusion, we direct the reader's attention to the interesting works of Tsien Hsue-Shen /XIII.5/, /X.20/, and /X,21/, in which he considers the determination of the critical load taking into account the rigidity of the experimental apparatus.

§ 41. Determination of the Reduction Coefficient of the Skin of a Supported Shell under Axial Compression

As has been shown in § 1 of /X. 9/, and also in § 31 of this work, for a flat supported plate the exact fulfillment of the conditions of clamping the points of the skin has, under axial compression, only a negligible effect on the value of the reduction coefficient φ. After the loss of stability, the cylindrical thin-walled shell buckles along a wavy surface consisting of a large number of shallow parts, each of which is almost flat. Therefore it may be assumed that the influence of the edge effect is also not large in the case under consideration, and that to the first approximation one can replace with sufficient accuracy the real supported shell by a shell with relaxation links whose skin is clamped to stiffening ribs only at the points of intersection of the latter. The skin of such a shell has more possibilities of absorbing the minimum of stress, transmitting the latter to the stiffening ribs, and therefore its reduction coefficient must be smaller than the reduction coefficient of the skin of an actual shell. Consequently, we should obtain a value for φ leading to a safety factor.

To solve the problem set we shall use the Ritz-Timoshenko method, assuming for the deflection the form*:

$$w = f_1 \sin m\alpha \sin n\beta + f_2 \sin^2 m\alpha \sin^2 n\beta, \tag{41.1}$$

where $\alpha = \frac{\pi x}{a}$, $\beta = \frac{\pi s}{b}$, m, n are integers, while a and b are the distances between the neighboring transverse and longitudinal ribs respectively.

Then the condition

$$w = 0$$

is satisfied at the stiffening ribs, and the clamping condition at the strip vertices, i.e., at $\alpha = l'\pi$, $\beta = l\pi$ (l and l' are integers). At the remaining points of the ribs there is no necessity to satisfy that last condition as ordinarily thin-walled stiffening ribs resist twisting weakly and hinder the rotation of the skin only slightly.

Introducing (41.1) in the equation of compatibility (40.3), we obtain the stress function where, without considering the edge effect, we set the arbitrary biharmonic function equal to $(p_1 y^2 + p_2 x^2)/2$, where p_1 is the mean axial stress, and p_2 the mean annular stress.

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* This buckle shape was assumed by A. S. Vol'mir in work /X. 8/ for a cylindrical strip. In our opinion, it is more suitable for the case we have considered of a supported cylindrical shell or a considerable portion thereof.

Thus we find the expressions for the stresses

$$\begin{aligned}
\frac{\partial^2\psi}{\partial x^2} = p_2 + \frac{\pi^2 Etn^2}{2b^2}\Big\{ -\frac{f_1^2}{4}\cos 2m\alpha + f_2^2\Big[-\frac{\cos 2m\alpha}{4} - \\
-\frac{\delta^4\cos 4m\alpha\cos 2n\beta}{(4\delta^2+1)^2} - \frac{\delta^4\cos 2m\alpha\cos 4n\beta}{4(\delta^2+4)^2} + \frac{\cos 4m\alpha}{16} + \\
+\frac{\delta^4\cos 2m\alpha\cos 2n\beta}{2(\delta^2+1)^2}\Big] + f_1 f_2 \delta^4\Big[\frac{2\sin m\alpha\sin n\beta}{(\delta^2+1)^2} - \frac{3\sin m\alpha\sin 3n\beta}{(\delta^2+9)^2} - \\
-\frac{27\sin 3m\alpha\cdot\sin n\beta}{(9\delta^2+1)^2}\Big] - \frac{2b^2}{\pi^2 Rn^2}\Big[-\frac{f_1\delta^4\sin m\alpha\sin n\beta}{(\delta^2+1)^2} + \\
+\frac{f_2}{4}\cos 2m\alpha - f_2\frac{\delta^4\cos 2m\alpha\cos 2n\beta}{4(\delta^2+1)^2}\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2\psi}{\partial s^2} = p_1 + \frac{\pi^2 Et\delta^2 n^2}{2b^2}\Big\{-\frac{f_1^2\cos 2n\beta}{4} - f_2^2\Big[\frac{\cos 2n\beta}{4} + \frac{\cos 4m\alpha\cos 2n\beta}{4(4\delta^2+1)^2} + \\
+\frac{\cos 2m\alpha\cos 4n\beta}{(\delta^2+4)^2} - \frac{\cos 4n\beta}{16} - \frac{\cos 2m\alpha\cos 2n\beta}{2(\delta^2+1)^2}\Big] + f_1 f_2\Big[\frac{2\sin m\alpha\sin n\beta}{(\delta^2+1)^2} - \\
-\frac{27\sin m\alpha\sin 3n\beta}{(\delta^2+9)^2} - \frac{3\sin 3m\alpha\sin n\beta}{(9\delta^2+1)^2}\Big] + \frac{2b^2}{\pi^2 Rn^2}\Big[\frac{f_1\sin m\alpha\sin n\beta}{(\delta^2+1)^2} + \\
+f_2\frac{\cos 2m\alpha\cos 2n\beta}{4(\delta^2+1)^2}\Big]\Big\},
\end{aligned} \qquad (41.2)$$

$$\begin{aligned}
\frac{\partial^2\psi}{\partial x\partial s} = \frac{\pi^2 Etn^2\delta^3}{4b^2}\Big\{f_2^2\Big[\frac{\sin 4m\alpha\sin 2n\beta}{(4\delta^2+1)^2} + \frac{\sin 2m\alpha\cdot\sin 4n\beta}{(\delta^2+4)^2} - \\
-\frac{\sin 2m\alpha\sin 2n\beta}{(\delta^2+1)^2}\Big] + f_1 f_2\Big[\frac{18\cos m\alpha\cos 3n\beta}{(\delta^2+9)^2} + \frac{18\cos 3m\alpha\cos n\beta}{(9\delta^2+1)^2} - \\
-\frac{4\cos m\alpha\cos n\beta}{(\delta^2+1)^2}\Big] - \frac{b^2}{\pi^2 Rn^2}\Big[4 f_1\frac{\cos m\alpha\cos n\beta}{(\delta^2+1)^2} + f_2\frac{\sin 2m\alpha\sin 2n\beta}{(\delta^2+1)^2}\Big]\Big\},
\end{aligned}$$

$$\delta = \frac{mb}{na}.$$

To satisfy the condition of clamping of the skin to the longitudinal ribs at the strip vertices it is necessary that the mutual axial approach of points of the skin, which, before the deformation, lie on the intersection of the longitudinal ribs with two neighboring transverse ribs, be equal to the contraction of the segment of the longitudinal rib between these transverse ribs, i. e., the condition

$$\int_0^a \frac{\partial u}{\partial x}dx = \int_0^a\Big[\frac{1}{Et}\Big(\frac{\partial^2\psi}{\partial s^2} - \nu\frac{\partial^2\psi}{\partial x^2}\Big) - \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Big]dx = a\varepsilon_{1c}^m. \qquad (41.3)$$

for $\beta = l\pi$ (l—integer),

should be satisfied where ε_{1c}^m is the mean relative elongation of the longitudinal rib between the neighboring transverse ribs. After simple computations, this condition yields the equation

$$-\varepsilon_{1c}^m = \frac{-p_1 + \nu p_2}{Et} + \frac{\pi^2}{8a^2}\Big(f_1^2 + \frac{3}{4}f_2^2\Big)m^2. \qquad (41.4)$$

The skin is also clamped to the transverse ribs. Consequently, the condition

$$\int_0^b \frac{\partial v}{\partial s}ds = \int_0^b\Big[\frac{1}{Et}\Big(\frac{\partial^2\psi}{\partial x^2} - \nu\frac{\partial^2\psi}{\partial s^2}\Big) - \frac{w}{R} - \frac{1}{2}\Big(\frac{\partial w}{\partial s}\Big)^2\Big]ds = b\varepsilon_{2t}^m,$$

should hold, where ε_{2t}^m is the mean relative elongation of the transverse rib.

We shall assume that the tensile and comressive rigidity of the transverse rib is large, so that one can take $\varepsilon_{2t}^m = 0$. Then, from the condition indicated, it follows that

$$p_2 = \frac{\pi^2 n^2 Et}{8b^2}\left(f_1^2 + \frac{3}{4} f_2^2 + \frac{2b^2}{\pi^2 R n^2} f_2\right) + \nu p_1 . \tag{41.5}$$

It should be noted that with the assumed shape of the buckle, the mutual approach of points of the skin lying on neighboring transverse ribs on the same generator are functions of β. This is explained by the fact that the given deformation has a period equal to twice the length and twice the width of the strip (without considering the influence of rigid bases). In view of this we should choose from the shell a portion of length 2a and width 2b. Then the relative axial displacement of the edge x = 2a in relation to the edge x = 0 equals

$$\Delta_1 = 2a\varepsilon_{1c}^{m} .$$

Therefore the work of the axial load, applied to these edges, is equal to

$$W_1 = \frac{b}{\pi}\int_0^{2\pi} \frac{\partial^2 \psi}{\partial s^2}\Delta_1\, d\xi = \frac{4ab}{Et}(p_1^2 - \nu p_1 p_2) - \tag{41.6}$$
$$- \frac{\pi^2 ab}{2b^2} p_1 \left(f_1^2 + \frac{1}{2} f_2^2\right) n^2 \delta^2 .$$

The total energy

$$\mathcal{E} = \mathcal{E}_{\text{м}} + \mathcal{E}_{\text{bend}} - W_1$$

is determined from the formulas (40.2), (41.6), (41.1), and (41.2). Setting

$$p_1^* = \frac{p_1 b^2}{Et^2}, \quad k^* = \frac{b^2}{Rt}, \quad \lambda = \frac{k^*}{\pi^2 n^2},$$
$$\frac{f_1}{t} = \xi_1 \lambda, \quad \frac{f_2}{t} = \xi_2 \lambda, \quad q_1^* = \frac{32 p_1^*}{k^* \lambda}, \tag{41.7}$$

and eliminating p_2 by (41.5), we find

$$\mathcal{E}_1 = \frac{32\,\mathcal{E} b^3}{a\pi^4 n^4 E t^5 \lambda^4} = \frac{\psi_1 \xi_1^4}{4} + \frac{\chi_1 \xi_2^4}{4} + \xi_2^2\left[\chi_2 + \frac{3}{8}(\nu + \delta^2) q_1^*\right] +$$
$$+ \frac{\psi_2}{2}\xi_1^2 \xi_2^2 - \psi_3 \xi_1^2 \xi_2 - \chi_3 \xi_2^3 + \xi_1^2\left[\psi_4 + \frac{q_1^*}{2}(\nu + \delta^2)\right] +$$
$$+ \nu q_1^* \xi_2 - \frac{64 p_1^2 (1 - \nu^2) b^4}{\pi^4 n^4 E t^4 \lambda^4},$$

where

$$\psi_1 = 6 + 2\delta^4, \quad \psi_3 = -6 - 32\delta^4 : (1 + \delta^2)^2,$$
$$\psi_2 = 5 + \delta^4\left[2 + \frac{32}{(\delta^2 + 1)^2} + \frac{72}{(9\delta^2 + 1)^2} + \frac{72}{(\delta^2 + 9)^2}\right],$$
$$\psi_4 = \psi_4^0 + \frac{1}{\lambda^2}\psi_4^1, \quad \psi_4^0 = \frac{16\delta^4}{(1 + \delta^2)}, \quad \psi_4^1 = \frac{4(1 + \delta^2)^2}{3(1 - \nu^2)}. \tag{41.8}$$
$$\chi_1 = \frac{35}{8} + \frac{17}{8}\delta^4 + \frac{\delta^4}{(\delta^2 + 4)^2} + \frac{\delta^4}{(4\delta^2 + 1)^2} + \frac{4\delta^4}{(\delta^2 + 1)^2},$$
$$\chi_3 = -5 - \frac{2\delta}{(\delta^2 + 1)^2}.$$
$$\chi_2 = \chi_2^0 + \frac{1}{\lambda^2}\chi_2^1, \quad \chi_2^0 = 6 + \frac{\delta^4}{(\delta^2 + 1)^2}, \quad \chi_2^1 = \frac{4(3\delta^4 + 2\delta^2 + 3)}{3(1 - \nu^2)}.$$

The extremum condition of the total energy gives the equations*:

$$\frac{\partial \mathcal{E}_1}{\partial \xi_1} = 0, \quad \frac{\partial \mathcal{E}_1}{\partial \xi_2} = 0$$

or

$$\psi_1 \xi_1^2 + \psi_2 \xi_2^2 - 2\psi_3 \xi_2 + 2\psi_4 + q_1^* (\nu + \delta^2) = 0. \tag{41.9}$$

$$-\chi_1 \xi_2^3 - \left[2\chi_2 + \frac{3}{4}(\nu + \delta^2) q_1^*\right]\xi_2 - \psi_2 \xi_1^2 \xi_2 + \psi_3 \xi_1^2 + 3\chi_3 \xi_2^2 - \nu q_1^* = 0. \tag{41.10}$$

Besides, at the boundary of stability the second variation of total energy should be zero, i. e., the equation

$$\frac{\partial^2 \mathcal{E}_1}{\partial \xi_1^2} \cdot \frac{\partial^2 \mathcal{E}_1}{\partial \xi_2^2} - \left(\frac{\partial^2 \mathcal{E}_1}{\partial \xi_1 \partial \xi_2}\right)^2 = 0.$$

should be satisfied, which, in view of (41.12), can be written in the form:

$$\begin{aligned} &3(\chi_1\psi_1 - \psi_2^2)\xi_2^2 - 6(\chi_3\psi_1 - \psi_2\psi_3)\xi_2 + 2(\chi_2^0\psi_1 - \psi_2\psi_4^0 - \psi_3^2) + \\ &+ 2(\chi_2^1\psi_1 - \psi_2\psi_4^1)\frac{1}{\lambda^2} + q_1^*\left[\frac{3}{4}\psi_1(\nu + \delta^2) - \psi_2(\nu + \delta^2)\right] = 0. \end{aligned} \tag{41.11}$$

Eliminating the quantities ζ_1 and q_1^*, from (41. 9)-(41. 11), we obtain a cubic equation in ζ_2, which can have one or three real roots. One has to choose that root among them for which the value of $|q_1^*|$ is minimal, and ζ_1^2, determined from the formula (41.9), satisfies the condition:

$$\xi_1^2 \geqslant 0. \tag{41.12}$$

Here one has to repeat the calculation, varying the integers m and n, and seek that form of buckling for which $|p_1^*|$ has the smallest value. Due to this, the necessary computations become very tedious. Therefore, we shall carry them out for the special case when

$$a = 2b. \tag{41.13}$$

One usually uses oblong strips for which

$$1.5b < a < 3b. \tag{41.14}$$

We shall assume that the results obtained for the intermediate case (41.13) will be also applicable in the entire interval (41.14). To simplify the calculations, instead of the simultaneous solution of the equations (41. 10), (41. 11) for ζ_2 and q_1^* we propose to take various positive values of ζ_2 and to determine the corresponding values of q_1^* and $1/\lambda^2$ from the equations, which are linear in these quantities.

Repeating the calculations for different m and n, we find the smallest of the quantities $|p_1^*/k^*|$. The results of these simple but rather tedious calculations are given in Table IV.

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* Note that here, in contradistinction to § 40, the total energy is not minimized with respect to m and n, as the virtual displacements have to satisfy the geometrical boundary conditions where the boundaries of the considered regions of variation of the parameters are given.

Table IV

k^*	19.4	27.3	41.2	49	65.7	87	109	165	196	262
$\frac{\lvert p^*_{1н}\rvert}{k^*}$	0.36	0.32	0.31	0.28	0.24	0.25	0.32	0.31	0.28	0.24
m	2	2	2	1	1	1	4	4	2	2
n	1	1	1	1	1	1	2	2	2	2

From the table we see that, for $k^* < 49$, to the lower critical stress corresponds a buckling with the formation of square waves, where one half-wave appears along the strip width. Further, with $49 < k^* < 87$, at the lower critical stress each half-wave occupies the entire length of the strip. With $k^* \geqslant 109$ the strip width begins to divide into two half-waves, where up to $k^* \leqslant 196$ square waves are formed and then there appear oblong pits, buckles, etc. The value of the lower critical stress is also repeated with a fourfold increase in k^*.

The mean axial elongation of the skin is equal to the mean elongation ε_{1c}^m of the longitudinal rib given by (41.4). In our case the mean elongation of the transverse rib is zero, as is the mean elongation of the skin of the transverse rib, and therefore the formula (41.5) is valid. In order for the skin to attain this elongation without buckling, an axial stress

$$T_1 = Et\,\varepsilon_{1c}^m : (1 - \nu^2),$$

is needed which in view of (41.4) and (41.5) is equal to

$$T_1 = p_1 - \frac{\nu E t f_2}{4R(1-\nu^2)} - \frac{Et\,\pi^2 n^2}{8b^2(1-\nu^2)}\left(f_1^2 + \frac{3}{4}f_2^2\right)(\nu + \delta^2).$$

Meanwhile, the mean stress in the skin as it buckles is p_1. Therefore, the reduction coefficient of the skin under axial compression is

$$\varphi = -\frac{p_1^*}{k^*} : \left\{-\frac{p_1^*}{k^*} + \frac{\lambda(\nu+\delta^2)}{8(1-\nu^2)}\left(\xi_1^2 + \frac{3}{4}\xi_2^2 + 2\nu\xi_2\right)\right\}. \tag{41.15}$$

Our problem is to determine the smallest value of this quantity for given values of the torsion and stress parameters. As is well known, at the upper critical stress $|p_1^*| : k^* \approx 0.6$. As had been found above, at the lower critical stress $|p_1^*| : k^* \geqslant 0.24$. We shall therefore determine φ at various stresses, beginning with those for which $|p_1^*| : k^* > 0.24$. The corresponding values of ξ_1 and ξ_2 have to be determined from (41.9) and (41.10). But the latter equation is cubic in ξ_2. Therefore we shall seek the solution of the problem by assuming the values of λ and ξ_2, and then determining the corresponding q_1^*, and therefore also p_1^* / k^* according to formula (41.10).

Thus we construct the following table of values of φ with

$$k^* = 61.6, \quad a = 2b, \tag{41.16}$$

where it turns out that the smallest values of φ are obtained for $n = 1$.

Table V

m	δ	$\lvert p_1^*/k^*\rvert$	φ	m	δ	$\lvert p_1^*/k^*\rvert$	φ
1	1/2	0.247 0.373 0.568	0.586 0.577 0.616	3	3/2	1.085 1.420 2.092	0.452 0.407 0.383
2	1	0.38 0.45 0.66 1.02 1.88	0.43 0.37 0.37 0.40 0.44	4	2	0.952 1.22 1.69 2.40	0.51 0.42 0.38 0.36

For a practical application, however, it is more convenient to express φ as a function of the mean elongation of the longitudinal rib. Therefore the data in the table are shown in graphical form, where the quantities $\varepsilon_{1c}^{m}R/t$ are taken as abscissas. The necessary computations were made using the relation

$$-\varepsilon_{1c}^{m}R/t=(1-\nu^2)\,p_1^*/k^*\varphi.$$

As can be seen from the table and from Figure 23, as the axial stress parameter increases from its lower limit, the first minimum of the reduction coefficient is reached at $\delta=\frac{1}{2}$ i.e., for the form of buckling with square pits. Then, at $|p_1^*/k^*|=0.38$ there is an abrupt passage to the form for which $\delta=1$, where the reduction coefficient abruptly falls from 0.57 to the value 0.43, and then to 0.37. Further, it increases to 0.41 as the load increases. At $|p/k^*|=1.32$, the smaller value of φ begins to correspond to the buckle form with $\delta=3/2$, and then with $\delta=2$. Hence we see that after the "snap", the apparent strength of the skin increases slowly with a rapid increase of longitudinal rib load, and the decrease of the reduction coefficient is rapid at first, and then slow.

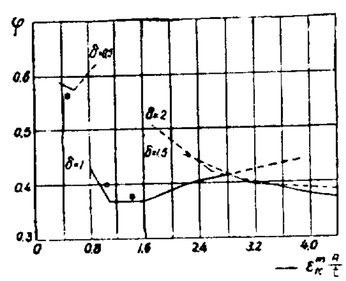

Figure 23

The corresponding experimental points of A. S. Vol'mir are given in the graph of Figure 23. They are close to the calculated points.

In conclusion we note that initially we had been considering a more general form of buckling than in (41.1), namely, we had been setting

$$w=f_1\sin m_1\alpha\sin n\beta+f_2\sin^2 m_2\alpha\sin^2 n\beta.$$

Here we do not give the results of these calculations, since it turned out that the case $m_1=m_2$ corresponds to the minimum of the lower bound of the stress, as well as to the minimum of φ.

§ 42. Determination of the Lower Critical Load of a Shell under Uniform Compression*

We shall consider a closed cylindrical shell supported by transverse ribs, which are rigid against bending and compression and weakly resistant to torsion, as in § 41. The skin is assumed to be clamped to the transverse ribs at the points $s = 0$, b/n, ..., $(2n-1)b/n$, where n is the integer to be determined. We shall take a to be the distance between the neighboring transverse ribs, p_1 and p_2 to be the mean values of the axial and annular stresses and p to be the external normal pressure, acting on the lateral surface and on the bottom of the shell, $\mu = \frac{\pi R}{an}$.

We shall determine the smallest value of p for the form of buckling given by the deflection

$$w = f_1 \sin\frac{\pi x}{a}\sin\frac{n\pi s}{b} + f_2 \sin^2\frac{\pi x}{a}. \tag{42.1}$$

We calculate the stress function ψ, the membrane stresses, and the axial displacement Δ_1 of the skin points lying on the transverse rib $x = a$, with respect to the points lying on the transverse rib $x = -a$, as in § 41. Then (42.1) gives the expression

$$\Delta_1 = -2a\left[\frac{-p_1 + \nu p_2}{Et} + \frac{\pi^2}{4a^2}\left(\frac{f_1^2}{2} + f_2^2\right)\right], \tag{42.2}$$

and the condition that the mean annular elongation of the skin and the transverse ribs be zero leads to the relation

$$p_2 = \frac{\pi^2 n^2 Et}{8b^2}\left(f_1^2 + \frac{4b^2}{n^2\pi^2 R} f_2\right) + \nu p_1. \tag{42.3}$$

The total energy of the shell per unit area of the middle surface is determined in the case under consideration from (46.2) and (40.12)

$$\begin{aligned} \partial_1 = \frac{32b^3\partial}{Eta^6} = \frac{\psi_1}{4}\zeta_1^4 + \zeta_1^2\left[\psi_2 + \frac{1}{2}q_1\left(1 + \frac{\nu}{\mu^2}\right)\right] + \\ + \zeta_2^2(\psi_3 + q_1) + \psi_4\zeta_1^2\zeta_2^2 - 2\psi_5\zeta_1^2\zeta_2 - 2\zeta_2 q_1(2-\nu) - \\ - \frac{64b^4}{E^2t^2a^4}p_1^2(1-\nu^2), \end{aligned} \tag{42.4}$$

in the derivation of which, besides (42.1), (42.2), we used the relation

$$p = -2p_1/R \tag{42.5}$$

and employed the notations

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* A more detailed exposition of this problem can be found in the work of F. S. Isanbaeva /X. 10/.

$$\zeta_1 = f_1 \frac{\pi b}{a^2}, \quad \zeta_2 = f_2 \frac{\pi b}{a^2}, \quad q_1 = \frac{32 p_1 b^2}{Eta^2},$$
$$\psi_1 = \frac{6}{\mu^4} + 2, \; \psi_2 = \frac{16\mu^4}{(1+\mu^2)^2} + \frac{16(1+\mu^2)\pi^4\theta^4}{3(1-\nu^2)\mu^4}, \tag{42.6}$$
$$\psi_3 = 24 + \frac{128\pi^4\theta^4}{3(1-\nu^2)}, \quad \psi_4 = \frac{16}{(1+\mu^2)^2} + \frac{16}{(1+9\mu^2)^2},$$
$$\psi_5 = -\frac{6}{\mu^2} - \frac{16\mu^2}{(1+\mu^2)^2}, \quad \theta = \frac{\sqrt{tR}}{a\sqrt{2}}.$$

From the conditions that the total energy be stationary

$$\frac{\partial Э_1}{\partial \zeta_1} = \frac{\partial Э_1}{\partial \zeta_2} = \frac{\partial Э_1}{\partial \mu} = 0$$

we obtain the equations

$$\psi_1\zeta_1^2 + 2\psi_2 + q_1\left(1 + \frac{\nu}{\mu^2}\right) + 2\psi_4\zeta_2^2 - 4\psi_5\zeta_2 = 0; \tag{42.7}$$

$$(2-\nu)\, q_1 + \psi_5\zeta_1^2 - \psi_4\zeta_1^2\zeta_2 - (\psi_3 + q_1)\,\zeta_2 = 0; \tag{42.8}$$

$$\psi_{1\mu}\frac{\zeta_1^2}{4} + \psi_{2\mu} - \frac{\nu}{\mu^3} q_1 + \psi_{4\mu}\zeta_2^2 - 2\psi_{5\mu}\zeta_2 = 0. \tag{42.9}$$

Here $\psi_{1\mu}, \ldots, \psi_{5\mu}$ are the derivatives of $\psi_1, \ldots, \psi_5$ with respect to μ.

Thus the problem reduces to the investigation of the interdependence between the parameters q_1, ζ_1, ζ_2, and μ, defined by the equations (42.7)-(42.9).

Choosing a parameter θ from this system of equations, one can find the smallest value of $|q_1|$ which will also be the pressure parameter of collapse for that θ. However, this method of solution of the problem involves very lengthy computations. Therefore we determine the lower critical load by the less tedious inverse method, without initially choosing the parameter θ. The essence of the method is the following:

1. We determine the quantity ζ_1^2 from (42.7) and eliminate it from the remaining equations. We thus obtain two linear equations in q_1 and θ, whose coefficients are known functions of ζ_2 and μ;

2. Taking particular values of μ, we calculate $\psi_1, \psi_2, \ldots, \psi_{5\mu}$ from (42.6);

3. Taking a set of values of $\zeta_2 < 0$ for a given μ, we find those values of q_1 and θ for which the ratio of p to the critical pressure p_0, determined from the linear theory, is minimum. We take this p to be the lower bound of critical pressure for the given θ. Here tthe condition $\zeta_1^2 \geqslant 0$ **should be used. The value of p_0 is determined from the formula:**

$$p_0 = \frac{8E}{[9(1-\nu^2)]^{3/4}} \frac{\pi R}{a}\left(\frac{t}{2R}\right)^{5/2} : [1 - 0.9\theta], \tag{42.10}$$

and we find the ratio p/p_0 from (42.5), (42.10), and (42.6):

$$-p/p_0 = q_1 \frac{[9(1-\nu^2)]^{3/4}}{64\pi^3\theta^3}(1 - 0.9\theta); \tag{42.11}$$

4. Repeating the calculations for other values of μ, we find the critical pressures for a series of values of θ. The calculated results are given in Table VI and shown in graphical form in Figure 24.

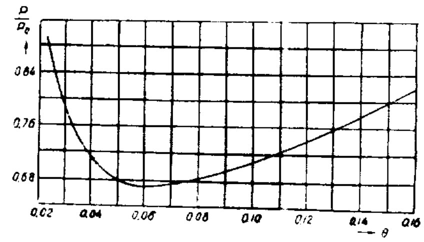

Figure 24

Table VI

μ	0.75	0.65	0.55	0.50	0.4	0.35	0.30
θ	0.171	0.137	0.105	0.091	0.062	0.048	0.038
p/p_0	0.829	0.772	0.714	0.698	0.678	0.681	0.720

On the basis of Table VI and the graph, we conclude that the smallest of the ratios p/p_0 is 0.678 and corresponds to $\theta = 0.062$, characterizing a class of short cylindrical shells. With the increase of shell length, θ decreases and the ratio p/p_0 increases, approaching unity. With decrease of the length of the shell, i.e., with $\theta > 0.062$, the ratio p/p_0 also increases. It would seem that here the influence of the non-linear factor should increase, but the results given in Table VI show the contrary. This is possibly explained by the fact that the form of bending chosen for very short shells does not entirely correspond to reality. From Table VI and the graph it can be seen that with the increase of θ from 0.048 to 0.105, the change in the ratio p/p_0 is insignificant. Therefore, the lower limit of the critical pressure for the parameter θ in the interval

$$0.048 < \theta < 0.105 \tag{42.12}$$

can be taken to be

$$p = 0.68 p_0, \tag{42.13}$$

and for smaller values of θ one can use Table VI.

Table VI illustrates the dimensions of shells which are included in the given interval of the parameter θ.

In order to verify the applicability of the inverse method, calculations have been carried out by the direct method for two values of θ lying at the boundaries of the interval (42.12). The results of the calculations have shown that the inverse method does actually give the minimal values of q_1 and p/p_0 for a given θ. In conclusion we note that from the solution given one does not obtain the exact von Mises formula for the upper critical pressure derived from the linear theory.

Table VII

θ	0.038	0.048	0.062	0.075	0.091	0.105
R/a for $R=900\,t$	1.6	2.04	2.63	3.17	3.85	4.45
R/a for $R=400\,t$	1,06	1.36	1.75	2.2	2.57	2.97

As is well known, the deformation of the shell before the loss of stability consists of a membrane part and an edge effect. Usually, the edge effect is neglected, assuming that before the loss of stability the transverse ribs do not resist compression, i. e., do not take any transverse load. Meanwhile, it is obvious that the shorter the shell, the greater the load which will be taken by the transverse ribs. After the loss of stability, the transverse ribs in carry increasingly large loads. In view of this, assuming the form of the bending given by (42. 1), we consider that at the instant of the loss of stability $f_1 = 0$, but $f_2 \neq 0$, i. e., the shell has an axially symmetric, barrel-shaped form. We approximate this state by the deflection $w = f_2 \sin\frac{2\pi x}{a}$, where the transverse ribs are considered as uncompressed. This assumption as **well as the assumption of non-resistance of the transverse ribs to compressions is an idealization** of the problem. The real situation lies somewhere in between. Therefore, both approaches to the problem are useful. To determine the values of f_2 at the appearance of the non-axially symmetric buckling, we set $\zeta_1^2 = 0$ in equation (48. 8). In view of the smallness of θ and of (42. 11), one can set $\psi_3 + q_1 \approx 24$. Consequently

$$\zeta_2 = \frac{(2-\nu)q_1}{\psi_3 + q_1} \approx \frac{(2-\nu)q_1}{24} \text{ or } f_2 \approx \frac{4(2-\nu)}{3}\frac{p_1 b}{\pi E t}.$$

Substituting for f_2 in formula (43. 3) with $f_1 = 0$, we find

$$p_2 = \left(\frac{4}{3} + \frac{1}{3}\nu\right)p_1 \text{ or } p_2 = 1.43 p_1 (\text{for } \nu = 0.3).$$

According to the membrane theory, $p_2 = 2p_1$, as is well-known.

Thus the mean annular stress in the skin at the moment of stability loss turns out in this case to be by 70% of that in a slightly compressed web frame. A further deflection, as can be seen from formula (42. 3), appreciably affects the mean annular stress, which was to be expected. Meanwhile, by taking $|p_2| : |p_1| = 2 =$ constant, we would not at all take into account the influence of the web frames even under large deflections.

Passing to the determination of the upper critical load, p_u, let us note that in the case of shells of medium length the solution given yields the same results as the von Mises formula. However, in the case of shells of smaller length the ratio of the upper critical stress (obtained from (42. 7) and (42. 8)) to the upper critical load (obtained from the membrane theory) is less than unity. As is apparent from Table VIII, for short shells the value of the upper critical load is approximately 11% less than the value obtained from the von Mises formula.

This result should not be considered as unexpected, if one takes into account that from the very beginning we have inward deflections due to which—in the case of a short shell—the axial compression must substantially reduce the stability of the shell with respect to transverse pressure.

Table VIII

μ	0.4	0.6	0.8
θ	0.080	0.151	0.197
p/p_0	0.917	0.903	0.893

§ 43. Determination of the Upper Critical Load of a Cylindrical Shell with Initial Imperfections under Axial Compression and Uniform External Pressure

The problem of the influence of initial imperfections of a cylindrical shell on its stability under axial compression was first investigated in the non-linear theory by L. Donnell /X. 17/. Later, together with K. Yan, he carried out a new, careful investigation of this problem /X. 4/, using the Papkovich-Ritz method. Here it was assumed that the deflection w due to the load is similar to the initial deflection w^0 and, consequently, for a given load

$$C = 1 + 2w^0/w = \text{const.} \tag{43.1}$$

If one uses this relation and omits the index "I" in the formulas of § 35, then the condition of compatibility (35.15) for a circular cylindrical shell can be written in the form given in /X. 17/:

$$\Delta\Delta\psi - Et\left\{C\left[\left(\frac{\partial^2 w}{\partial x \partial s}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial s^2}\right] - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}\right\} = 0. \tag{43.2}$$

We shall approximate the deflection due to the load by the function

$$w = at\left(\cos\frac{\mu n x}{R}\cos\frac{ns}{R} + b\cos\frac{2\mu n x}{R} + c\cos\frac{2ns}{R} + d\right). \tag{43.3}$$

The initial imperfection of the shell is characterized by: a) initial deflections from the ideal form, b) initial strains, c) anisotropy of the shell material, etc. Let us note that one should not neglect the possible effect of the anisotropy of a polycrystalline material as it can turn out that for a thin shell, only a small number of crystals are situated along its thickness. Following Donnell, we shall assume that all the imperfections indicated act like an initial deflection from the ideal form and that their total effect is defined by some given deflection w^0.

Obviously, the effect of the initial deflection depends not only on its amplitude, but also on the dimensions of the part of the shell under consideration. For example, if a rectilinear strip, whose length and thickness are equal respectively to l and t, has an initial deflection $a^0 t$ approximately given by the sinusoidal form

$$w^0 = a^0 t \sin \pi x/l, \tag{43.4}$$

then its initial relative curvature is

$$t\varkappa^0 = t\frac{d^2 w^0}{dx^2} = a^0\frac{\pi^2 t^2}{l^2}\sin\frac{\pi x}{l}.$$

The maximum of this quantity, characterizing the effect of the deflection on the deformation due to the load for a given value of a^0, is proportional to the ratio t^2/l^2.

Consequently,

$$a^0 = (U/\pi^2)(l/t)^2, \tag{43.5}$$

where U is the roughness factor of the strip, being approximately the same for all the strips irrespective of their dimensions if their manufacturing technique and the material are the same.

Analogous considerations also apply to a shell under axial compression, if the initial bending is defined by a function of the form (43.3). Here it is necessary to envisage the possibility of waves being formed along the length as well as along the circumference of the shell; therefore, if the buckling is equally probable in both directions, it is worthwhile to take, instead of (43.5), the relation

$$a^0 = (U/\pi^2)(l_x l_s/t^2),$$

where l_x and l_s are the corresponding lengths of the half-waves.

But in the majority of cases thin-walled cylinders are constructed by bending flat sheets, due to which buckling along the axial direction takes place before buckling along the circumference of the cylinder; therefore

$$a^0 = (U/\pi^2)\, l_x^{1+q}\, l_s^{1-q}/t^2 \quad q > 0.$$

It is obvious that $1-q$ cannot be negative. For the sake of definiteness we shall assume that $q = \frac{1}{2}$. Calculations show that this choice hardly affects the final result. Thus, let

$$\begin{gathered} w^0 = a^0 t\left(\cos\frac{\mu n x}{R}\cos\frac{ns}{R} + b\cos\frac{2\mu n x}{R} + c\cos\frac{2ns}{R} + d\right), \\ a^0 = (U/\pi^2)\, l_x^{1,5}\, l_s^{0,5}/t^2 = UR^2/(\mu^{1,5} n^2 t^2). \end{gathered} \tag{43.6}$$

We find ψ introducing (43.3) in (43.2) and then, using (40.2), (43.1) and (43.6), we set up the expression for the deformation energy. Here the mean relative shortening of the shell ε turns out to be

$$\varepsilon = \frac{\sigma}{E} + \frac{e\mu^2 at}{4R}(8b^2 + 1), \quad e = \frac{Cn^2 at}{2R}, \tag{43.7}$$

where σ is the modulus of the mean axial stress.

If for given σ and UR/t the value of ε does not change, as is the case for the testing of shells under compression by rigid testing machines, the work of the external load under variations of the quantities a, b, c, μ, and e will be zero. Consequently, in that case the total energy of the shell $\mathit{Э}$ is equal to its deformation energy. Determining the parameter d, as in § 40, from the periodicity condition of the annular displacement v, we set up for the determination of the equilibrium values of a, b, c, μ and e the equations:

$$\frac{\partial Э}{\partial a} = \frac{\partial Э}{\partial b} = \frac{\partial Э}{\partial c} = \frac{\partial Э}{\partial \mu} = \frac{\partial Э}{\partial e} = 0. \tag{43.8}$$

Solving them simultaneously with equation (43.7), one can construct a series of curves for various values of UR/t, which express the dependence between the quantities σ/σ_u and $\varepsilon/\varepsilon_u$ where σ_u and ε_u are the corresponding quantities found from the linear theory for an ideal shell. In Figure 25, taken from article /X. 4/, are given graphs for the values of UR/t from 0 to 0.4.

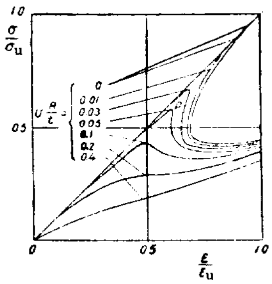

Figure 25

The necessary calculations (for small values of UR/t and for values of σ/σ_u greater than 0.75) were carried out with the help of the equations (43.8). The remaining graphs were obtained by minimizing the energy only with respect to the parameters a and n with fixed values of b = 0.18, c = 0.03, and μ = 0.728, taken without sufficient basis. This could have eliminated the most suitable roots of the system of non-linear algebraic equations (43.8). Therefore the numerical results of the investigation with $\sigma<0.75\ \sigma_u$ needs to be verified and made more precise. But the qualitative aspect of the phenomenon is apprently satisfactorily described by the solution just considered. As can be seen from Figure 25, for UR/t $\leqslant$ 0.2 every curve $\sigma - \varepsilon$ has a peak p, after which the further mutual approach of the shell edges can occur without increasing the load. If the shell is thin, then this peak is reached earlier than the stress in it reaches the plasticity limit σ_s of the shell material.

In the case of less thin shells, the loss of stability of the shell occurs after reaching the limit of plasticity, so that, in fact, we are dealing with the investigation of shell rigidity. In Figure 26 are shown graphs constructed on the basis of the graphs of Figure 25. Here the solid line is the plot of the peak value σ/σ_u against the quantity UR/t, and the dotted lines show the onset of plastic deformations for the corresponding values of the quantities UR/t and σ_s/UE. For example, if the shell material and the conditions of its preparation are such that σ_s/UE = 5, then with UR/t $\approx$ 0.076 the peak stress, equal to $\sigma = 0.5\,\sigma_u$, coincides with the onset of plastic flow; with 0<UR/t<0.076 the plastic flow starts after the loss of stability for the corresponding values of σ; for thinner shells, UR/t > 0.076, a purely elastic buckling occurs at the peak stresses, defined by the points of the solid line to the right of the point $\sigma = 0.5\sigma_u$.

Calculations carried out by the above method for the value U = 0.00015 have led in many cases to values of critical load which are considerably in excess of the values found experimentally. One of the possible reasons for this discrepancy between theory and experiment was indicated above. A second reason can be the fact that in the solution given we considered only such forms of buckling which are similar to the initial deflection from the ideal form of the cylindrical shell, which narrows down the class of admissible functions for w and may lead to excessive theoretical values for the critical load. In that connection, let us note that not every deviation from the form of a circular cylinder decreases the critical load. For example, in /0. 13/ it was demonstrated that a sinusoidal corrugation of a cylindrical shell along the circumferences of the cross-sections considerably increases the stability of the shell under axial compression. It is obvious that the corrugation of the generators which produces a middle surface formed by the rotation of a sine curve should increase the stability of a shell under external normal pressure. A search for such advantageous initial deviations from the circular cylinder, on the basis of the non-linear theory, was made in the article of Tsikal /VI. 13/, which is the first serious attempt to solve this important problem.

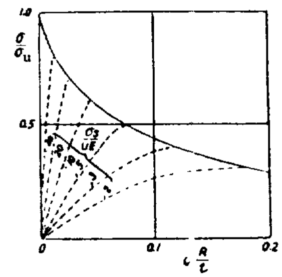

Figure 26

——— loss of stability under elastic deformations
- - - - - loss of stability starting from the limit of plasticity

Quite recently, Nash has carried out an investigation /X. 15/ on the stability of a cylindrical shell with initial imperfections under the action of uniform compression, starting from the equations of compatibility (43. 2) and the expression for the total energy (40. 12) and (40. 2). He approximated the desired deflection by the function:

$$w = at\left[\sin\frac{ns}{R}\sin\frac{\pi x}{L} + d\left(1 - \cos\frac{2\pi x}{L}\right)\right], \tag{43. 9}$$

which coincides, in fact, with the form of the deflection (42. 1), utilized in /X. 10/ for the determination of the lower critical pressure with uniform compression.

The second term of the right-hand member of this expression represents a symmetrical buckling toward the center of curvature of the shell, which lowers the potential energy of the shell which is increased by the transverse compression, while the sinusoidal part corresponds to the form of buckling of an ideally shaped shell, if one takes for n the number of waves calculated in § 36 from the linear theory. Since the critical uniform pressure p calculated from the linear theory is greater than the experimental values by at most 30-35%, one can hope that the form of the deflection (43. 9) will turn out to be suitable as an approximation of the buckling of a shell whose initial deviations from the ideal form are not large (for example, $a_0 \leqslant 0.5$). The form of the initial bending will be assumed to be similar to the deflection under load, i. e., we shall assume

$$w^0 = a^0 t\left[\sin\frac{ns}{R}\sin\frac{\pi x}{L} + c\left(1 - \cos\frac{2\pi x}{L}\right)\right]. \tag{43. 10}$$

Here one satisfies the geometrical boundary condition w = 0 for x = 0 and x = L, which in the case under consideration—in contradistinction to the case of axial compression—is very important, as the smallest critical load corresponds to the formation of buckles and cavities of lengths equal to the length of the shell. Besides, in solving the problem one has to satisfy the equation (42. 3) characterizing the condition that the mean annular elongation of the transverse rib skin be zero. Thus, one constructs the expression for the total energy $\mathcal{E}$ as a function of the quantities p, a, n, and d, and minimizes it with respect to the parameters a and d, i. e., one sets up the equations

$$\partial Э/\partial a = 0, \ \partial Э/\partial d = 0. \tag{43.11}$$

We shall not give all these computations here. Let us merely point out that it is advisable to determine the roughness factor U^0 in the case under consideration from the formula

$$a^0 = U^0 \{l_x l_s / [(l_x + l_s) t]\}^2, \tag{43.12}$$

where $l_x = L$ is the shell length and $l_s = \pi R/n$ is the length of a half-wave along the circumference. Here, on the basis of numerical calculations carried out with the equations (43.11) for various shells for which experiments were made by Windenburg and Trilling /X. 18/, Nash proposed the value $U^0 = 3 \cdot 10^{-4}$. The number of circular waves was to be determined for the first approximation from the theory for small deflections. Then this value of U^0 was used to determine the critical hydrostatic pressure for a shell having the following characteristics:

$$L = 20.3 \text{ cm}, \quad R = 20.3 \text{ cm}, \quad t = 0.229 \text{ cm}, \quad E = 2.02 \cdot 10^6 \text{ kg/cm}^2, \quad \nu = 0.3.$$

Although the parameters of the shell were not taken into account in the indicated processing of the experimental data, the critical pressure found theoretically turned out to be, for $a^0 = 0.15$--0.20, as can be seen from Figure 20, approximately 20 kg/cm^2, while in the experiment the shell buckled under a pressure of 19.6 kg/cm^2. It is possible that with the assumptions made, such an excessively good agreement between the theoretical and experimental results is to some extent accidental and in other cases the error of a given solution will turn out to be somewhat larger. Doubt as to the reliability of this solution is also caused by the fact that in the minimization of the energy functional the equation $\partial Э/\partial n = 0$ was not used; instead of this the critical pressure was minimized with respect to the parameter n.

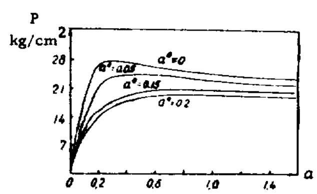

Figure 27

However, for uniform compression the shell does not have such a variety of possible buckle shapes as in the case of axial compression, in which for small deflections the ratio μ of the frequencies of the buckles along the length and along the circumference remains undetermined. Therefore Nash's solution can be considered to be satisfactory as a first approximation.

Unfortunately, in article /X. 15/ the author limits himself to the consideration of the one example indicated above, and does not give any tables or graphs which facilitate calculation. Besides, the values used for U^0, found from a certain set of experiments, can lead to considerable errors in the calculation of the critical pressure of a specimen prepared under other conditions. We therefore propose to determine the relationship between the load p and the parameter a^0 without using the expression (43.12), making use of the expressions (43.9) and (43.10) and setting up the equations (43.11). Here, one can take n to be the number of waves along the circumference, using the formulas (36.13), (36.15) and (36.17), in which one has to set $m = \pi R/L$, $\lambda = 1/2$. Let p be the pressure corresponding to the maximal deflection w from the load, p_0 be the critical pressure determined from (42.10) for an ideal shell. We also introduce the notations

$$\zeta^0 = a^0 \frac{r^4}{R}, \quad w' = w \frac{n^2}{R}.$$

Then the results of calculations carried out by F. S. Isanbaeva for various values of ζ^0 for $\theta = 0.07$ are given in graphical form in Figure 28. As can be seen from these graphs, to each value of $\zeta^0 \leqslant 0.2$ there corresponds a pressure maximum, after reaching which the further increase in deflection occurs with the falling off of pressure or without a noticeable increase of it This value of pressure may be considered as the critical pressure for a shell having, at least on part of its surface, initial irregularities of the form (43.10). It turns out that in the region of maximum pressure, the curves ζ^0 = const almost coincide with the curves w^0 =const, and therefore, together with the values of ζ^0 the corresponding values of w^0 are also shown in the graph. This allows one to determine the peak value of p/p_0 for a given initial irregularity w^0.

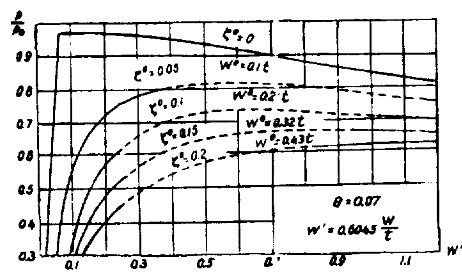

Figure 28

Note that here one attempts to determine the critical load with a form of the initial irregularity which is similar to the desired form of the deflection. For example, in the formula (43.10), instead of a fixed parameter d^0 characterizing the shape of the initial irregularity, one fixes the parameter d at the very beginning of the calculations, although this substitution should only be carried out after setting up the expression for the total energy and its minimization with repsect to d.

The error thus introduced into equations like the second of (43.11) cannot lead, in our opinion, to a considerable error in the value of the critical load, as in the region of the extremum point of the load vs. deflection curve, the changes in the load are slight even for considerable change in the deflection. These considerations were confirmed in the case considered in § 44 by the corresponding calculations of N. I. Krivosheev /X. 19/.

§ 44. The Stability of a Shell Having Initial Irregularities under Torsion

To supplement § 38 we shall adopt the following notations:

$$l = L/\cos\vartheta,\ H = 1/4\,\theta^2,\ A = (1-\nu^2)\frac{l^2}{\rho^2}\,\frac{\tau}{E}. \tag{44.1}$$

$\tau^* = 16\pi^2 A/H^2$ is the dimensionless stress parameter.

$$\mu = \pi R/ln,\ \rho = \sin^2\vartheta,$$

where ϑ is angle of inclination of the waves to the generators of the cylinder. We shall refer the shell to oblique axes x', y', which are connected with the cylindrical coordinates x, s = y by (26.17). We assume that the x' axis is parallel to the wave crests which form with the buckling, i.e., in the (26.17) we set

$$s = y = y',\ \varphi = \vartheta.$$

The expressions for the potential energy of the middle surface and the bending energy in the x', y' coordinates are

$$\begin{aligned}
\mathcal{E}_{\text{м}} &= \frac{1}{2Et\cos\vartheta}\int_0^l\int_0^{2\pi R}\Big\{(\Delta_k\psi)^2\cos^2\vartheta + \\
&+2(1+\nu)\Big[\Big(\frac{\partial^2\psi}{\partial x'\partial y'}\Big)^2 - \frac{\partial^2\psi}{\partial x'^2}\,\frac{\partial^2\psi}{\partial y'^2}\Big]\Big\}\,dx'\,dy',\\
\mathcal{E}_{\text{bend}} &= \frac{D}{2\cos\vartheta}\int_0^l\int_0^{2\pi R}\Big\{(\Delta_k w)^2\cos^2\vartheta + \\
&+2(1-\nu)\Big[\Big(\frac{\partial^2 w}{\partial x'\partial y'}\Big)^2 - \frac{\partial^2 w}{\partial x'^2}\,\frac{\partial^2 w}{\partial y'^2}\Big]\Big\}\,dx'\,dy'.
\end{aligned} \tag{44.2}$$

The length of the arc described by the end point x = L of the shell rotating about the end point x = 0, is approximately equal to

$$\int_0^L\Big(\frac{\partial u}{\partial s}+\frac{\partial v}{\partial x}\Big)dx = -\int_0^L\Big\{\frac{2(1+\nu)}{Et}\,\frac{\partial^2\psi}{\partial x\partial s} + \frac{\partial w}{\partial x}\Big(\frac{\partial w}{\partial s}+\frac{\partial w^0}{\partial s}\Big)+\frac{\partial w}{\partial s}\,\frac{\partial w^0}{\partial x}\Big\}\,dx.$$

Then the work of the applied load is given by

$$\begin{aligned}
W_c = \tau\int_0^l\int_0^{2\pi R}\Big\{&\frac{2(1+\nu)}{Et}\Big(\frac{\partial^2\psi}{\partial x'\partial y'}+\sin\vartheta\,\frac{\partial^2\psi}{\partial y'^2}\Big)+\\
+&\Big(\frac{\partial w}{\partial x'}+\sin\vartheta\,\frac{\partial w}{\partial y'}\Big)\Big(\frac{\partial w}{\partial y'}+\frac{\partial w^0}{\partial y'}\Big)+\frac{\partial w}{\partial y'}\Big(\frac{\partial w^0}{\partial x'}+\sin\vartheta\,\frac{\partial w^0}{\partial y'}\Big)\Big\}\,dx'dy'.
\end{aligned} \tag{44.3}$$

The total potential energy of the shell is equal to:

$$\mathcal{E} = \mathcal{E}_{\text{м}} + \mathcal{E}_{\text{bend}} - W_c.$$

With a hinged support the following boundary conditions should be satisfied at $x' = 0$ and $x' = l$:

$$w = 0, \frac{\partial^2 w}{\partial x'^2} + 2\sin\theta \frac{\partial^2 w}{\partial x' \partial y'} + [(1-\nu)\sin^2\theta + \nu]\frac{\partial^2 w}{\partial y'^2} = 0, \tag{44.4}$$

$$\frac{\partial^2 \psi}{\partial y'^2} = 0, \frac{\partial^2 \psi}{\partial x' \partial y'} + \sin\theta \frac{\partial^2 \psi}{\partial y'^2} = -\tau\cos\theta. \tag{44.5}$$

Let the ends of the shell be supported by transverse ribs rigid in their plane and weakly resisting to torsion and bending out of their plane. We shall also assume that they can be considered as inextensible, i.e.,

$$\varepsilon_t = 0. \tag{44.6}$$

We shall consider a shell with the initial irregularity

$$w^0 = f_0 t\left(\sin\frac{\pi x'}{l}\sin\frac{ny'}{R} + f\sin^2\frac{\pi x'}{l}\sin^2\frac{ny'}{2R}\right). \tag{44.7}$$

For an approximate solution of the problem we shall use the Ritz-Papkovich method, taking the form of the deflection to be

$$w = f_1 t\left(\sin\frac{\pi x'}{l}\sin\frac{ny'}{R} + f\sin^2\frac{\pi x'}{l}\sin^2\frac{ny'}{2R}\right). \tag{44.8}$$

Thus, just as in § 43, we assume that the effect of the torsion will be to increase irregularity while retaining its form. In the right-hand member of the equation (26.19), introducing (44.7) and (44.8) and integrating, we obtain an expression for ψ. The biharmonic part of the solution is taken to be of the form

$$\cos^2\theta\left[\frac{Px'^2}{2} + \frac{\tau}{\cos\theta}(x'y' - x'^2\sin\theta)\right]. \tag{44.9}$$

The distances P are determined from (44.6), which we shall, as before, satisfy only in the mean, setting

$$\varepsilon_t = \frac{1}{2\pi R}\int_0^{2\pi R}\frac{\partial v}{\partial y'}dy' = 0 \text{ at } x' = 0 \text{ and } x' = l.$$

Substituting for ψ, and using (44.7) and (44.8), we obtain

$$\frac{P}{Et} = \frac{1}{4n^2}(\zeta - 2\xi_0)\left\{\frac{1}{2}\zeta\left(1 + \frac{3}{16}f^2\right) - f\right\}. \tag{44.10}$$

where

$$\zeta = \xi + 2\xi_0, \ \xi = f_1 tn^2/R, \ \xi_0 = f_0 tn^2/R. \tag{44.11}$$

(44.8) satisfies the geometrical condition for $w = 0$, and satisfies the static conditions (44.4) and (44.5) in the mean. This is admissable, as in solving the problem by the Ritz-Papkovich method, static conditions are no the essential boundary conditions.

Using (44.2), (44.3), (44.7)-(44.11) and the expression obtained for ψ, we calculate the quantity

$$
\begin{aligned}
\vartheta_1 = \frac{128\, n^4 \vartheta}{\pi t R L E} &= (\zeta - 2\xi_0)^2 \{\zeta^2 (f^4 \varphi_1 + f^2 \varphi_2 + \varphi_3) + \zeta (f^3 \varphi_4 + f \varphi_5) + \\
&+ f^2 \varphi_6 + \varphi_7\} - \zeta(\zeta - 2\xi_0)\left(1 + \frac{3}{16} f^2\right) \sqrt{\rho(1-\rho)}\, \frac{\tau^*}{\mu^2}.
\end{aligned} \qquad \textbf{(44.12)}
$$

where we have set

$$
\begin{gathered}
\varphi_1 = \frac{35}{256} + \mu^4 \left(\frac{17}{16} + \frac{1}{32}\frac{b_{31}}{a_{11}} + 2\frac{b_{12}}{a_{12}} + \frac{1}{2}\frac{b_{11}}{a_{14}}\right), \quad \varphi_3 = 3 + \mu^4, \\
\varphi_2 = \frac{5}{4} + \frac{1}{8}\mu^4 \left(64\frac{b_{11}}{a_{11}} + 121\frac{b_{21}}{a_{21}} + 64\frac{b_{12}}{a_{12}} + \frac{b_{23}}{a_{23}} - 4\right), \\
\varphi_4 = -\frac{5}{2} - 4\rho\mu^2 - \frac{4\mu^2}{a_{12}}[(\rho + 4\mu^2) b_{12} - 32\rho\mu^2(1 + 4\mu^2)], \\
\varphi_5 = -12 - 32\frac{\mu^2}{a_{11}}[(\rho + \mu^2) b_{11} - 8\rho\mu^2(1 + \mu^2)], \\
\varphi_6 = 12 + 4\rho^2 + \frac{2}{a_{12}}\{[(\rho + 4\mu^2)^2 + 16\rho\mu^2] b_{12} - \\
- 64\rho\mu^2(1 + 4\mu^2)(\rho + 4\mu^2)\} + \frac{4\pi^4}{6H^2\mu^4}\left(\frac{3}{16} + \rho\mu^2 + \frac{1}{2}\mu^2 + 3\mu^4\right), \\
\varphi_7 = \frac{32}{a_{11}}\{[(\rho + \mu^2)^2 + 4\rho\mu^2] b_{11} - 16\rho\mu^2(1 + \mu^2)(\rho + \mu^2)\} + \frac{4\pi^4}{6H^2\mu^4} b_{11}. \\
b_{ik} = (i^2 + k^2\mu^2)^2 + \rho(2ik\mu)^2, \quad (i = 1, 2;\ k = 1, 2, 3), \\
a_{ik} = \{(i^2 + k^2\mu^2)^2 - \rho(2ik\mu)^2\}^2.
\end{gathered} \qquad \textbf{(44.13)}
$$

With the given form of the initial irregularity, the values of the parameters ξ_0, ρ, and μ are fixed, and therefore the expression for the total energy contains only two arbitrary parameters ζ and f and the minimum condition for ϑ yields the equations

$$
\partial\vartheta_1/\partial f = 0, \quad \partial\vartheta_1/\partial\zeta = 0.
$$

Developing these equations and eliminating τ^* from the second we obtain

$$
\sqrt{\rho(1-\rho)}\,\frac{\tau^*}{\mu^2} = \frac{8}{3}(\zeta - 2\xi_0)\left\{(4\varphi_1 f^2 + 2\varphi_2)\zeta + 3\varphi_4 f + \frac{\varphi_5}{f} + \frac{2\varphi_6}{\zeta}\right\}. \qquad \textbf{(44.14)}
$$

$$
\begin{gathered}
(3\varphi_2 - 32\varphi_1)\zeta^3 f^3 + 3(\varphi_5 - 8\varphi_4)\zeta^2 f^2 + (6\varphi_3 - 16\varphi_2)\zeta^3 f + \\
+ (3\varphi_7 - 16\varphi_6)\zeta f - 8\varphi_5\zeta^2 - \xi_0\Big\{(3\varphi_2 - 32\varphi_1)\zeta^2 f^3 + \\
+ \left(\frac{3}{2}\varphi_5 - 24\varphi_4\right)\zeta f^2 + (6\varphi_3 - 16\varphi_2)\zeta^2 f - \frac{3}{2}\varphi_4\zeta f^4 - \\
- (16f + 3f^3)\varphi_6 - 8\varphi_5\zeta\Big\} = 0.
\end{gathered} \qquad \textbf{(44.15)}
$$

We shall determine the values of ρ and μ by appropriate choice from the minimum condition of stress, i.e., we shall determine the stress for the most unsatisfactory form of the initial irregularity. In this formulation, the solution of the system (44.14), (44.15) can be obtained as follows:

a) we take values of ρ and μ for a particular value of ξ_0;

b) we calculate φ_1, ..., φ_7 from (44.13) and introduce it in (44.14) and (44.15); thus we obtain τ^* as a function of ζ, f and H, and a relation between the last two;

c) we construct curves $\tau^*(\zeta)$ for a series of values of H;

d) keeping ρ fixed, we carry out analogous calculations for other values of μ;

e) from the obtained family of curves $\tau^*(\zeta)$ we choose that curve which defines the smallest of the quantities τ^*: τ^*_λ, where τ^*_λ is the value of the stress parameter according to the linear theory.

Thus, for a given value of ρ we select the corresponding value of H and the curve $\tau^*(\zeta)$. To reduce the calculations, the values of ρ and μ, corresponding to a given H, may be taken in the first approximation from the linear theory. Calculations show that such a solution leads to an insignificant error.

The curves $A(\zeta)$, obtained in this way, have a form similar to those given in Figure 29 for H = 400.

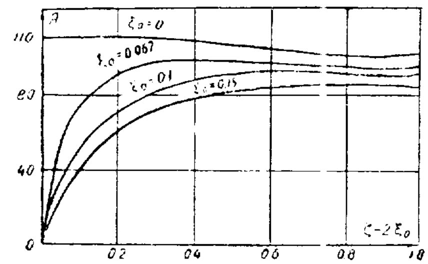

Figure 29

As can be seen from the graphs, at a certain value of load—which we shall call critical— there is a sharp increase in the deflection without noticeable increase in load. The corresponding values of the initial deflection w^0, at which one reaches the critical load, are given in Table IX.

Table IX

ξ_0	H=46.6	ξ_0	H=95.4	ξ_0	H=400	ξ_0	H=2080
	$\frac{w^0_{max}}{t}$		$\frac{w^0_{max}}{t}$		$\frac{w^0_{max}}{t}$		$\frac{w^0_{max}}{t}$
0.07	0.115	0.05	0.085	0.067	0.17	0.015	0.074
0.14	0.244	0.1	0.101	0.1	0.267	0.03	0.147
0.2	0.38	0.2	0.384	0.15	0.435	0.06	0.306

Figure 30 gives a comparison of the critical values of A with the values of the quantities A_λ found from (38.17) in the linear theory, and with the experimental data given in the work of Donnell /IX. 10/.

From the graph it is obvious that almost all the experimental points lie between the theoretical curves constructed for the values of $w^0 = 0$ and $w^0 = 0.4t$.

The critical values of the quantity A for shells whose middle surface has an initial irregularity $w^0_{max} \leqslant 0.25t$, can be determined from the formula

$$A = A_\lambda(1 - 0.6w^0_{max}/t),$$

which, with an error of 2 %, holds also for

$$(w^0_{max}/t) = 0.4.$$

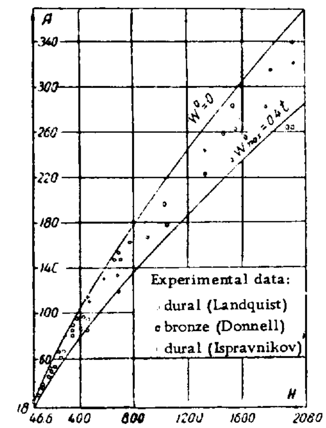

Figure 30

We shall further assume that the middle surface has a symmetrical irregularity of the form

$$w^0 = f_0 t \sin^2 \frac{\pi x'}{l} \sin^2 \frac{ny'}{2R}. \tag{44.16}$$

Under the action of a load such an irregularity will increase, retaining a symmetrical form until this equilibrium form becomes unstable and the appearance of a new unsymmetrical form of equilibrium becomes likely. To determine the critical load corresponding to this point, we shall represent the deflection by the expression

$$w = f_1 t \sin \frac{\pi x'}{l} \sin \frac{ny'}{R} + f_2 t \sin^2 \frac{\pi x'}{l} \sin^2 \frac{ny'}{2R}. \tag{44.17}$$

We consider that f_1 is an infinitesimal quantity, and therefore in the expression for the total energy we omit all terms containing f_1 in powers higher than the second. Thus, analogously to the preceding, we obtain the relations

$$\frac{\sqrt{\rho(1-\rho)}}{\mu^2}\tau^* = \varphi_2\eta^2 + \eta(2\eta_0\varphi_2 + \varphi_5) + \eta_0^2\left(\varphi_2 - \frac{5}{4} + \frac{1}{2}\mu^4\right) + \\ + \eta_0(\varphi_5 + 12) + \varphi_7, \tag{44.18}$$

$$\eta^3\left(4\varphi_1 - \frac{3}{8}\varphi_2\right) + \eta^2\left\{3(4\eta_0\varphi_1 + \varphi_4) - \frac{9}{8}\eta_0\varphi_2 - \frac{3}{8}\varphi_5\right\} + \\ + \eta\left\{4\eta_0(2\eta_0\varphi_1 + \varphi_4) - \frac{9}{8}\eta_0^2\varphi_2 - \frac{3}{4}\eta_0(\varphi_5 + 6) + \right. \\ \left. + \frac{3}{16}\left(\frac{5}{2} - \mu^4\right)\eta_0^2 + 2\varphi_6 - \frac{3}{8}\varphi_7\right\} - \\ - \frac{3}{8}\eta_0\left\{\eta_0^2\left(\varphi_2 - \frac{5}{4} + \frac{1}{2}\mu^4\right) + \eta_0(\varphi_5 + 12) + \varphi_7\right\} = 0, \tag{44.19}$$

where

$$\eta = f_2 t n^2/R, \quad \eta_0 = f_0 t n^2/R. \tag{44.20}$$

The critical load is determined from (44.18) and (44.19) in the same way as from the system (44.14) and (44.15). The difference consists only in that here one does not construct every curve $\tau^*(\tau_i)$, but τ_K^* is determined immediately. Here, from the values of τ^* found for a given ρ, we select the one for which the ratio τ^*/τ_Λ^* takes the minimal value.

Calculations show that a symmetrical irregularity has little effect on the value of the critical load. For example, for the amplitude of the irregularity $0.5t\,(\eta_0 = 0.4)$, the critical value of A is reduced by 9% for H = 95.4, and for H = 2.080 with an amplitude $\approx 2t$ by 9%.

It is probable that, among the various irregularities which occur in actual shells, there also exist those which can lower the apparent strength even more than the irregularities considered above.

Here we have confined ourselves to an exposition of the problem as given in the article of N. I. Krivesheev /X. 19/.

In the article of Nash /X. 15/ it is mentioned that the problem under consideration had been tackled in the Doctor's thesis of Loo, an excerpt of which has recently been published /X. 16/. Judging from this short exposition, the form of deflection taken by Loo is, in our notation of §§ 38, 44,

$$w = at\left[\sin\frac{n(s+\gamma x)}{R}\sin\frac{\pi x}{L}+\frac{d}{2}\left(\cos\frac{2\pi x}{L}-1\right)\right],\quad 0\leqslant x\leqslant L$$

with an initial irregularity

$$w^0 = wa^0/a.$$

Varying the total energy of the shell with respect to deflection parameters a and d, Loo obtains two algebraic equations in τ, a, d, n, and γ, where γ is the tangent of the angle of inclination of the wave. In what follows, the author assumes that

$$m=\frac{\pi R}{nl}=0.722\left(\frac{Rt}{L^2}\right)^{1/4},\quad \gamma=1.732\left(\frac{Rt}{L^2}\right)^{1/4} \tag{44.21}$$

instead of looking for the values of n and γ which correspond to the most unsatisfactory form of w^0.

The values of m and γ, close to the quantities taken by Loo, were found by Donnell for sufficiently long shells by neglecting γ^2 in comparison with unity. Loo makes use of (44.21) also for short shells, as a result of which he obtains excessively large values for the upper critical stress in comparison with Donnell's solution, which in turn gives somewhat excessive values of τ_l, as shown in § 38.

Having thus simplified the problem Loo derives after some calculations the approximate relation:

$$1-\frac{\tau}{\tau_\Lambda}=1.14\left[a^0\left(\frac{Rt}{L^2}\right)^{0.6}\right]^{0.5}, \tag{44.22}$$

where τ_Λ is the critical stress according to the linear theory.

Further, there is an attempt (following Donnell) to relate a^0 with the most probable form of buckling of shells, assuming that

$$a^{\circ} = U_0\left[\left(\frac{m}{1+m}\right)\frac{L}{t}\right]^2.$$

Finally, choosing for U_0 the arbitrary value $U_0 = 5 \cdot 10^{-5}$, Loo obtains the required relation.

In view of the above, we believe that Loo's "solution" given here should be taken with caution.

§ 45. Non-Linear Theory of the Edge Effect in a Cylindrical Shell. The Influence of an Initial Edge Deformation upon the Stability of a Shell under Axial Compression

We shall consider a shell having initial deviations w^0 from the cylindrical surface and corresponding initial stresses and moments $T_1^0, \ldots, M_{12}^0$. The additional deflection due to the applied load will be denoted by w, and the function of the additional stresses by ψ.

Let

$$w = w^b + w^\kappa, \quad \psi = \psi^b + \psi^\kappa, \quad T_1 = T_1^b + T_1^\kappa, \tag{45.1}$$

where w^b, $T_1^b, \ldots$ are quantities determined from the membrane theory. For a cylindrical shell referred to the lines of curvature, in equations (20.19) and (20.21) one should set

$$\begin{gathered} k_{11}^1 = -\frac{\partial^2 w^0}{\partial x^2} - \frac{\partial^2 w}{\partial x^2}, \quad k_{12}^1 = -\frac{\partial^2 w^0}{\partial x \partial s} - \frac{\partial^2 w}{\partial x \partial s}, \\ k_{22}^1 = k - \frac{\partial^2 w^0}{\partial s^2} - \frac{\partial^2 w}{\partial s^2}, \quad T_1^0 = \frac{\partial^2 \psi^0}{\partial s^2}, \quad T_1^\kappa = \frac{\partial^2 \psi^\kappa}{\partial s^2}, \end{gathered} \tag{45.2}$$

where ψ^0 and w^0 satisfy the equations (20.8) and (20.10):

$$D\Delta\Delta w^0 - \frac{\partial^2 \psi^0}{\partial s^2}\frac{\partial^2 w^0}{\partial x^2} + 2\frac{\partial^2 \psi^0}{\partial x \partial s}\frac{\partial^2 w^0}{\partial x \partial s} - \frac{\partial^2 \psi^0}{\partial x^2}\left(\frac{\partial^2 w^0}{\partial s^2} - k\right) = 0, \tag{45.3}$$

$$\Delta\Delta\psi^0 - Et\left[\left(\frac{\partial^2 w^0}{\partial x \partial s}\right)^2 + \frac{\partial^2 w^0}{\partial x^2}\left(k - \frac{\partial^2 w^0}{\partial s^2}\right)\right] = 0. \tag{45.4}$$

Besides, neglecting the changes in curvature of the membrane deflection, we shall set $\partial^2 w/\partial x^2 = \partial^2 w^\kappa/\partial x^2$.

Thus, we shall obtain the following equations for the determination of ψ^k and w^k

$$\begin{gathered} D\Delta\Delta w^\kappa - (T_1^b + T_1^\kappa)\frac{\partial^2 w^0}{\partial x^2} - (T_1^0 + T_1^b + T_1^\kappa)\frac{\partial^2 w^\kappa}{\partial x^2} - \\ - 2(T_{12}^b + T_{12}^\kappa)\frac{\partial^2 w^0}{\partial x \partial s} - 2(T_{12}^0 + T_{12}^b + T_{12}^\kappa)\frac{\partial^2 w^\kappa}{\partial x \partial s} + T_2^\kappa k - \\ - (T_2^b + T_2^\kappa)\frac{\partial^2 w^0}{\partial s^2} - (T_2^0 + T_2^b + T_2^\kappa)\frac{\partial^2 w^\kappa}{\partial s^2} = 0, \end{gathered} \tag{45.5}$$

$$\begin{gathered} \Delta\Delta\psi^\kappa - Et\left[\left(\frac{\partial^2 w^\kappa}{\partial x \partial s}\right)^2 + 2\frac{\partial^2 w^\kappa}{\partial x \partial s}\frac{\partial^2 w^0}{\partial x \partial s} + \right. \\ \left. + \frac{\partial^2 w^\kappa}{\partial x^2}\left(k - \frac{\partial^2 w^0}{\partial s^2} - \frac{\partial^2 w^\kappa}{\partial s^2}\right) - \frac{\partial^2 w^\kappa}{\partial s^2}\frac{\partial^2 w^0}{\partial x^2}\right] = 0. \end{gathered} \tag{45.6}$$

Let the moment part of the deformation due to the load and the initial deformation be characterized by quantities which do not depend on s, or else change very slowly with s, but change rapidly as functions of x. Then

$$\frac{\partial^2 w^0}{\partial s^2} << \frac{\partial^2 w^0}{\partial x^2}, \quad T_1^\kappa << T_2^\kappa, \quad \Delta\Delta\psi^0 = d^4\psi^0/dx^4, \ldots$$

Here the equations (45. 3) and (45. 4) can be replaced by the approximate equations

$$\frac{d^4\psi^0}{dx^4} - Etk\frac{d^2w^0}{dx^2} = 0, \quad D\frac{d^4w^0}{dx^4} + k\frac{d^2\psi^0}{dx^2} = 0.$$

Thus, setting

$$x = \xi L/2, \quad 16\omega^4 = L^4\sqrt{3(1-\nu^2)}/t^2R^2, \tag{45.7}$$

we obtain the equations

$$\frac{d^2\psi^0}{dx^2} = T_2^0 = Etkw^0, \quad \frac{d^4w^0}{d\xi^4} + 4\omega^4w^0 = 0. \tag{45.8}$$

Analogously we find

$$\frac{d^2\psi^\kappa}{dx^2} = T_2^\kappa = Etkw^\kappa, \quad T_1^b = -T, \tag{45.9}$$

$$\frac{d^4w^\kappa}{d\xi^4} + 4\omega^2\mu^2\left(\frac{d^2w^\kappa}{d\xi^2} + \frac{d^2w^0}{d\xi^2}\right) + 4\omega^4w^\kappa = 0, \quad 4\mu^2\omega^2 = \frac{TL^2}{4D}. \tag{45.10}$$

The middle term of the latter equation characterizes the influence of the membrane part of the deformation on the edge effect. Its presence shows that the superposition of the solutions of the membrane theory and the usual linear theory of the edge effect is not admissable in the given case in view of the nonlinearity of the original equations.

For example, let the shell be of the shape of an ideal circular cylinder before clamping the skin to the transverse ribs. Let the inner radius of the skin exceed the outer radius of the transverse rib by the quantity f^0t. We shall assume that when the ends of the skin are clamped to the transverse ribs (for example, by means of closely spaced rivets) this gap is reduced, due to which the skin gets an initial symmetric deflection w^0 equal to f^0t at the transverse ribs, and corresponding initial stresses. We shall determine the influence of this initial deformation on the deformed state of the shell under the action of compressive axial stresses T = const.

Of course, the initial deflection considered is an edge effect. From (45. 8) we find for it the expression

$$w^0 = A_1 \operatorname{ch}\omega\xi\cos\omega\xi + B_1 \operatorname{sh}\omega\xi\sin\omega\xi,$$

where A_1 and B_1 are constants determined by the initial conditions

$$w^0 = -f^0t, \quad dw^0/d\xi = 0 \quad \text{for } \xi = \pm 1. \tag{45.11}$$

If the shell is not very short, then $\operatorname{ch}\omega \approx \operatorname{sh}\omega$. In the zone of the edge $\xi = 1$ we can also set $\operatorname{sh}\omega\xi \approx \operatorname{ch}\omega\xi$. Thus, after slight calculations, we obtain the expression of w^0 for this portion of the shell, satisfying the boundary conditions (45. 11):

$$w^0 = -f^0t \operatorname{ch}\omega\xi\,[\sin\omega(1-\xi) + \cos\omega(1-\xi)]/\operatorname{ch}\omega. \tag{45.12}$$

An analogous expression for the second half of the shell is obtained by replacing ξ by $-\xi$ in (45. 12).

Further, for $\xi \geqslant 0$ we find the approximate expression for the integral of the equation (45. 10)

$$w^\kappa = \operatorname{ch}\alpha\xi\,(A_2\cos\beta\xi + B_2\sin\beta\xi) - w^0,$$

where

$$\alpha = \omega\sqrt{1-\mu^2}, \quad \beta = \omega\sqrt{1+\mu^2}. \qquad (45.13)$$

For $\xi = 1$ the conditions $w = 0$, $dw/d\xi = 0$ have to be satisfied. Calculating the corresponding A_2 and B_2 we obtain

$$w^{\kappa} = -(tf^{\circ} + w^{b})\frac{\operatorname{ch}\alpha\xi}{\operatorname{ch}\alpha}\left[\frac{\alpha}{\beta}\sin\beta(1-\xi) + \cos\beta(1-\xi)\right] - w^{\circ}. \qquad (45.14)$$

Since

$$T_2^b = 0, \text{ thus } \nu\varepsilon_1^b = -\varepsilon_2^b,$$
$$T_1 = T_1^b = Etw^b/\nu R, \quad w^b = \nu TR/Et. \qquad (45.15)$$

We determine the annular stress from (45.9). We set

$$T = \eta E t^2 k, \qquad (45.16)$$

where η is a numerical factor. For the lower critical stress it is approximately equal to 0.19. Introducing this expression into (45.10) and taking (45.7) into consideration, we find

$$\mu^2 = \eta\sqrt{3(1-\nu^2)}. \qquad (45.17)$$

According to (45.8), (45.9), and (45.14) the total circumferential stress is

$$T_2^0 + T_2^{\kappa} = -Etk(tf^{\circ} + w^{b})\frac{\operatorname{ch}\alpha\xi}{\operatorname{ch}\alpha}\left[\frac{\alpha}{\beta}\sin\beta(1-\xi) + \cos\beta(1-\xi)\right]. \qquad (45.18)$$

In the linear theory of the edge effect

$$T_2^0 + T_2^{\kappa} = -Et^2k\cos\omega\xi\,[\sin\omega(1-\xi) + \cos\omega(1-\xi)] : \operatorname{ch}\omega.$$

The difference between these values is large unless μ^2 is small.

We shall calculate the mean circumferential stress along the shell length:

$$(T_2^0 + T_2^{\kappa})_c = \int_0^1 (T_2^0 + T_2^{\kappa})\,d\xi = -2Etk(tf^{\circ} + w^{b})\,\alpha/(\alpha^2 + \beta^2).$$

Denoting by p_c the mean pressure given by (45.13) we find

$$-(T_2^0 + T_2^{\kappa})_c = p_c R = Etk(tf^{\circ} + w^{b})\sqrt{1-\mu^2}/\omega. \qquad (45.19)$$

In solving the problem one can assume, as a first approximation, that a non-uniform pressure of mean density p_c has the same effect on the shell stability as a uniformly distributed pressure p_c, as in both cases, in the anticipated form of buckling, it is essential that one half-wave be formed along the shell length.

Consequently, if $p_c \geqslant p_k$, where p_k is the critical external pressure given by (36.22), the shell can lose its stability under the combined action of the axial compression T and the pressure referred to, arising as a result of the initial deformation.

According to formulas (45.15) and (45.16), $w^b = \nu\eta t$. Equating p_c and p_k and using (36.17), we shall obtain from (45.19) and (36.22) equations for the determination of the critical stress:

$$\eta = 2.15(f^0 + \nu\eta)\sqrt{1 - \eta\sqrt{3(1-\nu^2)}}\,\theta\lambda, \tag{45.20}$$

$$\eta = 1.205\theta\lambda : \left\{(1-\nu^2)^{1/2}\left[1 - 1.81\,(1-\lambda)\,\theta\right]\right\}, \tag{45.21}$$

where

$$\lambda = T/p_{\kappa}R.$$

For given f^0 and θ one can find λ and the critical value of η from these equations.

Let, for example, $\eta = 0.19$, $\omega = 10$, or $\theta = 0.0464$. From equations (45.20) and (45.21) we find $\lambda = 4.1$, $f^0 \approx 0.51$.

Thus, in the case under consideration, with $f \geqslant 0.51$ the shell loses stability when the axial compressive stress reaches its lower critical value.

The above considerations can explain, in the relevant cases, the phenomenon of premature loss of stability of actual cylindrical shells having an initial symmetrical deformation of the edge type.

Chapter XI

LARGE DEFLECTIONS OF SHALLOW CYLINDRICAL STRIPS

§ 46. The Stability and Large Deflections of a Long Cylindrical Plate under a Uniformly Distributed Transverse Load*

We shall consider, within the limits of the theory of shallow shells, the exact solution of the problem of equilibrium of a circular cylindrical plate under the action of an external normal pressure $p = \text{const} > 0$. The results thus obtained will apply to shallow beams if one replaces the flexural rigidity $D = Et^3/12(1-\nu^2)$ by EI.

For very long plates with uniform boundary conditions along the length all the quantities characterizing the deformation depend only on the coordinate s. For the sake of brevity we shall introduce simpler notations as follows instead of those of § 35:

$T_2^I = T$ is the membrane stress along the arc, b is the strip width, $\xi = 2s/b$, $u_2^I = v$ is the displacement along the tangent to the arc before deformation, $w^I = w$ is the normal displacement, w_0 is the initial irregularity.

Let a dot over a letter denote differentiation with respect to ξ. According to equation (35. 2)

$$T = \text{const} = -c_1 = Et\varepsilon/(1-\nu^2) \tag{46. 1}$$

After introducing the dimensionless quantities

$$q = \frac{pRb^2}{4D}, \quad \mu = \frac{b}{2}\sqrt{\frac{c_1}{D}}, \quad W = \frac{4wR}{b^2}, \qquad V = \frac{8R^2v}{b^3}, \quad k^* = \frac{b^2}{Rt}, \tag{46. 2}$$

we shall obtain from (35. 1) and (35. 3) the equations

$$\overset{....}{W} + \mu^2(-1 + \ddot{W} + \ddot{W}_0) + q = 0 \tag{46. 3}$$

$$-\frac{4}{3}\mu^2/k^{*2} = \dot{V} + W + \frac{1}{2}\dot{W}^2 + \dot{W}\dot{W}_0. \tag{46. 4}$$

The retaining of the quadratic term in the expression (46. 4) will allow us to consider, in the following, larger displacements as well.

If the edges are hinged, one has to satisfy the conditions:

$$w = 0, \ \ddot{w} = 0, \ v = 0 \ \text{for} \ \xi = \pm 1. \tag{46. 5}$$

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* See article /XI. 1/. For a sinusoidal beam this question has been investigated in detail in /XI. 2/.

The general solution of the equation (46. 3) for $w_0 = 0$ has the form

$$W = -\frac{c_2}{\mu^2}\cos\mu\xi - \frac{c_3}{\mu^2}\sin\mu\xi + c_4\xi + c_5\cdot\left(\frac{q}{\mu^2} - 1\right)\frac{\xi^2}{2}. \qquad (46.6)$$

Determining c_2, c_3, c_4, c_5 from the boundary conditions (46.5), we obtain

$$W = Q\left(\frac{\cos\mu\xi}{\mu^2\cos\mu} + \frac{\xi^2}{2} - \frac{1}{\mu^2} - \frac{1}{2}\right),\ Q = 1 - \frac{q}{\mu^2}. \qquad (46.7)$$

Introducing this expression in (46.4), integrating and utilizing the boundary conditions for v, we shall obtain a relation between the dimensionless parameter of the external load q and the dimensionless parameter of the compressive stress μ in the form

$$\frac{1}{2}A_1Q^2 - B_1Q + \frac{4}{3}\left(\frac{1}{k^*}\right)^2\mu^2 = 0. \qquad (46.8)$$

Here

$$A_1 = \frac{1}{3} - \frac{2.5}{\mu^2}\operatorname{tg}\mu + 0.5\,\frac{\operatorname{tg}^2\mu}{\mu^2} + \frac{2.5}{\mu^2};\ B_1 = \frac{\operatorname{tg}\mu}{\mu} - \frac{1}{\mu^2} - \frac{1}{3}. \qquad (46.9)$$

We shall denote the deflection parameter at the vertex of the panel vertex by ζ. According to (46. 4)

$$\zeta = Q\left(\frac{1-\cos\mu}{\mu^2\cos\mu} - \frac{1}{2}\right). \qquad (46.10)$$

The relations (46. 8) and (46. 10) allow one to investigate the behavior of an infinitely long shallow panel with hinged edges. For any given value of the curvature parameter k^*, the critical values of the parameter of the external load q_1 and q_2 (pressures of snapping and of collapse) can be found from (46. 8), from (46. 10) we can obtain the corresponding values of the deflection parameter ζ_1, ζ_2 at the strip vertices. When the curvature parameter k^* is sufficiently large, by neglecting in equation (46. 8) the term $4\,\mu^2/3k^{*2}$ we shall obtain

$$Q = 0, \qquad (46.11)$$

or

$$\frac{1}{2}A_1Q = B_1. \qquad (46.12)$$

The relation (46. 11) corresponds to the membrane state of the strip before the snapping. In fact, from (46. 11) and (46. 2) we have $c_1 = pR$.

The equation (46. 12) describes the behavior of the strip after snapping. As the calculations show, for $k^* > 70$ one may use the equations (46. 11) and (46. 12) which do not depend on k^* instead of (46. 8).

For the following investigation we shall write equation (46. 8) in the form

$$Q = (B_1 \pm \sqrt{\Delta_1}) : A_1;\quad \Delta_1 = B_1^2 - \frac{8}{3}A_1\mu^2/k^{*2}. \qquad (46.13)$$

From (46. 8) and (46. 9) it is seen that for $\mu = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ the quantity Q has multiple roots equal to zero. The corresponding values of the pressure parameter will be: $q = \frac{\pi^2}{4}$, ... However, for a given k* only those values of the parameter μ for which $\Delta_1 \geqslant 0$ have a real meaning.

According to (46.13) to every given μ correspond two values of Q, and consequently, two values of q. One of these values of q corresponds to the state before the buckling. Depending on the value of the curvature parameter k* two cases can occur:

1. $\Delta_1 = 0$ with $\mu < \pi/2$. Then there is only one multiple root, corresponding to the value of μ obtained from the condition $\Delta_1 = 0$. In that case, the snapping phenomenon does not exist, and the curves of (46.8) and (46.10), which, for brevity, will be denoted by $F(q, \mu) = 0$, $\Phi(q, \zeta) = 0$, will have the forms shown in Figure 31.

2. $\Delta_1 = 0$ with $\mu > \pi/2$. In that case there are two multiple roots, one of which corresponds to the value $\mu = \pi/2$, and the second to the value obtained from the condition $\Delta_1 = 0$.

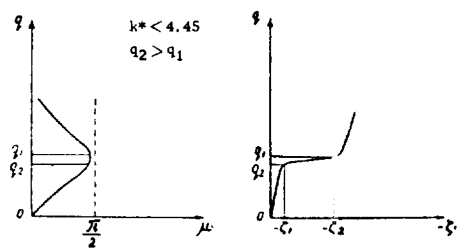

Figure 31

In this case the dependence curves $F(q, \mu) = 0$, $\Phi(q, \zeta) = 0$, have the forms shown in Figure 32. The strip loses its stability with snapping.

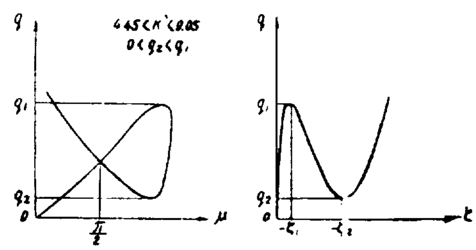

Figure 32

The maximum of the compressive stress parameter μ which is possible in strips with hinged edges will be the one for which $\Delta_1 \geqslant 0$ when $k^* \to \infty$.

As $A_1 > 0$ always holds, the value of μ found from the equation $B_1 = 0$ will be the greatest. It is $\mu \approx 4.685$.

The smallest value of k* for which snapping occurs is determined by the condition $\Delta_1 > 0$ at $\mu \to \pi/2$. Hence, $k^* > 4.45$.

From Figure 32 it is apparent that snapping is possible only when the compression parameter μ becomes greater than $\pi/2$. The value of $\mu = \pi/2$ corresponds to the Eulerian compressive force for a strip of length b with hinged ends. As a matter of fact, from (46. 2), for $\mu=\pi/2$, $c_1=\pi^2D/b^2=T_*$.

Determining the critical values of q_1 and q_2 from (46. 13) is equivalent to determining the maxima and minima of the curve $F(q, \mu) = 0$. Owing to the tedious calculations involved in expressing q in terms of μ, we find the quantities q_1 and q_2 for various values of k* by constructing curves through the points. The quantities p_1 and p_2 thus obtained may be expressed as

$$p_1=\alpha_1(k^*)\frac{4D}{Rb^2},\quad p_2=\alpha_2(k^*)\frac{4D}{Rb^2}. \tag{46.14}$$

So far we have confined ourselves to the consideration of only the symmetric form of strip buckling, whereas the general expression (46. 6) for the deflection function also contains an asymmetrical term. For hinged rectilinear edges the boundary conditions (46. 5) have to be satisfied, from which it follows that for all $\mu\neq\pi$, $c_3 = c_4 = 0$, and the symmetric form of the deflection is the only possible one. For $\mu=\pi$, along with the symmetrical form of the deflection, an asymmetrical form also becomes possible. This testifies to the fact that parts of the curves

$$F(q, \mu)=0,\ \Phi(q, \zeta)=0 \tag{46.15}$$

correspond, for the values $\mu>\pi$, to unstable states of equilibrium, and when determining the loading capacity of the strip they should be dropped.

Asymmetrical buckling appears only in strips whose curvature parameters satisfy the condition $\Delta_1>0$ for $\mu=\pi$. Calculations give the value $k^*>9.04$.

Thus, the process of buckling of a strip with a large curvature parameter is described by the solution obtained in the following way. Initially, with increase of the load, the panel deflections are symmetrical; when the compressive stress parameter μ reaches the value π, an asymmetrical buckling occurs (by jump) to the equilibrium state which, for large deflections, will again by symmetrical.

Taking into account the asymmetrical form of buckling, we obtain a table for the values of α_1 and α_2 for hinged edges.

Table X

k^*	4.45	10	20	30	50	70	∞
α_1	2.47	5.81	9.08	9.55	9.74	9.80	9.87
α_2	2.47	-0.75	-3.97	-4.45	-4.62	-4.69	-4.75

Let the boundary conditions of the problem have the form

$$W=\frac{\delta}{2}Q;\quad W'=\pm\gamma Q;\quad V=\pm\beta\cdot\frac{4}{3}\cdot\frac{\mu^2}{k^{*2}}\ \text{for}\ \xi=\pm1. \tag{46.16}$$

The coefficients δ, γ, and β characterize the flexibility of the supports along the normal, with respect to rotation and in the tangent plane respectively.

Applying the boundary conditions (46. 16), (46. 4), and (46. 6), we find

$$W=Q\left[\frac{(1+\gamma)\cos\mu\xi}{\mu\sin\mu}+\frac{\xi^2}{2}-\frac{(1+\gamma)\cos\mu}{\mu\varepsilon}-\frac{1}{2}(1+\delta)\right], \tag{46. 17}$$

$$\frac{1}{2}AQ^2-BQ+C=0, \tag{46. 18}$$

where

$$A=\frac{1}{3}+\frac{(1+\gamma)^2}{2\sin^2\mu}+\frac{(1+\gamma)(3-\gamma)}{2}\,\frac{\cos\mu}{\mu\sin\mu}-\frac{2(1+\gamma)}{\mu^2};$$

$$B=\frac{1+\gamma}{\mu^2}-\frac{2+3\delta}{6}-\frac{1+\gamma}{\mu}\operatorname{ctg}\gamma;\ C=\frac{4}{3}\,\frac{\mu^2}{k^{*2}}(1+\beta). \tag{46. 19}$$

These relations allow us to consider the influence of the various deviations from the conditions of rigid and hinged edges on the carrying capacity of the strip. As an illustration, let us see this for a strip with a curvature parameter $k^* = 40$.

A. Let the supports be flexible in the normal direction. In that case, $\beta=\gamma=0$, $\delta\neq 0$.

Let us find the minimum of the absolute value of δ for which snapping no longer occurs. This may be determined with sufficient accuracy from the condition

$$B^2-2AC\leqslant 0, \tag{46. 20}$$

if one sets $\mu = 3.2$. It is $\delta\approx -7.91$. Here, if $t = 1$ mm, then $w(s = \pm b/2) = 0.415$ mm, and the value of the load parameter of the buckling drops from $q_1(\delta = 0) = 19.14$ to $q_1(\delta = -7.91)\approx \pi^2$.

Hence it can be seen that the loading capacity of the strip decreases when, under load, the supports undergo an additional displacement from the center of gravity.

B. Let the supports be flexible in the tangent plane. In that case, $\delta = \gamma = 0$ $\beta\neq 0$.

The coefficient β can take various values depending on the degree of flexibility of the supports. We shall find the value of β for which the strip with $k^* = 40$ no longer snaps. This value of β will be obtained with sufficient accuracy from equation (46. 20) with $\mu = 3.2$. It is $\beta = 10.77$.

Consequently, at $\beta\approx 11$ the buckling load becomes $q_1 = \pi^2$, instead of 19.14. Since in shallow strips and beams there is a considerable thrust even for small external pressures, then, due to the flexibility of the supports in the tangent plane, the buckling load can turn out to be less than half of its value for the case of rigid fastening.

C. Let the supports be flexible with respect to rotation. In that case, $\delta=\beta=0$, $\gamma\neq 0$.

Calculations show that for $\gamma = -1.0536$ the panel snapping does not occur and the loading capacity is reduced from $q_1 = 19.14$ to $q_1 \approx \pi^2$.

Thus, the impossibility of realizing ideal boundary conditions of hinged or rigid fastening in experiment and in actual structures can be one of the reasons for the fact that the observed critical loads sometimes turn out to be much smaller than the theoretical ones.

Another reason for this state of affairs is the influence of the initial irregularities on the behavior of the strip.

Our analysis of the exact solution of the equilibrium equation for the rigid fastening of the panel edges having symmetrical irregularities of the form $w_0 = -\alpha t \cos \pi\xi/2$ and $w_0 = -\alpha t(1 + \cos \pi\xi)$ or an asymmetrical irregularity of the form $w_0 = -\alpha t \sin \pi\xi/2$ shows that for $\alpha = 1$ such irregularities reduce the snap pressure by 10-30%, while with increasing k* the influence of the irregularities decreases. At the same time, these irregularities increase somewhat the collapse pressure p_2.

Of considerably greater influence is an antisymmetric irregularity of the form

$$W_0 = -\alpha t \sin n\pi\xi \qquad (46.21)$$

In Table XI are given the results of calculations of the corresponding pressure parameter of the snapping q_1^1 for $\alpha = 1$, and also, for comparison, the values of q_1^0 for $\alpha = 0$.

Table XI

k^*	40	60	80	100
q_1^0	19.16	19.78	19.88	20.08
q_1^1 $n = 1$	10.33	12.18	12.68	13.80
$q_1^1(n=1):q_1^0$	0.54	0.615	0.635	0.69
$q_1^1(n=2)$	9.5	11.07	12.27	13.07
$q_1^1(n=2):q_1^0$	0.49	0.59	0.61	0.65

Calculations show that for $n>2$, the quantity q_1 begins to increase rapidly, approaching q_1^0. This indicates that the antisymmetrical irregularity is less dangerous for a high frequency (n = 3; 4; 5) than for a low frequency (n = 1; 2).

The results of calculating q_2 for certain values of k* for the irregularities (46.21) with n = 1 and $\alpha = 1$ are given in Table XII.

Table XII

k^*	40	60	80	100
q_2^0	4.88	4.35	4.17	4.18
q_2^1	9.66	8.25	7.42	7.02
$q_2^1 : q_2^0$	1.98	1.90	1.78	1.7

From Table XII it can be seen that the antisymmetrical irregularity increases the lower critical load, and its influence on q_2 turns out to be just as strong as on q_1. Therefore, von Karman's assertion about the weak influence of initial imperfections of shape on the value of the lower critical load, expressed by him in /XI. 3/, is erroneous. An antisymmetrical irregularity can, by lowering the upper critical load and raising the lower one, eliminate the possibility of snapping for a certain amplitude.

§47. The Convergence of Galerkin's Method for the Solution of the Problem of §46

In view of the fact that an exact solution of non-linear problems of the theory of shells is possible only in very rare cases, it is interesting to clarify the effectiveness of using approximate methods for the solution of such problems by comparing the approximate solutions with the exact ones. We shall carry out this comparison on the example of the problem of §46, whose approximate solution will be found by Galerkin's method /XI. 4/. Let

$$W=\sum_{m=1}^{\infty}\zeta_m\cos\frac{m\pi\xi}{2},\quad m=1,\ 3,\ \ldots, \tag{47.1}$$

where ζ_1, ζ_2, ... are arbitrary parameters, yet to be determined.

Of course, the boundary conditions (46.5) for W are satisfied here.

Introducing (47.1) in equation (46.4), we obtain V, and setting $V=0$ for $\xi=\pm 1$, we obtain the relation

$$\mu^2=-\frac{6k^{*2}}{4\pi}\sum_{m=1}^{\infty}(-1)^l\frac{\zeta_m}{m}-\frac{\pi^2}{54}k^{*2}\sum_{m=1}^{\infty}(m\zeta_m)^2, \tag{47.2}$$

where

$$m=1,\ 3,\ 5,\ 7,\ \ldots;\ l=2 \text{ for } m=1,\ 5,\ 9,\ \ldots;$$
$$l=1 \text{ for } m=3,\ 7,\ 11,\ \ldots.$$

Introducing (47.1) in equation (46.3) and integrating the latter by Galerkin's method we shall obtain an infinite system of equations:

$$-\zeta_m\left(\frac{m\pi}{2}\right)^4+\mu^2\left[(-1)^l\frac{4}{m\pi}+\left(\frac{m\pi}{2}\right)^2\zeta_m\right]=(-1)^l\frac{4}{m\pi}q, \tag{47.3}$$

where m and l take the same values as in (47.2). The relations (47.2), (47.3) allow one to determine the symmetric deformation of the strip with any degree of accuracy.

In (46.3) and (46.4), substituting the quantities $W+W_H$, $V+V_H$, and $\mu^2+\delta\mu^2$ for W, V, and μ^2, where W_H, V_H, and $\delta\mu^2$ are infinitesimal increments, we obtain the equations of neutral equilibrium

$$\ddot{\ddot{W}}_H+\mu^2\ddot{W}_H+\delta\mu^2(\ddot{W}-1)=0, \tag{47.4}$$

$$-4\delta\mu^2/3k^{*2}=\dot{V}_H+W_H+W\dot{W}_H. \tag{47.5}$$

As is well known, for very shallow strips the smallest value of the critical load corresponds to the loss of stability in asymmetrical form with the formation of two half-waves; therefore we set

$$W_H=\zeta_H\sin\pi\xi. \tag{47.6}$$

Substituting in (47.5), we obtain V_H, and setting $V_H=0$ for $\xi=\pm 1$, we obtain: $\delta\mu^2=0$.

Here, from equation (47.4) it follows that $\mu^2=\pi^2$, i.e., the asymmetric form of buckling appears when μ becomes equal to π. The exact solution gives us the same value for μ.

Substituting $\mu=\pi$ in the equations (47.2) and (47.3) one can determine the critical values of the deflection parameter ζ_{m1}, ζ_{m2} and the corresponding values of the load parameter q_1, q_2 for various k^* (see Table XIII).

Table XIII

k^*		$W=\zeta_1 \cos \pi\xi/2$	$W=\zeta_1 \cos \pi\xi/2 + \zeta_3 \cos \frac{3\pi\xi}{2}$	Exact solution
20	q_1	9.085	9.08	9.08
	q_2	-4.15	-3.97	-3.97
70	q_1	9.810	9.806	9.800
	q_2	-4.873	-4.698	-4.690

As can be seen from the table, the solution by Galerkin's method almost coincides, in the second approximation, with the exact solution, whereas the first approximation gives a larger value for q_1 and a smaller value for q_2.

For strips with rigidly fixed edges we take the expression for the deflection in the form of the series

$$W=\sum_{n=1}^{\infty}\zeta_n[1+(-1)^{n-1}\cos n\pi\xi], \quad n=1, 2, \ldots . \tag{47.7}$$

Proceeding analogously to the above, we shall obtain

$$\mu^2=-\frac{3k^{*2}}{4}\sum_{n=1}^{\infty}\zeta_n-\frac{3\pi^2k^{*2}}{16}\sum_{n=1}^{\infty}n^2\zeta_n^2, \tag{47.8}$$

$$q=\mu^2+\frac{1}{2}(n\pi)^2[\mu^2-(n\pi)^2]\zeta_n . \tag{47.9}$$

Let

$$W_n=\zeta_n \sin \pi\xi \cos \frac{\pi\xi}{2} . \tag{47.10}$$

According to (47.5) we again obtain $\delta\mu^2=0$, whereupon, integrating equation (47.4) by the Galerkin method, we find $\mu^2\approx 20.23$. The exact solution gives the value of $\mu^2\approx 20.25$.

Setting $\mu^2\approx 20.23$ in equations (47.8) and (47.9), we obtain relations for the determination of the critical values of the deflection parameters ζ_{n1}, ζ_{n2} the corresponding values of the load parameter q_1, q_2. Calculations for $k^*=100$ yield:

1-e approximation	$q_1=20.09$;	$q_2=-0.38$;
2-e "	$q_1=20.08$;	$q_2=3.67$;
3-e "	$q_1=20.08$;	$q_2=4.04$;
4-e "	$q_1=20.08$;	$q_2=4.13$;
Exact solution	$q_1=20.08$;	$q_2=4.22$.

Thus, in the case of fixed edges as well, the second approximation coincides, within the limits of accuracy of the calculations, with the exact solution for q_1, and the fourth approximation gives almost the exact value of q_2. This case illustrates the process of approximating the exact value of q_2 from below in a particularly convenient way.

The calculations listed show that the application of the Galerkin method enables one to find with sufficient accuracy the upper critical load as well as the lower one. For the determination of the lower critical load one requires a higher approximation.

We do not have a formal proof of the convergence of Galerkin's method for the solution of non-linear problems. However, it can be explained why, in using this method, we approach monotonically the real value of the upper critical load from above, and the real value of the lower critical load from below. As a matter of fact, when the condition of compatibility is satisfied, Galerkin's method follows from the principle of virtual displacements. According to this, by taking an increased number of terms from the series which approximates the deflection, we increase the number of degrees of freedom of the system. This facilitates the snapping of the strip, as well as the return collapse. Here it is necessary to note that in our investigation we assume the completeness of the system of approximating functions.

The essential nature of that condition is confirmed by actual experience in solving non-linear problems. For example, in determining the lower critical value of the axial compressive stress for a complete cylindrical shell (works /X. 41/ and the same form of deflection was taken as was defined by the formulas (40. 5) and (40. 6). But the first of these works an additional constraint was imposed on the possible amplitudes by the assumption that $g_2 = g_3$. Owing to this, the complete system of approximating functions

$$g_0, \; g_1 \cos mkx \cos nks, \quad g_2 \cos 2mkx, \quad g_3 \cos 2nks, \ldots$$

was turned into an incomplete system of functions

$$g_0, \; g_1 \cos mkx \cos nks, \; g_2(\cos 2mkx + \cos 2nks), \ldots,$$

which led to an increased value (by 7%) for the lower critical stress (40. 20), instead of the value (40. 22), found by Kempner by varying the total energy with respect to all the parameters g_0, g_1, g_2, and g_3.

It should also be noted that the successive approximations to the actual expression for the deflection, obtained by including in the expression of the approximating function new terms of the series of the complete system of functions with the corresponding new deflection parameters, give monotonically varying approximations to the lower critical stress only when the chosen deflection function for the first approximation characterizes sufficiently well the actual shape of the deformation. For example, our calculations have shown that if in solving the problem of § 46 one approximates the deflection function in the first approximation by a sine-form with three half-waves along the strip width, then the pressure vs. deflection graph obtained is situated much higher than the actual graph, and with increasing deflection the pressure increases monotonically. Later on, with the introduction in the expression for deflection of terms which give sine-form with two and one half-waves along the strip width, one begins to obtain graphs with lower and upper extremum points, and then, by passing to the consideration of a sine-form with 4, 5, ... half-waves, the extremum points from above and from below, corresponding to the upper and the lower critical value of pressure, begin to approach each other monotonically.

§ 48. Cylindrical Strip Supported on Ribs, Flexible in the Tangent Plane, under the Action of Normal Pressure

We shall consider the problem of determining the large deflections of a rectangular cylindrical strip, freely supported at the edges on ribs /XI.6/. Here it is assumed that the transverse sections of these ribs have a very large moment of inertia about the axis passing through the center of gravity of the section parallel to the plate surface, and a very small one about the axis perpendicular to the plate surface. We shall therefore consider that the ribs do not allow the plate edges to be displaced in the direction perpendicular to its surface, but do not at all hinder displacements in directions tangential to its surface and perpendicular to the rib.

It is also assumed that the ribs are inextensible. For the longitudinal ribs of the strip, at s = 0 and s = b, the following boundary conditions should be directly satisfied

$$\varepsilon_1 = \frac{1}{Et}\left(\frac{\partial^2\psi}{\partial s^2} - \nu\frac{\partial^2\psi}{\partial x^2}\right) = 0, \quad T_2 = \frac{\partial^2\psi}{\partial x^2} = 0. \tag{48.1}$$

The first of these conditions signifies that the strip next to the rib, as well as the rib itself, does not expand in the direction of the rib. The second condition signifies that the rib does not resist bending in the direction tangential to the strip.

From the preceding two equations it follows that with s = 0 and s = b

$$T_1 = \frac{\partial^2\psi}{\partial s^2} = 0. \tag{48.2}$$

Analogously, one can obtain the boundary conditions satisfied by the function ψ in the neighborhood of the transverse ribs

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial s^2} = 0 \quad \text{for } x = 0, \quad x = a. \tag{48.3}$$

In view of the free support of the strip on the ribs, further conditions have to be satisfied which ensure the absence of bending moments at the strip edges:

$$\begin{aligned} w &= \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{for } x = 0,\ x = a; \\ w &= \frac{\partial^2 w}{\partial s^2} = 0 \quad \text{for } s = 0,\ s = b. \end{aligned} \tag{48.4}$$

We shall seek the approximation to the deflection function in the form

$$w = \sum_{m=1}^{N}\sum_{n=1}^{N} C_{mn} w_{mn}, \quad w_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi s}{b}. \tag{48.5}$$

It is obvious that every term of that series satisfies the conditions (48. 4). Intro ducing this expression for w into the equation (40. 3) we shall obtain

$$\Delta\Delta\psi = -Et\sum_{mnpq} C_{mn}C_{pq}\left(\frac{\partial^2 w_{mn}}{\partial x^2}\cdot\frac{\partial^2 w_{pq}}{\partial s^2} - \frac{\partial^2 w_{mn}}{\partial x\partial s}\cdot\frac{\partial^2 w_{pq}}{\partial x\partial s}\right) + \tag{48.6}$$
$$+\frac{Et}{R}\sum_{mn} C_{mn}\frac{\partial^2 w_{mn}}{\partial x^2}.$$

We shall solve this equation by the Fourier method. We shall utilize the representation in the form of a double Fourier series of the function

$$\frac{\partial^2 w_{mn}}{\partial x^2}\cdot\frac{\partial^2 w_{pq}}{\partial s^2} - \frac{\partial^2 w_{mn}}{\partial x\partial s}\cdot\frac{\partial^2 w_{pq}}{\partial x\partial s} = \sum_{kl}\frac{4}{ab}A^{kl}_{mnpq}w_{kl} \tag{48.7}$$

and its odd continuation beyond the region $0<x\leqslant a$, $0\leqslant s\leqslant b$, where k, l = 1, 2, 3,...

$$A^{kl}_{mnpq} = \int\int\left(\frac{\partial^2 w_{mn}}{\partial x^2}\cdot\frac{\partial^2 w_{pq}}{\partial s^2} - \frac{\partial^2 w_{mn}}{\partial x\partial s}\cdot\frac{\partial^2 w_{pq}}{\partial x\partial s}\right)w_{kl}\,dx\,ds. \tag{48.8}$$

Here and in what follows the integrals are taken over the entire surface of the strip. Introducing (48.7) in (48.6) we obtain

$$\Delta\Delta\psi = -Et\sum_{mnpqkl}\frac{4}{ab}A^{kl}_{mnpq}w_{kl}C_{mn}C_{pq} - \tag{48.9}$$
$$-\frac{Et}{R}\sum_{mn}C_{mn}\left(\frac{m\pi}{a}\right)^2 w_{mn}.$$

We shall seek the solution of this equation in the form of the series

$$\psi = \sum_k\sum_l a_{kl}w_{kl}, \tag{48.10}$$

each of whose terms satisfies the boundary conditions (48.1) and (48.3), and the coefficients a_{kl} satisfy the system of equations

$$\left[\left(\frac{k\pi}{a}\right)^2+\left(\frac{l\pi}{b}\right)^2\right]^2 a_{kl} = -Et\sum_{mnpq}\frac{4}{ab}A^{kl}_{mnpq}C_{mn}C_{pq} - \frac{Et}{R}\left(\frac{k\pi}{a}\right)^2 C_{kl}. \tag{48.11}$$

Here the quantities C_{pq}, C_{mn}, C_{kl} are zero if at least one of the indices k, ..., q exceeds N. Further, introducing the expressions (48.5) and (48.10) in (40.4), multiplying both members of the obtained equation by w_{rs} and integrating over the entire surface of the strip, we arrive, after some simple calculations, at the system of equations:

$$-D\left[\left(\frac{\pi r}{a}\right)^2+\left(\frac{\pi s}{b}\right)^2\right]^2 C_{rs} + \sum_{klmn}4a_{kl}C_{mn}\frac{1}{ab}\int\int\left[\frac{\partial^2 w_{kl}}{\partial x^2}\cdot\frac{\partial^2 w_{mn}}{\partial s^2} + \right. \tag{48.12}$$
$$\left. + \frac{\partial^2 w_{kl}}{\partial s^2}\cdot\frac{\partial^2 w_{mn}}{\partial x^2} - 2\frac{\partial^2 w_{kl}}{\partial x\partial s}\cdot\frac{\partial^2 w_{mn}}{\partial x\partial s}\right]w_{rs}\,dx\,ds +$$
$$+\frac{1}{R}\left(\frac{\pi r}{a}\right)^2 a_{rs} = \frac{4}{ab}\int\int p w_{rs}\,dx\,ds, \quad r,\ s=1,\ 2,\ \ldots,\ N.$$

Integrating by parts and taking into account that at the strip edges the functions w_{rs} are zero, we have

$$\iint \frac{\partial^2 w_{kl}}{\partial x^2} \cdot \frac{\partial^2 w_{mn}}{\partial s^2} w_{rs} dxds =$$

$$= \iint w_{kl} \left(\frac{\partial^4 w_{mn}}{\partial x^2 \partial s^2} w_{rs} + 2 \frac{\partial^3 w_{mn}}{\partial x \partial s^2} \frac{\partial w_{rs}}{\partial x} + \frac{\partial^2 w_{mn}}{\partial s^2} \cdot \frac{\partial^2 w_{rs}}{\partial x^2} \right) dxds;$$

$$\iint \frac{\partial^2 w_{kl}}{\partial s^2} \cdot \frac{\partial^2 w_{mn}}{\partial x^2} w_{rs} dxds =$$

$$= \iint w_{kl} \left(\frac{\partial^4 w_{mn}}{\partial x^2 \partial s^2} w_{rs} + 2 \frac{\partial^3 w_{mn}}{\partial x^2 \partial s} \frac{\partial w_{rs}}{\partial s} + \frac{\partial^2 w_{mn}}{\partial x^2} \frac{\partial^2 w_{rs}}{\partial s^2} \right) dxds.$$

Consequently,

$$B^{rs}_{klmn} = A^{rs}_{klmn} + A^{rs}_{mnkl} = A^{kl}_{mnrs} + A^{kl}_{rsmn} =$$

$$= \iint w_{kl} \left(\frac{\partial^2 w_{mn}}{\partial s^2} \cdot \frac{\partial^2 w_{rs}}{\partial x^2} + \frac{\partial^2 w_{mn}}{\partial x^2} \frac{\partial^2 w_{rs}}{\partial s^2} - 2 \frac{\partial^2 w_{mn}}{\partial x \partial s} \frac{\partial^2 w_{rs}}{\partial x \partial s} \right) dxds.$$

Introducing the latter and the expressions (48.11) in (48.12), we obtain for the coefficients C_{rs} the system of equations

$$\begin{gathered}
-D\left[\left(\frac{\pi r}{a}\right)^2 + \left(\frac{\pi s}{b}\right)^2\right]^2 C_{rs} - \sum_{klmn} \sum_{\alpha\beta\gamma\delta} \frac{16Et\, B^{kl}_{rsmn} A^{kl}_{\alpha\beta\gamma\delta} C_{mn} C_{\alpha\beta} C_{\gamma\delta}}{[(k\pi/a)^2 + (l\pi/b)^2]^2 a^2 b^2} - \\
- \sum_{klmn} \frac{4Et}{abR} \cdot \frac{B^{kl}_{rsmn} (k\pi/a)^2 C_{kl} C_{mn}}{[(k\pi/a)^2 + (l\pi/b)^2]^2} - \frac{Et (\pi r/a)^4 C_{rs}}{R^2 [(\pi r/a)^2 + (\pi s/b)^2]^2} - \\
- \frac{4Et}{abR} \left(\frac{\pi r}{a}\right)^2 \sum_{mnpq} \frac{A^{rs}_{mnpq} C_{mn} C_{pq}}{[(\pi r/a)^2 + (\pi s/b)^2]^2} = \frac{4}{ab} \iint p w_{rs} dxds \\
(r,\ s = 1,\ 2,\ \ldots,\ N).
\end{gathered} \tag{48.13}$$

In order to obtain simpler formulas we shall consider the first approximation to the solution, when N = 1. In that case, assuming that the pressure is distributed uniformly over the whole surface of the strip, after some simple calculations and the introduction of the notations

$$k^* = b^2/Rt, \quad H = b^2/8R, \quad \gamma = b^2/a^2, \tag{48.14}$$

we find the following dependence of the pressure on the deflection at the center of the strip:

$$\begin{gathered}
p = \frac{\pi^2}{64R^2} \cdot \frac{Etb^2\gamma^2}{(1+\gamma)^2} \left\{ A(\gamma) \left(\frac{C_{11}}{H}\right)^3 - \left(\frac{C_{11}}{H}\right)^2 - \right. \\
\left. - \left[\frac{1}{2} + \frac{4.46\gamma^2}{k^{*2}} (1+\gamma)^4\right] \frac{C_{11}}{H} \right\}.
\end{gathered} \tag{48.15}$$

Here

$$A(\gamma) = (\gamma + 1)^2 \sum_{kl} \frac{1}{(k^2\gamma + l^2)^2} \left(\frac{1}{k} \cdot \frac{1}{4 - l^2} + \frac{1}{l} \cdot \frac{k}{4 - k^2} \right)^2. \tag{48.16}$$

We give the values of this quantity for some values of the elongation parameter of the strip:

$b/a = 1$	0.75	0.5	0.3
$A = 0.46$	0.47	0.49	0.5

To solve the problem in the second approximation we shall take the deflection function in the form

$$w = C_{11}\sin\frac{\pi x}{a}\sin\frac{\pi s}{b} + C_{13}\sin\frac{\pi x}{a}\sin\frac{3\pi s}{b}. \tag{48.17}$$

Here, from (48. 13) we find two equations for the determination of the deflection coefficients C_{11} and C_{13}. The coefficients of these equations contain the quantities A^{kl}_{mnpq}, calculated from (48. 8), and also the quantities:

$$B^{kl}_{mnpq} = A^{kl}_{mnpq} + A^{kl}_{pqmn} = B^{kl}_{pqmn} = B^{lk}_{nmpq} =$$
$$= \frac{8\,klmnpq\,[2(m^2q^2+n^2p^2)-(k^2-m^2-p^2)(l^2-n^2-q^2)]\,\pi^2}{[(k^2-m^2-p^2)^2-4m^2p^2]\,[(l^2-n^2-q^2)^2-4n^2q^2]\,ab}, \tag{48.18}$$
$$k,\ l=1,\ 3,\ \ldots;\ m,\ n,\ p,\ q=1\ \text{or}\ 3.$$

Utilizing the notation (48.14) and introducing the new notations

$$\zeta = C_{11}/t,\ \zeta_{13} = C_{13}/C_{11},\ \ p^* = pb^4/Et^4, \tag{48.19}$$

we shall obtain a system of two cubic equations in ζ and ζ_{13}:

$$\zeta^3(\sigma_{11}+\sigma_{21}\zeta_{13}+\sigma_{31}\zeta_{13}^2+\sigma_{41}\zeta_{13}^3)+$$
$$+\zeta^2(\sigma_{51}+\sigma_{61}\zeta_{13}+\sigma_{71}\zeta_{13}^2)\frac{k^*}{8}+\zeta\left(\sigma_{81}+\sigma_{91}\frac{k^{*2}}{64}\right)=-1{,}621p^*, \tag{48.20}$$

$$\zeta^3(\sigma_{12}+\sigma_{22}\zeta_{13}+\sigma_{23}\zeta_{13}^2+\sigma_{43}\zeta_{13}^3)+\zeta^2(\sigma_{53}+\sigma_{63}\zeta_{13}+\sigma_{73}\zeta_{13}^2)\frac{k^*}{8}+$$
$$+\zeta\left(\sigma_{83}+\sigma_{93}\frac{k^{*2}}{64}\right)\zeta_{13}=-0{,}5404p^*. \tag{48.21}$$

The values of the coefficients σ_{ik} for some γ are listed in Table XIV*.

Table XIV

$\frac{a}{b}$	σ_{11}	$\sigma_{21}=3\sigma_{13}$	$\sigma_{31}=\sigma_{23}$	$\sigma_{41}=3\sigma_{33}$	σ_{51}	σ_{61}	σ_{71}	σ_{81}
2	33.35	−135.7	487	−75.29	81.92	−86.02	172.9	223.0
1	14.92	−38.10	117.9	−48.72	32 00	−31.91	56.28	35.68
0.5	2.497	−6.20	35.63	−10.63	5 120	−5.052	8.643	13.94
0.3	0.472	−1.256	3.833	−2.614	0 8726	−0.859	1.462	10.60

$\frac{b}{a}$	σ_{91}	σ_{43}	σ_{53}	σ_{63}	σ_{73}	σ_{83}	σ_{93}
2	40.96	1032	−43.01	345.8	16.35	1507	6.059
1	16	388	−15.96	112.6	3.84	892	0.64
0.5	2.56	77.34	−2.526	17.29	0.2805	763.2	0.0467
0.3	0.436	18.81	−0.430	2.924	0.0376	737	0.00627

Eliminating p^* from equations (48.20) and (48.21) we find a quadratic equation in ζ, whose coefficients are polynomials of at most the third degree in ζ_{13}

Taking various values of ζ_{13} and determining the corresponding values of ζ from (48. 21), one can construct a graph of the change of the pressure parameter p^*.

Example. Let $\gamma = b^2/a^2 = 1/4$.

----

* See article /IX.8/, which has a misprint in formula (3.2).

The values of relative deflection ζ_1^0, ζ_2^0 at the center of the plate, the upper critical pressure parameter p_1^* and of the lower critical pressure parameter p_2^*, calculated from the first approximation formula in M. A. Koltunov's article /XI. 7/ and from our formula, are given in Table XV.

Table XV

	a/b	k*	ζ_1^0	ζ_2^0	p_1^*	p_2^*
First approximation according to /XI. 7/...	2	40	2.124	5.380	44.8	20.2
From (48.15)	2	40	2.30	3.50	45.8	37.6
From /XI. 7/	2	80	3.41	11.64	256	-127
From (48.15)	2	80	3.67	9.93	262	54
From (48.21)	2	80	3.59	9.92	253	68

As can be seen, the results calculated according to the formula of article /XI. 7/ and from (48. 15) differ considerably. Both of these formulas were derived taking one and the same form of the deflection for the same boundary conditions, but the first of them, in contradistinction to the second, was obtained by integrating the conditions of compatibility by Galerkin's method. Let us note that in the example considered, the difference in the value of the upper critical pressure, determined from these formulas, is not large. The collapse pressure (and also the strip deformation after the snapping)is determined incorrectly, by Koltunov's method in the first approximation. Hence it follows that in using the Bubnov-Galerkin method for the integration of the equations of compatibility, it is necessary to determine the strain function more exactly by taking for it an expression which contains some varying parameters.

The critical pressures are determined by us in the second approximation only for the value of the curvature parameter k* = 80. From Table XV we see that the second approximation reduces by 3.5% the value of the upper critical pressure and increases by almost 25% the lower critical pressure in comparison with the first approximation according to (48. 15).

§ 49. Influence of an Asymmetrical Irregularity on the Deformation of A Shallow Strip under a Transverse Load

We shall consider the deflection of a strip with hinged edges. In that case, the exact fulfillment of the geometrical boundary conditions is of considerable importance

$$u = v = w = 0 \qquad \text{for } x = \pm a/2,\ s = \pm b/2. \tag{49.1}$$

Besides, the static conditions

$$\begin{aligned} &\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial s^2} = 0 \ \text{ for } x = \pm a/2; \\ &\frac{\partial^2 w}{\partial s^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{for } s = \pm \frac{b}{2} \end{aligned} \tag{49.2}$$

should be satisfied.

Taking into account the geometrical character of the boundary conditions, we shall solve the problem in the displacement components

$$u_1^I = u,\ \ u_2^I = v,\ \ w^I = w.$$

We obtain the corresponding equilibrium equations from the equations (35.3), (35.4), and (35.5), dropping the index I and using the formulas (35.1).

The conditions (49.1) and (49.2) are satisfied if

$$w = (w_1 \cos y_1 + w_1^H \sin 2y_1) \cos x_1; \quad x_1 = \pi x/a,\ \ y_1 = \pi s/b;$$

$$\begin{aligned} u = (u_1 \cos y_1 + u_3 \cos 3y_1 + u_1^H \sin 2y_1 - u_3^H \sin 4y_1) \sin 2x_1 + \\ + (u_2 \cos y_1 + u_2^H \sin 2y_1) \sin 4x_1; \end{aligned} \tag{49.3}$$

$$\begin{aligned} v = (v_1 \sin 2y_1 + v_3 \sin 4y_1 + v_5 \sin 6y_1 + v_1^H \cos y_1 + v_3^H \cos 3y_1 + \\ + v_5^H \cos 5y_1) \cos x_1 + (v_2 \sin 2y_1 + v_4 \sin 4y_1 + \\ + v_2^H \cos y_1 + v_4^H \cos 3y_1) \cos 3x_1. \end{aligned} \tag{49.4}$$

(49.5)

Starting from the principle of virtual displacements, we shall apply the generalized variational equation (22.5)* in order to solve the problem.

In the case under consideration, the contour integrals in (22.5) are equal to zero, as in varying the quantities $u_1, \ldots, w_1^H$ on the contour, everywhere $\delta u = \delta v = \delta w = 0$ everywhere. The fact that the integral taken over the surface of the strip is zero gives the equations

$$\iint\limits_{(\sigma)} \left(\frac{\partial T_1}{\partial x} + \frac{\partial T_{12}}{\partial s} \right) \delta u \, d\sigma = 0, \quad \iint\limits_{(\sigma)} \left(\frac{\partial T_{12}}{\partial x} - \frac{\partial T_2}{\partial s} \right) \delta v \, d\sigma = 0 \tag{49.6}$$

----

* See the derivation of the equation for a cylindrical shell in /0. 6/.

$$
\iint_{(\sigma)} \left\{ D\Delta\Delta w - T_1\left(\frac{\partial^2 w^0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) - 2T_{12}\left(\frac{\partial^2 w^0}{\partial x \partial s} + \frac{\partial^2 w}{\partial x \partial s}\right) - \right.
$$

$$
- T_2\left(\frac{\partial^2 w^0}{\partial s^2} + \frac{\partial^2 w}{\partial s^2} - k\right) + p -
\tag{49.7}
$$

$$
\left. \underline{-\left(\frac{\partial T_1}{\partial x} + \frac{\partial T_{12}}{\partial s}\right)\frac{\partial w}{\partial x} - \left(\frac{\partial T_{12}}{\partial x} + \frac{\partial T_2}{\partial s}\right)\frac{\partial w}{\partial s}} \right\} \delta w \, d\sigma = 0.
$$

The first two of these equations are the equations of Galerkin's method applied to the equilibrium equations (35.4) and (35.5), and the equation (49.7) differs from the Galerkin equation for the equilibrium equation (35.3) by the presence of the additional underlined terms. On exact solution of the problem these additional terms vanish. For an approximate solution, as the calculations show, they are small in comparison with the remaining terms, and therefore we shall also integrate the third one of the equilibrium equations by Galerkin's method.

Thus, substituting the expressions (49.4), (49.5) and equating the coefficients of $\delta u_1, \ldots, \delta w_1^H$ to zero*. we obtain a system of 18 equations in 18 parameters $u_1, \ldots, w_1^H$. Because of the orthogonality of the functions by which one approximates the displacement components u and v, the system of equations for expressing u_i, u_i^H, v_k, v_k^H in terms of w_1, w_1^H divides into two independent systems, one of which contains only the parameters of the symmetrical deformation, and the other only the parameters of asymmetrical deformation, where v_k, v_k^H are easily expressed in terms of u_i, u_i^H. Even though this simplifies considerably the succeeding computations, they remain sufficiently tedious, and therefore we shall carry out the further analysis of the solution for the special case where

$$
b/a = 1/2. \tag{49.8}
$$

Let the strip have an initial irregularity

$$
w_0 = \zeta_0 t \sin(2\pi s/b). \tag{49.9}
$$

We also introduce the notations

$$
q = pRb^2/4D; \quad \zeta = w_1/t; \quad \zeta^H = w_1^H/t. \tag{49.10}
$$

Then we obtain the fundamental relations

$$
\frac{16}{\pi^2} q = -\left(2.428k^* + \frac{38.05}{k^*} + \frac{258.9}{k^*}\zeta^{H2} + \frac{596.8}{k^*}\zeta_0\zeta^H + \frac{87.53}{k^*}\zeta_0^2\right)\zeta -
$$

$$
- 25.09\zeta^2 - \frac{62.52}{k^*}\zeta^3 - 33.89\zeta^{H2} - 78.58\zeta_0\zeta^H; \tag{49.11}
$$

$$
(5.439\zeta^H + 5.998\zeta_0)\zeta^2 + (1.41\zeta^H + 1.638\zeta_0)\zeta k^* + 62.08\zeta_0\zeta^H +
$$

$$
+ 18.4\zeta^{H3} + (0.00361k^{*2} + 48.4\zeta_0^2 + 9.164)\zeta^H = 0. \tag{49.12}
$$

With $\zeta_0 = 0$, $\zeta^H \neq 0$, from the last equation we find the relation:

$$
18.4\zeta^{H2} = -5.44\zeta^2 - 1.41\zeta k^* - 0.00361k^{*2} - 9.16 > 0. \tag{49.13}
$$

Hence it follows that the asymmetrical form of stability loss of the panel takes place at k* >10.22.

Solving (49.13) for various values of k*, we shall find the states in which the asymmetrical deflection component is present or absent respectively, and

----

* See article /XI.5/.

from equation (49.11) in which $\zeta_0 = \zeta^H = 0$, we shall obtain the corresponding critical values of the load parameter. To illustrate the process of solution we shall first consider a strip with a small curvature parameter k* = 20. Here, utilizing the condition dq/d = 0, we shall find for symmetrical deformation the pressures of bang and collapse q_1 = 18.56, q_2= 0.323, and the corresponding deflection parameters

$$\zeta_1 = -1.343;\ \zeta_2 = -4.007.$$

The dependence of q vs. ζ is given by the solid line in Figure 33.

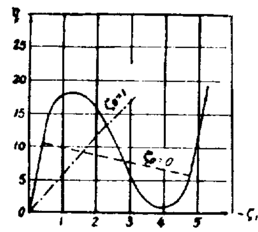

Figure 33

Dependence of the load parameter on the deflection parameter at the strip vertex (λ = 0.5; k* = 20)

Solving (49.13) with k*=20, we find that it takes place when $0.408 < -\zeta < 4.776$. This inequality determines the existence region of the asymmetrical deformation component. Setting $\zeta = -0.408$ and $\zeta = -4.776$ we find from (49.11) the critical values of the pressure parameter for which the asymmetrical deformation is respectively present and absent: $q_1^H = 10.26$, $q_2^H = 5.68$. The dependence of q ζ for the case $\zeta > -0.408$ and $\zeta < -4.776$ coincides with the preceding, and in the segment $0.408 < -\zeta < 4.776$ is shown by a dashed line.

With $\zeta_0 \neq 0$, by taking various values of ζ_1^H we find from (49.12)the corresponding values of ζ, after which we determine from (49.11) the value of the pressure parameter q. The dependence curve q vs. ζ thus obtained for a strip with the initial irregularity $\zeta_0 = -1$ is shown by the dash-and dot line in Figure 33.

As an example of a shallow strip with an upward slope, we shall consider a wtrip with the curvature parameter k* = 100.

Carrying out calculations analogous to the preceding, we obtain for the case

a) for the symmetrical deformation of an ideal strip

$$\zeta_1 = -6.358,\quad \zeta_2 = -20.39,\quad q_1 = 427.4,\quad q_2 = -105.7; \qquad (49.13b)$$

b) with the presence or absence of asymmetrical deformation:

$$\zeta_1 = -0.329,\quad \zeta_2 = -25.59,\quad q_1^H = 47.58,\quad q_2^H = 167.5.$$

The dependence of curves q vs. ζ for these cases and for the case $\zeta_0 = -1$ are given in Figure 34 by a solid, dashed, and dot-and-dash line respectively.

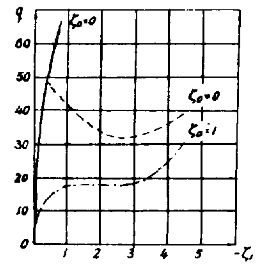

Figure 34

Dependence of the load parameter on the deflection parameter at the strip vertex ($\delta = 0.5$; $k^* = 100$)

The calculations given show that the asymmetrical form of stability loss with hinged edges appears already in shallow strips with $k^* > 10.22$. Therefore, when the curvature parameter $k^* > 10$, the determination of the critical load for the symmetrical form of stability loss along one half-wave is of no practical interest.

This should be remembered all the more, since the initial shape of the strip is not ideally symmetrical, and the deformation of the strip occurs as a simultaneous development of the symmetrical and asymmetrical components of deflection. With an initial asymmetry in the shape of the middle surface of the strip, it becomes, in the first stage of loading, more flexible to bending, due to which the deflection exceeds by several times the deflection of an ideal circular strip with the very same load (see Figure 34). In that case, the phenomenon of stability loss does not take place in the usual sense, but from a practical point of view, one can take as the critical load that for which either the deflection begins to grow fast without considerable increase in load, or else becomes inadmissibly large.

§ 50. Bending of a Shallow Cylindrical Strip with Freely-Supported Edges under a Uniform Transverse Load

As an example of the application of separate integration by Galerkin method of the equation of compatibility (35. 15), and of the equation of equilibrium (35. 3), we shall consider the deflection of a strip under the action of a uniform transverse load with freely-supported edges*.

The boundary conditions

$$T_1 = \frac{\partial^2 \psi}{\partial s^2} = 0,\quad T_{12} = -\frac{\partial^2 \psi}{\partial x \partial s} = 0,\quad \frac{\partial^2 w}{\partial x^2} = 0 = w \quad \text{for } x = \pm\frac{a}{2}, \tag{50.1}$$

$$T_2 = \frac{\partial^2 \psi}{\partial x^2} = 0 = \frac{\partial^2 \psi}{\partial x \partial s} = T_{12},\quad \frac{\partial^2 w}{\partial s^2} = 0 = w \quad \text{for } s = \pm\frac{b}{2}$$

are satisfied exactly, provided

$$\psi = \psi_1\left(1+\cos\frac{2\pi x}{a}\right)\left(1+\cos\frac{2\pi s}{b}\right) + \psi_2\left(1-\cos\frac{4\pi x}{a}\right)\left(1+\cos\frac{2\pi s}{b}\right) + \psi_3\left(1+\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{4\pi s}{b}\right), \tag{50.2}$$

$$w = w_1 \cos\frac{\pi x}{a}\cos\frac{\pi s}{b} + w_2\cos\frac{\pi x}{a}\cos\frac{3\pi s}{b} + w_3\cos\frac{3\pi x}{a}\cos\frac{\pi s}{b}. \tag{50.3}$$

Here $\psi_1 \ldots w_3$ are the parameters being varied and which are to be determined.

Introducing (50. 2) and (50. 3) in the equations (35. 3) and (35. 15) and integrating them by the Galerkin method, we obtain the following fundamental relations

$$\begin{aligned}
&f_1[2\lambda^4 + 2 + (\lambda^2+1)^2] + 2f_2 + 2\lambda^4 f_3 = \\
&= -0.0625\lambda^2(2\zeta_1^2 + 9\zeta_2^2 + 9\zeta_3^2 + 6\zeta_1\zeta_2 + 6\zeta_1\zeta_3 + 16\zeta_2\zeta_3) - \\
&\quad - 0.1643k^*\lambda^2(0.1111\zeta_1 + 0.0222\zeta_2 + 0.20\zeta_3), \\
&2f_1 + f_2[32\lambda^4 + 2 + (4\lambda^2+1)^2] = \\
&= 0.0625\lambda^2(-\zeta_1^2 - 9\zeta_3^2 - 2\zeta_1\zeta_2 - 9\zeta_1\zeta_3 + 25\zeta_2\zeta_3) - \\
&\quad - 0.6572k^*\lambda^2(0.0222\zeta_1 + 0.0044\zeta_2 - 0.1429\zeta_3), \\
&2\lambda^4 f_1 + f_3[2\lambda^4 + 32 + (\lambda^2+4)^2] = \\
&= 0.0625\lambda^2(-\zeta_1^2 - 9\zeta_2^2 + 9\zeta_1\zeta_2 - 2\zeta_1\zeta_3 + 25\zeta_2\zeta_3) - \\
&\quad - 0.6572k^*\lambda^2(0.0222\zeta_1 - 0.0159\zeta_2 + 0.040\zeta_3).
\end{aligned} \tag{50.4}$$

$$\begin{aligned}
p^* = &-5.50(\lambda^2+1)^2\zeta_1 + \lambda^2 k^*(17.5\,3f_1 + 14.035f_2 + 2.807f_3) + \\
&+ 240.34\lambda^2[f_1(\zeta_1 + 1.5\zeta_2 + 1.5\zeta_3) + f_2(0.5\zeta_1 + 0.5\zeta_2 - 2.25\zeta_3) + \\
&+ f_3(0.5\zeta_1 + 0.5\zeta_3 - 2.25\zeta_2)].
\end{aligned} \tag{50.5}$$

$$\begin{aligned}
&\lambda^2[f_1(5.5\zeta_1 + 15\zeta_2 + 13.5\zeta_3) + f_2(0.875\zeta_1 + 0.5\zeta_2 - 6.937\zeta_3) + \\
&+ f_3(-1.187\zeta_1 + 1.125\zeta_2 - 4.187\zeta_3)] - k^*\lambda^2(0.1168f_1 + 0.0934f_2 + \\
&+ 0.1368f_3) - 0.0229(\lambda^2+1)^2\zeta_1 - 0.0687(\lambda^2+9)^2\zeta_2 = 0.
\end{aligned} \tag{50.6}$$

$$\begin{aligned}
&\lambda^2[f_1(5.5\zeta_1 + 13.5\zeta_2 + 15\zeta_3) + f_2(-1.187\zeta_1 - 4.187\zeta_2 + 1.125\zeta_3) + \\
&+ f_3(0.875\zeta_1 - 6.937\zeta_2 + 0.5\zeta_3)] + k^*\lambda^2(0.4672f_1 - 1.0679f_2 + \\
&+ 0.3271f_3) - 0.0229(\lambda^2+1)^2\zeta_1 - 0.0687(9\lambda^2+1)^2\zeta_3 = 0.
\end{aligned} \tag{50.7}$$

Here we have set

$$f_i = \psi_i : Et^3,\quad \zeta_i = w_i : t, \text{ where } i = 1, 2, 3;$$

$k^* = b^2 : Rt$ is the parameter of curvature; $p^* = pb^4 : Et^4$ is the load parameter.

* The investigation given below was carried out at our request by M.S. Kornishyn and is being published here for the first time.

If we consider $f_2, f_3, \zeta_2, \zeta_3$ to be zero, we shall obtain the solution of the problem to the first approximation

$$p^* = -5.50(\lambda^2+1)^2\zeta_1 - 0.3202\alpha k^{*2}\zeta_1 - 6.5797\alpha k^*\zeta_1^2 - 30.10\alpha\zeta_1^3, \qquad (50.8)$$

where

$$\alpha = \frac{\lambda^4}{3\lambda^4+2\lambda^2+3}. \qquad (50.9)$$

In the case of a flat plate $k^* = 0$,

$$p^* = -5.50(\lambda^2+1)^2\zeta_1 - 30.10\alpha\zeta_1^3. \qquad (50.10)$$

In the following are given the calculated results for some particular values of the parameters λ and k^*.

A. Strip with $\lambda = 0.5$. We shall first consider the solution in the first approximation. Setting $\lambda = 0.5$ in (50.8), we obtain

$$p^* = -8.60\zeta_1 - 0.00543k^{*2}\zeta_1 - 0.1115k^*\zeta_1^2 - 0.510\zeta_1^3. \qquad (50.11)$$

Calculations show that for strips with $k^* < 55.2$ the stresses grow monotonically, with increasing deflections, and with $k^* > 55.2$ the snapping takes place.

In particular, for $k^* = 80$, and the collapse at $p_1^* = 36.14$ and $\zeta_{1,1} = -3.45$, and the exhaustion at $p_2^* = 36.14$ and $\zeta_{1,2} = -8.23$.

If, instead of (50.2) we take for the stress function another expression, namely,

$$\psi = \sum_{m,n} \psi_{mn} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \ldots, \; m, n = 1, 3, 5\ldots, \qquad (50.12)$$

as is done in /XI. 6/ and /XI. 10/, in the first approximation we shall obtain for a similar strip the corresponding values $p^*_{1\,[6]} = 262$; $p^*_{2\,[10]} = 256$.

Note that according to (50.12) the stresses T_1 and T_2 are zero at the contour, and the shearing stress T_{12} is zero in the mean.

Thus, for our case, the critical load would turn out to be one-fourth as much. This is explained by the difference in the boundary conditions which, on the face of it, does not seem to be so considerable. Actually, in the works cited, the strip had been more rigid, with edges supported by incompressible ribs which are easily bent in their plane (see § 48).

Here, we do not compare the lower critical loads corresponding to considerably larger deflections, as for their determination the first and even the second approximations are not always sufficient.

Using the concrete example of a strip with $\lambda = 0.5$ and $k^* = 80$, we shall exhibit the influence of higher order approximations on the value of the upper and lower critical loads.

On the basis of equations (50.4)-(50.7), with $\zeta_1 = -3.5$, we obtain $\zeta_2 = -0.05$, $\zeta_3 = +0.026$, $p^* = 67.07$. Here the deflection at the center of the strip is $\zeta_c = -3.524$.

If in these equations one sets $\zeta_2 = \zeta_3 = 0$, retaining all three terms for ψ, then for the same deflection at the center we obtain $p^* = 68.51$.

From equation (50.11) with $\zeta_1 = \zeta_c = -3.524$ we have $p^* = 64.16$.

From a comparison of the obtained results it is apparent that in determining the upper critical load for a strip with $\lambda = 0.5$ and $k^* = 80$ one can limit oneself to the first approximation for the stress function, as well as for the deflection function. Such a conclusion is even more valid with respect to strips with $\lambda < 0.5$ and $k^* < 80$, as the convergence of the process then improves.

Setting $\zeta_1 = -8$, from equations (50.4)-(50.7) we find: $\zeta_2 = -0.18$, $\zeta_3 = -1.10$, $p^* = 31.57$, $\zeta_c = -9.28$.

If in (50.4)-(50.7) one sets $\zeta_2 = \zeta_3 = 0$, then for $\zeta_1 = \zeta_c = -9.28$ we shall obtain $p^* = 45.64$.

From equation (50.11) with $\zeta_1 = \zeta_c = -9.28$ we have $p^* = 40.71$.

Consequently, in the region of the lower critical load the influence of the higher approximations on the stress function as well as on the deflection function is substantial.

B. The case of a square strip and plate. From (50. 8), for a square strip we have:

$$p^* = -22.01\zeta_1 - 0.04003k^{*2}\zeta_1 - 0.8225k^*\zeta_1^2 - 3.76\zeta_1^3. \qquad (50.13)$$

On the basis of this relation, we find that the snapping occurs in the strip when $k^* > 33.5$. For $k^* = 50$, we find for the critical state:

$$\zeta_{1,1} = -2.07 \text{ and } p_1^* = 109.90, \ \zeta_{1,2} = -5.22 \text{ and } p_2^* = 51.48.$$

Utilizing the relations (50.4)-(50.7) we convince ourselves that with $k^* = 50$, one can limit onself, in the case of a square panel, to the first approximation for the upper as well as for the lower critical loads.

As the value of the curvature parameter k^* increases, the convergence of the process deteriorates.

For example, for $k^* = 80$, from (50. 13) we have $\zeta_{1,1} = -2.77$ and $p_1^* = 345.6$, $\zeta_{1,2} = -8.89$ and $p_2^* = -85.32$.

Setting $\zeta_1 = -8$, from the equations (50.4)-(50.7) we find $\zeta_2 = -0.68$, $\zeta_3 = 0.22$, $p^* = +5.50$, $\zeta_c = -8.46$. From (50.13) with $\zeta_1 = \zeta_c = -8.46$ we have $p^* = -79.13$.

Thus the load has changed its sign where, as the calculations show, the fundamental role is now played by ζ_2.

We shall consider the influence of the higher approximations, in particular of the term with ζ_2, on the upper critical load.

In (50.4)-(50.7) setting $\zeta_3 = 0$ and $\zeta_1 = -3$, we find

$$\zeta_2 = -0.40, \ \zeta_c = -3.40, \ p^* = 290.39.$$

From (50.13) with $\zeta_1 = \zeta_c = -3.40$ we obtain $p^* = 333.06$.

Consequently, the correction of the second approximation is also considerable in the region of the upper critical load--about 13 %.

For a square plate the equations (50. 5) (50. 7) take the form:

$$\begin{gathered}
-p^{*}=22.01\,\zeta_1+3.82\,\zeta_1^{3}+203.05\,\zeta_2^{3}+129.28\,\zeta_1\zeta_2^{2}+30.08\,\zeta_1^{2}\zeta_2.\\
0.0916\,\zeta_1+6.868\,\zeta_2+0.0843\,\zeta_1^{3}+7.421\,\zeta_2^{3}+\\
+0.9396\,\zeta_1^{2}\zeta_2+4.10\,\zeta_1\zeta_2^{2}=0.
\end{gathered} \qquad (50.14)$$

The results calculated according to this formula are close to the results found by another method in /XI. 11/, as can be seen from Table XVI.

Table XVI

$\zeta_c=\zeta_1+2\zeta_2$	-1	-3	-4
p^{*} [XI.11]	26.7	169.5	346.2
p^{*} (50.14)	26.7	174.9	347.8

The method used in /XI. 11/ is universal and has allowed its authors to investigate a series of important cases of fastening of the plate edges, but its utilization requires very cumbersome calculations.

In those cases when at the contour static conditions are given for the stress function, as is the case with our problem, separate integration of the original equations by the Galerkin method gives the fastest results, which are also entirely satisfactory.

§ 51. Experimental Investigation of the Bending of Strips under a Transverse Load

We know of only one work, /XI. 9/, devoted to the testing of thin cylindrical strips under transverse loads in which 60 specimens with $\lambda = \frac{a}{b} = 1; 2; 3$ were prepared from st. 2 and D16T* and were shaped to values of the curvature parameter $25 < k^* < 125$. The testing was carried out in a special set-up; the loading was done by compressed air and was measured by a mercury manometer with a mirrored millimeter scale; the deflections at a series of points were measured by indicators with scale division of 0.01 mm; the deformations were measured by wire resistances.

In preparing such shallow cylindrical strips with the ratio $t:R \sim 1:1{,}000$ it is very difficult to preserve the regular geometrical shape. The specimens therefore had initial irregularities.

Some of the specimens were tested under hinged edges, others under rigid fastening. Together with strips attached rigidly and hinged at all four edges, strips with free curvilinear edges were also tested.

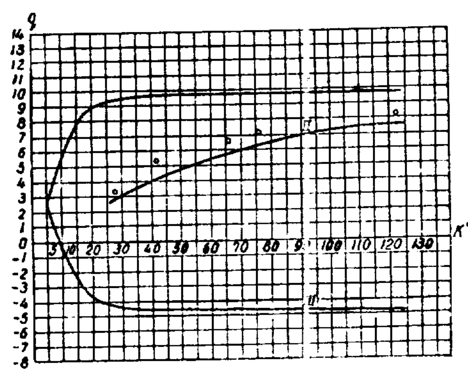

Figure 35

I—theoretical curve of the values of α_1 with asymmetrical buckling; II—experimental curve of the values of α_1, taking into account the deviation of the boundary conditions of the samples tested from the conditions of ideal hinged fastening; III—theoretical curves of the values of α_2 with asymmetrical buckling

As shown by experiment, the strip deflections can be monotonic or be accompanied by a snapping, depending on the value of the curvature parameter k*, and also of the quantity λ. In the latter case, the load, having reached some maximal

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* Translator's note: Russian symbols.

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value p_1, starts to drop suddenly to the value p_2 with simultaneous increase of the deflection. Such a form of stability loss is characteristic for strips with a curvature parameter $k^* \geqslant 40$ and $\lambda > 2$.

Below are given graphs of the theoretical values of α_1 and α_2 from formula (46. 14), and the values of α_1 obtained from the experiment for long strips with rigid and hinged fastenings of straight edges.

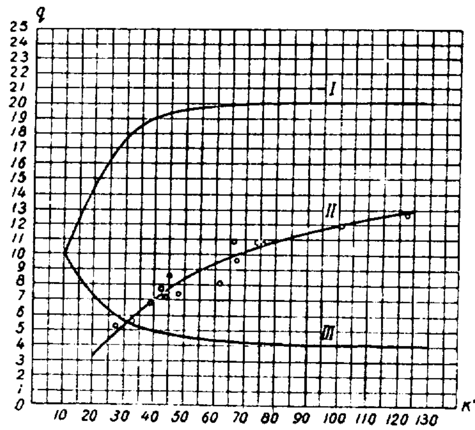

Figure 36

I--theoretical curve of values of α_1 with asymmetrical buckling; II--experimental curve of values of α_1; III--theoretical curve of values of α_2

From the graphs it can be seen that in reality the strips lose their stability under a load considerably smaller than that given by the theoretical solution on the assumption of ideal shape and ideal boundary conditions. The discrepancy between the experimental and the theoretical values of α_1 are all the greater the smaller the parameter of curvature.

Thus, in the given case we have a picture analogous to that observed when cylindrical shells are tested for axial stress and spherical shells tested for external pressure, when the loss of stability also occurs under a load considerably smaller than the predicted theoretical solution under ideal conditions. The discrepancy shown is explained mainly by the deviations existing in the experimental specimens from the regular geometrical shape and partially by deviations in the boundary conditions. This is confirmed by the theoretical analysis of the influence of these factors upon the strip stability, given in §46.

Proceeding from the experimental results in long strips in /XI. 9/, the following empirical relations were obtained for the parameter α_1 from the curvature parameter k^*:

$$\alpha_1^* = 16.5 - \frac{500}{k^*} + \frac{4650}{k^{*2}}. \qquad (51.1)$$

with hinged edges:

$$\alpha_1^{\text{sh}} = 0.6\,\alpha_1^{\text{з}}, \qquad (51.2)$$

for values $25 < k^* < 125$.

The dependences (51. 1), (51. 2) are shown graphically in Figures 35 and 36.

The curve α_1^h is situated somewhat lower than the corresponding experimental points, which is explained by the fact that in testing the specimens, the hinge at the edges was not ideal, and to take this fact into account in (51. 2) a numerical coefficient was taken which is somewhat smaller than its experimental value.

For strips of finite length, a semi-empirical formula for the critical pressure is obtained in the form

$$p_{cr} = \beta_1 \frac{4D}{Rb^2}, \qquad (51.3)$$

where α_1 is the coefficient depending on the curvature parameter k^* of the panel, given by the relations (51.1), (51.2); β_1 is the coefficient depending on the ratio of the sides $\lambda = \frac{a}{b}$. For fixed edges:

$$\beta_1^{\text{з}} = 1 - \frac{0.2}{\lambda} + \frac{1.2}{\lambda^2} \qquad (51.4)$$

for hinged edges:

$$\beta_1^{\text{sh}} = 1 + \frac{2}{\lambda^2}, \qquad (51.5)$$

where $\lambda \geqslant 1$.

The empirical relations given here are obtained from the results of testing a comparatively small number of samples and could be made more precise at a later date on the basis of more comprehensive experimental data.

Chapter XII

THE STABILITY AND LARGE DEFLECTIONS OF CIRCULAR CONICAL SHELLS

§ 52. The Stability of a Conical Shell with Circular Section under Longitudinal Compression

We shall specify the position of a point on the middle surface σ of a conical shell by the distance r from that point to the cone vertex, measured along the generatrix, and the angle φ between the axial plane passing through the point and the axial plane of the origin of coordinates. Then in the formula of § 25 one has to set

$$\alpha = r,\ \beta = \varphi,\ k_1 = 0,\ k_2 = \operatorname{ctg}\gamma/r,\ B = r\sin\gamma, \tag{52.1}$$

where γ is half the cone angle.

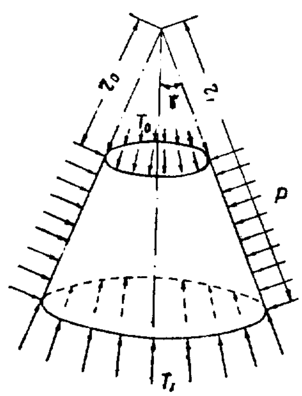

Figure 37

Let the shell be in equilibrium under a uniform external pressure p, and compressive and shearing stresses uniformly distributed along the end sections. We shall denote the compressive stress by T_0 and the shearing stress applied to end $r = r_0$ by τ_0 (Figure 37). Then the internal stresses in the membrane state of equilibrium, determined from (25.11) and (25.12) by neglecting the bending terms, will be respectively equal to

$$T_1^1 = -\frac{pr\operatorname{tg}\gamma}{2} - \frac{T_0 r_0}{r},$$

$$T_2^1 = -pr\operatorname{tg}\gamma,\ T_{12}^1 = -\frac{\tau_0 r_0^2}{r^2}. \tag{52.2}$$

As one is considering a shell in the form of an ideal circular cone, and the state before the stability loss is considered to be a membrane state, in the equations of neutral equilibrium (25.27) and (25.33) one should set

$$\varkappa_1^0 = \cdots = \varkappa_2^1 = 0.$$

The additional stresses which appear with buckling are expressed, according to formulas of the form (25.32), by the stress function ψ:

$$\begin{gathered} T_1 = \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi_1^2}, \quad T_2 = \frac{\partial^2 \psi}{\partial r^2}, \\ T_{12} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \varphi_1}\right), \quad \varphi_1 = \varphi \sin \gamma. \end{gathered} \tag{52.3}$$

The equations for determining ψ and the deflection w take the form

$$\begin{gathered} D\Delta\Delta w + \frac{1}{r}\operatorname{ctg}\gamma\frac{\partial^2 \psi}{\partial r^2} - T_1^1\frac{\partial^2 w}{\partial r^2} - T_2^1\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi_1^2}\right) - \\ -2T_{12}^1\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \varphi_1}\right) = 0, \quad \Delta\Delta\psi - Et\frac{\operatorname{ctg}\gamma}{r}\frac{\partial^2 w}{\partial r^2} = 0, \end{gathered} \tag{52.4}$$

where

$$\Delta(\ldots) = \frac{\partial^2}{\partial r^2}(\ldots) + \frac{1}{r}\frac{\partial}{\partial r}(\ldots) + \frac{1}{r^2}\frac{\partial^2(\ldots)}{\partial \varphi_1^2}. \tag{52.5}$$

If the ends of the shell are freely supported, then at the edges $r = r_0$ and $r = r_1$ one has to satisfy the conditions

$$\begin{gathered} w = 0, \quad T_1 = 0, \quad \varepsilon_2 = 0, \\ \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \varphi_1^2}\right) = 0. \end{gathered} \tag{52.6}$$

In the case of pure longitudinal compression

$$p = \tau = 0. \tag{52.7}$$

This problem has been investigated by I. Ya. Shtaerman /XII.1/ on the assumption that an axially symmetrical buckling occurs with the loss of stability of the shell buckling*. With this assumption

$$\Delta(\ldots) = \frac{1}{r}\frac{d}{dr}\left[r\frac{d}{d}(\ldots)\right].$$

Consequently, introducing the notation

$$\omega = dw/dr \tag{52.8}$$

and integrating the equations (52.4) once, we shall obtain

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)\right] = Et(\omega + c)\operatorname{ctg}\gamma, \tag{52.9}$$

$$Dr\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\omega)\right] + \left(\frac{d\psi}{dr} + c'\right)\operatorname{ctg}\gamma + T_0 r_0 \omega = 0, \tag{52.10}$$

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* Non-axially symmetric buckling has been considered in § 20 of /0.13/. See also article /IX.8/.

where c and c' are arbitrary constants. Eliminating the quantity $d\psi/dr$ from (52.9) with the help of (52.10), we arrive at the equation

$$D\Delta_2\Delta_2\omega + T_0 r_0 \Delta_2\omega + Et\omega \operatorname{ctg}^2\gamma + Etc \operatorname{ctg}^2\gamma - \frac{c' \operatorname{ctg}\gamma}{r} = 0, \tag{52.11}$$

where

$$\Delta_2(\ldots) = r\frac{d^2(\ldots)}{dr^2} + \frac{d(\ldots)}{dr} - \frac{(\ldots)}{r}. \tag{52.12}$$

The homogeneous part of equation (52.11) can be represented in the form

$$(\Delta_2 + m_1)(\Delta_2 + m_2)\omega_0 = 0, \quad m_1 m_2 = \frac{Et \operatorname{ctg}^2\gamma}{D}, \quad m_1 + m_2 = \frac{T_0 r_0}{D}. \tag{52.13}$$

Here m_1 and m_2 are roots of the equation

$$Dm^2 - T_0 r_0 m + Et \operatorname{ctg}^2\gamma = 0.$$

The absolute minimum of T_0 is reached when

$$m = \sqrt{Et \operatorname{ctg}^2\gamma/D} \tag{52.14}$$

and is equal to

$$T_0 = 2\sqrt{DEt \operatorname{ctg}^2\gamma/r_0^2}. \tag{52.15}$$

Consequently,

$$m_1 = m_2 = T_0 r_0/2D.$$

Thus, instead of equation (52.13) we have

$$(\Delta_2 + m)(\Delta_2 + m)\omega_0 = 0. \tag{52. 13A}$$

The integrals of the equation

$$(\Delta_2 + m)\omega_0' = r\frac{d^2\omega_0'}{dr^2} + \frac{d\omega_0'}{dr} + \left(m - \frac{1}{r}\right)\omega_0' = 0 \tag{52.16}$$

are also integrals of the equation (52.13a). By the substitution

$$x = 2\sqrt{mr} \tag{52.17}$$

the latter is reduced to a second-order Bessel equation

$$\frac{d^2\omega_0'}{dx^2} + \frac{1}{x}\frac{d\omega_0'}{dx} + \left(1 - \frac{4}{x^2}\right)\omega_0' = 0,$$

whose general integral is

$$\omega_0' = C_1 I_2(x) + C_2 N_2(x), \tag{52.18}$$

where I_2 and N_2 are Bessel functions of the first and second kind.

It is not difficult to convince oneself that the function $r d\omega_0'/dr$ is also an integral of the equation (52.13a). Besides, equation (52.11) has a particular integral of the form $C_5 + C_6/r$. Thus the general integral of that equation has the form

$$\omega = \frac{dw}{dr} = C_1 I_2(2\sqrt{mr}) + C_3 N_2(2\sqrt{mr}) + C_3 r \frac{d}{dr}[I_2(2\sqrt{mr})] + \\ + C_4 r \frac{d}{dr}[N_2(2\sqrt{mr})] + C_5 + \frac{1}{r} C_6 = \omega_0 + C_6 + \frac{1}{r} C_6, \\ C = -C_5, \ C' = -T_0 r_0 C_5 \operatorname{tg}\gamma + Et \operatorname{ctg}\gamma C_4. \qquad (52.19)$$

Integrating once more, we shall obtain the expression for w where an arbitrary constant C_7 will appear.

In the case of a symmetrical deformation, as can be seen from (25. 24), upon satisfying the boundary condition $w = 0$ the boundary condition $\varepsilon_2 = 0$ becomes equivalent to the condition $u = 0$, which can contradict the condition $T_1 = 0$. We shall assume that the skin can slide along the transverse ribs and, consequently, it is not obligatory to satisfy the boundary condition $\varepsilon_2 = 0$. Thus, in the case under consideration, three boundary conditions ought to be satisfied at each end.

The computations are simplified if, as in /XII. 1/, one limits oneself to the case of a dome with a very small opening, as with $r \to 0$, $N_2 \to \infty$ and to keep the solution finite it is necessary to set $C_2 = C_4 = C_6 = 0$, where it is sufficient to satisfy the boundary conditions at only one end $r = r_1$. Besides, it is useful to take into account that according to (52. 14), for large values or r the quantity $2\sqrt{mr}$ is of the order of $\sqrt{R/t}$. Consequently, I_2 and its derivatives can be approximately replaced by their asymptotic expressions

$$I_2(x)\sqrt{\frac{1}{2}\pi x} = \cos\left(x - \frac{5}{4}\pi\right) - \frac{15}{8x}\sin\left(x - \frac{5}{4}\pi\right); \ldots \qquad (52.20)$$

in the case of a symmetrical deformation we have, according to (25. 24) and (54. 4)

$$\varepsilon_1 = \frac{du}{dr}, \quad \varepsilon_2 = \frac{u}{r} + \frac{w \operatorname{ctg}\gamma}{r}.$$

But for the membrane part of w, equal to C_7, one has to satisfy the conditions $T_2 = 0$, $T_1 r \sin\gamma =$ const or $\varepsilon_2 + \nu_1 = 0$, $r(\varepsilon_1 + \nu\varepsilon_2) =$ const, $u + w \operatorname{ctg}\gamma =$ const. Consequently, with $w = C_7$ we should have $u =$ const., $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, i. e., the deflection $w = C_7$ produces a displacement of the shell as a rigid body, and does not influence the deformed state and the boundary conditions. Therefore C_7 may be taken to be zero.

Thus, it remains to choose C_1, C_3, C_5 in such a way as to satisfy the boundary conditions.

From (52.10), taking (52.3) and (52.16) into account, we find

$$T_1 \operatorname{ctg}\gamma = \frac{DC_5}{r^3} - \frac{m D \omega_0}{r}.$$

Consequently, in order that the condition $T_1 = 0$ be satisfied when $r = r_1$, we should have

$$C_5 = m r_1 \omega_0 \ \text{ at } \ r = r_1. \qquad (52.21)$$

To set up the equations expressing the boundary conditions $w = 0$ and $d\omega/dr + \nu\omega/r = 0$, with $r = r_1$ or $x = x_1 = \sqrt{2mr_1}$, it is necessary to carry out considerable computation work. Here, besides expressions of the form (52.20), we use the formulas for the differentiation of Bessel functions with $p = 1$ and $p = 2$:

$$\frac{d}{dx} I_p(x) = -\frac{p}{x} I_p(x) + I_{p-1}(x), \quad \int x^{p+1} I_p(x)\, dx = x^{p+1} I_{p+1}(x).$$

To simplify the computations we shall neglect quantities of the order of unity in comparison with quantities of the order of x_1^2. Thus, after eliminating C_5 by (52. 21), we bring the geometrical boundary conditions into the form

$$c_3 (I_0 - \nu I_2)_{x=x_1} + \nu \left(c_1 I_2 + \frac{c_2}{2} x_1 I_1 \right)_{x=x_1} = 0, \tag{52.22}$$

$$\left(c_1 I_2 + \frac{c_2 x_1}{2} I_1 \right)_{x=x_1} = 0. \tag{52.23}$$

In order for these equations to be consistent, we should have, for $c_3 \neq 0$

$$I_0 (x = x_1) = \nu I_2 (x = x_1) \text{ or } \operatorname{tg}\left(2\sqrt{mr_1} - \frac{\pi}{4} \right) = \frac{16}{17}\sqrt{mr_1}. \tag{52.24}$$

Simultaneously with this equation, (52. 24) must be satisfied, and therefore the equation

$$\sqrt{Et \operatorname{ctg} \gamma / D} = m_0,$$

should be satisfied, where m_0 is a root of the equation (52.24).

This will not be satisfied exactly, but one can choose two successive roots of (52.24), namely, m_0' and m_0'', between which m_0 will be included, and as these numbers are rather large and their difference is small in comparison with m_0, the value of T_0 at m determined from equation (52. 24) will be only slightly larger than the absolute minimum of T_0 given by (52. 15).

In conclusion let us note that from (52.15) one cannot obtain the formula for the critical compressive stress in a circular plate by setting $\gamma = \pi/2$, or a formula for a very shallow conical shell, as for small values of ctg γ the quantity $\sqrt{mr_1}$ is no longer much larger than unity, and the asymptotic expansions of the Bessel functions, which we had used, are inapplicable.

53. The Stability of a Conical Shell under the Combined Action of Longitudinal Compression and External Normal Pressure

The stability of a circular conical shell under uniform compression had been considered, apparently for the first time, in article /XII. 2/, where no attention was paid to the fulfillment of the boundary conditions. Later, I. I. Trapezin in article /XII. 3/ gives the solution of that problem by the Galerkin method, satisfying only part of the boundary conditions—this work, like the article /XII. 2/, not being carried through to computational formulas. Therefore, here we follow our article /IX. 8/, where the question has been treated generally with the proper consideration of the boundary conditions.

Let the shell be under a uniform compression p and a longitudinal compressive stress whose absolute value at the end $r = r_0$ is T_0. Then, in the formulas (52. 2) which define the stresses before the loss of stability, one should set $\tau_0 = 0$. Introducing the substitutions

$$z = \ln\frac{r}{r_0}, \quad \psi = F\cos n_1\varphi_1, \quad w = e^{\nu_1 z} w_1 \cos n_1\varphi_1,$$
$$\nu_1 = \frac{1-\nu}{2}, \quad n_1 = \frac{n}{\sin\gamma}, \quad \varphi = \varphi_1/\sin\gamma \tag{53.1}$$

and neglecting unity in comparison with n_1^2 according to the theory of shallow shells, we bring the equations (52.4) into the form

$$\frac{d^4F}{dz^4} - 4\frac{d^3F}{dz^3} - 2n_1^2\frac{d^2F}{dz^2} + 4n_1^2\frac{dF}{dz} + n_1^4F -$$
$$- \frac{Etr_0}{\mathrm{tg}\,\gamma} e^{(1+\nu_1)z}\left[\frac{d^2w_1}{dz^2} + (2\nu_1 - 1)\frac{dw_1}{dz} + (\nu_1^2 - \nu_1)\, w_1\right] = 0, \tag{53.2}$$

$$e^{(\nu_1-4)z}\left\{\frac{d^4w_1}{dz^4} - 4(1-\nu_1)\frac{d^2w_1}{dz^2} - 2n_1^2\frac{d^2w_1}{dz^2} + 4n_1^2(1-\nu_1)\frac{dw_1}{dz} +\right.$$
$$+ n_1^4 w_1 + \frac{r_0}{D\,\mathrm{tg}\,\gamma} e^{(1-\nu_1)z}\left(\frac{d^2F}{dz^2} - \frac{dF}{dz}\right) + \frac{p\,\mathrm{tg}\,\gamma}{D} r_0^3 e^{3z}\left[\frac{1}{2}\frac{d^2w_1}{dz^2} +\right.$$
$$\left. + \left(\nu_1 + \frac{1}{2}\right)\frac{dw_1}{dz} - n_1^2 w_1\right] +$$
$$\left. + \frac{T_0r_0^2e^z}{D}\left[\frac{d^2w_1}{dz^2} + (2\nu_1 - 1)\frac{dw_1}{dz} + (\nu_1^2 - \nu_1)\, w_1\right]\right\} = 0. \tag{53.3}$$

The boundary conditions (52.6) take the form

$$\left.\begin{aligned} &w_1 = 0, \quad \frac{d^2w}{dz^2} = 0, \qquad &(53.4)\\ &\frac{dF}{dz} - n_1^2F = 0, \quad \frac{d^2F}{dz^2} - \frac{dF}{dz} = 0 \qquad &(53.5)\end{aligned}\right\} \text{ for } z = 0,\; z = \zeta = \ln(1 + L/r_0).$$

We want to solve the boundary value problem, taking the waves along the shell length to be of the form

$$w_1 = A\sin m_1 z, \quad m_1 = m\pi/\zeta. \tag{53.6}$$

Here the boundary conditions (53.4) are satisfied if m is an integer*.

(53.7)

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* Translator's note: the numbering of formulas in the Russian text inadvertently introduces (53. 7) where no formula seems to be referred to.

The initial equations are applicable only in the case of a short, thin shell which buckles with a formation of a large number of waves. We shall assume that one is considering shells of medium length, for which

$$r_0 \sin\gamma \sim L, \tag{53.8}$$

where, just as before, the symbol $\sim$ indicates that the two quantities compared are of the same order of magnitude.

For shells of small angles

$$\zeta = \ln(1 + L/r_0) \sim \sin\gamma,$$

and we have $\zeta \sim 1$, if $\sin\gamma \sim 1$. From equation (53. 2) we find

$$\begin{gathered} F = A_1 e^{n_1 z} + A_2 e^{-n_1 z} + B_1 e^{(n_1+2)z} + B_2 e^{(2-n_1)z} - \\ - AEtr_0 \operatorname{ctg}\gamma e^{(1+\nu_1)z} (\Phi \sin m_1 z + \chi \cos m_1 z), \end{gathered} \tag{53.9}$$

where A_1, A_2, B_1, B_2, are arbitrary constants.

$$\Phi \approx \frac{m_1^2 + \nu_1 - \nu_1^2}{(m_1^2 + n_1^2)^2}, \quad \chi \approx \frac{m_1(1 - 2\nu_1)(m_1^2 + n_1^2) + 4\nu_1 m_1^3}{(m_1^2 + n_1^2)^2}. \tag{53.10}$$

Satisfying the conditions (53.5) we arrive at a system of equations for determining A_1, A_2, B_1, and B_2. When approximately determining the hyperbolic terms of the expression (53.19), one can set

$$\operatorname{sh} n_1\zeta \approx \operatorname{ch} n_1\zeta,$$

and all the more so since, as will become evident in the following, these terms have a negligible influence on the value of the critical load. Thus, we find

$$\begin{gathered} A_1 \approx \frac{\Phi_6 + n_1\Phi_5}{2(n_1^2 - n_1)\operatorname{sh} n_1\zeta} e^{(1+\nu_1)\zeta}; \quad B_1 \approx -\frac{\Phi_5}{2(n_1+1)\operatorname{sh} n_1\zeta} e^{(\nu_1-1)\zeta}; \\ A_2 \approx \frac{n_1\Phi_5 - \Phi_6}{n_1^2 + n_1}; \quad B_2 \approx -\frac{\Phi_5}{n_1 - 1}, \end{gathered} \tag{53.11}$$

where

$$\begin{gathered} \Phi_5 = 0{,}25(\Phi_4 - n_1^2\chi), \quad \Phi_6 = 0{,}5(2\Phi_2 - n_1^2\chi - \Phi_4), \\ \Phi_4 = (1+\nu_1)\Phi_2 + m_1\Phi_1, \quad \Phi_3 = (1+\nu_1)\Phi_1 - m_1\Phi_2, \\ \Phi_1 = (1+\nu_1)\Phi - m_1\chi, \quad \Phi_2 = (1+\nu_1)\chi + m_1\Phi. \end{gathered} \tag{53.12}$$

As the geometrical and the static boundary conditions are satisfied, we can integrate the equations (53.3) by the Bubnov-Galerkin method, multiplying its left-hand member by $wrdrd\varphi$. The characteristic equation obtained has the form

$$P - Q - M + N = 0, \tag{53.13}$$

where

$$\begin{gathered} P \approx m_1 e^2 (m_1^2 + n_1^2)^2/(m_1^2 + 1 - \nu_1^2), \\ Q \approx \frac{pm_1}{D} \frac{n_1^2 + \lambda_1[m_1^2 + 3\nu_1(1-\nu_1) - 0.5]}{m_1^2 + 0.25(1+2\nu_1)^2}, \\ M = \frac{4\nu_1}{e^{2\nu_1} - 1}\left\{\frac{\Phi_6 - n_1\Phi_5}{m_1^2 + (n_1 + \nu_1 - 1)^2} - \frac{(\Phi_6 + n_1\Phi_5)e^{2\nu_1 z}}{m_1^2 + (n_1 + \nu_1 - 1)^2} + \right. \\ \left. + \frac{\Phi_5 e^{2\nu_1 z}(n_1 + 2)}{m_1^2 + (n_1 + 1 + \nu_1)^2} + \frac{\Phi_5(n_1 - 2)}{m_1^2 + (n_1 - 1 - \nu_1)^2}\right\}, \\ N = [m_1(\Phi_1 - \Phi_3) + \nu_1(\Phi_4 - \Phi_2)]/(m_1^2 + n_1^2) \end{gathered} \tag{53.14}$$

with the notations

$$\rho = 1 + \frac{L}{r_0}, \quad \zeta = \ln \rho, \quad \nu_1 = \frac{1-\nu}{2}, \quad \nu_2 = 1 - 2\nu_1^2,$$

$$\varepsilon^2 = \frac{\nu_1 t^2 (1 - \rho^{2\nu_1 - 2})}{12 r_0^2 \operatorname{ctg}^2 \gamma (1-\nu^2)(\rho^{2\nu_1} - 1)(1-\nu_1)}, \quad \tilde{D} = \frac{(1+2\nu_1) Et (\rho^{2\nu_1} - 1)}{2\nu_1 r_0 \operatorname{tg}^3 \gamma (\rho^{1+2\nu_1} - 1)},$$

$$\lambda = \frac{T_0 (1+2\nu_1)(1-\rho^{-\nu})}{\nu p r_0 \operatorname{tg} \gamma (\rho^{1+2\nu_1} - 1)}, \quad \lambda_1 = 0{,}5 + \frac{\lambda [m_1^2 + 0.25(1+2\nu_1)^2]}{m_1^2 + 0.25\nu^2},$$

$$\tilde{K} = \tilde{D} \frac{[m_1^2 + 0.25(1+2\nu_1)^2](m_1^2 + \nu_2)}{m_1^2 (m_1^2 + \nu_1^2)}, \quad m_1 = \frac{m\pi}{\zeta},$$

$$\eta^2 = \frac{\varepsilon^2 (m_1^2 + \nu_1^2) m_1^2}{[m_1^2 + (1-\nu_1)^2](m_1^2 + \nu_2)}, \quad \delta_1 = \frac{\sqrt[4]{3}}{\sqrt{\eta}} \tag{53.15}$$

(m is an integer).

This equation can be considerably simplified for an extensive class of thin shells, satisfying the condition:

$$\zeta = \ln \rho \leqslant 1, \quad L \leqslant 1.72 r_0. \tag{53.16}$$

Then, from (53. 15) we have $m_1^2 > \pi^2$. Retaining only the principal terms in the expressions (53. 12) and (53. 14), we obtain

$$\frac{M}{N} \approx \frac{4\nu_1 [\nu n_1^4 - 2\nu n_1^3 - m_1^2 n_1^2 + 2(1-\nu) m_1^4]}{m_1^2 [m_1^2 + (n_1 + \nu_1 + 1)^2][m_1^2 + (n_1 + \nu_1 - 1)^2]}.$$

Calculations made with this formula by taking account of the expression for P and of the following solution show that, admitting an overestimate of 2-3% in the value of the critical load, one may neglect the quantity M in equation (53. 13) and thus somewhat simplify the expressions for Q and N. Let us note that the maximum error occurs at the boundary of the region when he shell becomes arbitrarily long and it is desirable to support it by elastic transverse ribs. With the decrease of ζ, this error, as well as the error from neglecting unity in comparison with n_1^2 (which we had been doing systematically) falls sharply to 1-2%. Thus, we shall determine the critical load from the approximate equation

$$\frac{p}{\tilde{D}} \frac{n_1^2 + \lambda_1 m_1^2}{m_1^2 + 0.25(1+2\nu_1)^2} = \frac{\varepsilon^2 (m_1^2 + n_1^2)^2}{m_1^2 + (1-\nu_1)^2} + \frac{m_1^4 + \nu_2 m_1^2}{(m_1^2 + n_1^2)^2 (m_1^2 + \nu_1^2)}. \tag{53.17}$$

In the special case of longitudinal compression p = 0, $p\lambda$ is given in terms of T_0 by (53.15), where it is not difficult to convince oneself that to the critical load corresponds the value of $m^2 > 1$. Consequently, equation (53.17) may be replaced by the approximate equation

$$\frac{T_0 (1+2\nu_1)(1-\rho^{-\nu})}{\nu \tilde{D} (\rho^{1+2\nu_1} - 1)} = \frac{\varepsilon^2 (m_1^2 + n_1^2)^2}{m_1^2} + \frac{m_1^2}{(m_1^2 + n_1^2)^2}$$

and the critical value of the compressive stress is equal to

$$T_{0,\kappa} = \frac{\nu \sqrt{KD(1-\nu^2)(\rho^{2\nu_1} - 1)(1 - \rho^{2\nu_1 - 2})}}{\sqrt{\nu_1 (1-\nu_1)} R_0 (1-\rho^{-\nu})}, \quad R_0 = r_0 \operatorname{tg} \gamma. \tag{53.18}$$

The deviation of this formula from the analogous exact formula (52. 15) results in an overestimate of 4-5% with $\zeta \leqslant 1$.

Utilizing the notations (53.15) and setting, in addition,

$$\delta = (m_1^2 + n_1^2)/m_1, \tag{53.19}$$

we bring equation (53.17) into the form

$$p = \bar{K}(\eta_1^2\delta^2 + 1/\delta^2) : (\lambda_1 - 1 + \delta/m_1). \tag{53.20}$$

By replacing the quantities m_1, η, by m and ε respectively, this equation takes the form of the corresponding equation (36. 14) for a cylindrical shell, and therefore, without repeating the rest of the analysis, we give only the final formulas for the determination of the critical value of m_1 and p:

a)

$$m = 1, \quad m_1 = \pi/\ln\rho, \quad \rho = 1 + L/r_0; \tag{53.21}$$

b)

$$p_\kappa = \frac{1.74\,\bar{K} m_1 \eta^{3/2}}{1 + (\lambda_1 - 1)\, m_1/\delta_1}, \quad \delta_1 = \sqrt[4]{3}/\sqrt{\eta} \tag{53.22}$$

for shells satisfying the condition

$$0.49 \geqslant \theta_1 \geqslant -0.83, \quad \theta_1 = 2(1 - \lambda_1)\, m_1/\delta_1; \tag{53.23}$$

c) for shells satisfying the condition

$$0.86 \geqslant \theta_1 \geqslant -2.93 \tag{53.24}$$

the approximate value of the critical pressure is equal to

$$p_\kappa = 1.31\bar{K}\eta^{3/2} m_1 (1.33 + 2\beta^2) : \left[1 + \frac{m_1}{\delta_1}(\lambda_1 - 1)(1 + \beta)\right], \tag{53.25}$$

where β is the smallest-absolute-value root of the equation (36. 19). With $\lambda = 0$, $\lambda_1 = 0.5$, one obtains the formula for uniform compression.

With $0.25 \leqslant \nu \leqslant 0.33$ (as is usually the case for metals) after simple calculations neglecting the second degree terms, (53. 22) may be written in the form in the form

$$\begin{gathered} p_\kappa \simeq 1.41\,\frac{(\rho^{1-\nu} - 1)^{1/4}(1 - \rho^{-(1+\nu)})^{3/4}\, E\,\mathrm{tg}\,\gamma\,(t/R_0)^{3/2}}{(\rho^{2-\nu} - 1)\,[1 + \theta(\lambda - 0.5)]\ln\rho}, \\ \rho = 1 + \frac{L}{r_0}, \ \lambda = \frac{(2 - \nu)\,T_0 (1 - \rho^{-\nu})}{p R_0^{\nu} (\rho^{2-\nu} - 1)}, \quad R_0 = r_0\,\mathrm{tg}\,\gamma, \\ \theta = 1.13\left\{\frac{1 - \rho^{-(1+\nu)}}{\rho^{1-\nu} - 1}\right\}^{1/4}\sqrt{\frac{t}{R_0}}\,\frac{\mathrm{tg}\,\gamma}{\ln\rho}, \end{gathered} \tag{53.26}$$

when the condition (53.26) is observed, or

$$0.49 \geqslant (1 - 2\lambda)\theta \geqslant -0.83. \tag{53.27}$$

Let us note that in passing from formula (53.22) to the simplified formula (53.26) we have reduced the critical pressure by at most 5-7 % (with $\ln\,\rho = 1$), but formula (53.22) in its turn had been derived from the characteristic equation (53.13) by simplifications which increase it by 2-3%. Besides, by approximating the solution from (53. 6) by one term of the series, we had increased the value of the critical load, where for axial compression this increase turned out to be 4-5%, and therefore our last transformation of formula (53. 22) results in an improvement. Thus, as a result of all the simplifications, formula (53. 26) gives a value of the upper critical pressure close to the actual one.

§ 54. The Stability of a Conical Shell of Varying Thickness under Uniform External Pressure

Let the shell thickness vary linearly along the length

$$t = \alpha r \quad (\alpha = t_0/r_0), \tag{54.1}$$

where t_0 is the shell thickness at $r = r_0$.

The investigation of the stability of such a shell is of interest, mainly because it is a uniformly strong, thin-walled structure. Besides, the equation of its neutral equilibrium can be integrated exactly /XII. 7/ for this case.

The stress in the membrane state of equilibrium are determined, just as for a shell of constant thickness, from (52. 2)

$$T_1^1 = -\frac{1}{2} pr \operatorname{tg} \gamma, \quad T_2^1 = -pr \operatorname{tg} \gamma. \tag{54.2}$$

We determine the additional stresses and moments appearing with the stability loss of the shell from (25. 24) and (25. 25)

$$\begin{gathered} T_1 = K_0 r(\varepsilon_1 + \nu\varepsilon_2), \quad T_{12} = K_0 r(1-\nu)\varepsilon_{12}, \\ M_1 = D_0 r^3(\varkappa_1 + \nu\varkappa_2), \quad M_{12} = D_0 r^3(1-\nu)\varkappa_{12}, \quad \overrightarrow{\underleftarrow{1,\ 2}} \end{gathered} \tag{54.3}$$

where

$$\begin{gathered} K_0 = \frac{E\alpha}{1-\nu^2}, \quad D_0 = \frac{E\alpha^3}{12(1-\nu^2)} = \frac{D}{r^3}, \quad \varepsilon_1 = \frac{\partial u}{\partial r}, \\ \varepsilon_2 = \frac{\partial v}{r\partial\varphi_1} + \frac{1}{r}(u + w \operatorname{ctg}\gamma); \quad 2\varepsilon_{12} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial\varphi_1} - v\right) \end{gathered} \tag{54.4}$$

$$\varkappa_1 = -\frac{\partial^2 w}{\partial r^2}, \quad \varkappa_2 = -\frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}, \quad \varkappa_{12} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\varphi_1}\right). \tag{54.5}$$

Eliminating u and v from (54.4), we find the equation of compatibility:

$$r\frac{\partial^2}{\partial r^2}(r\varepsilon_2) - r\frac{\partial\varepsilon_1}{\partial r} + \frac{\partial^2\varepsilon_1}{\partial\varphi_1^2} - \frac{\partial^2}{\partial r\partial\varphi_1}(2r\varepsilon_{12}) = r \operatorname{ctg}\gamma\, \frac{\partial^2 w}{\partial r^2}.$$

Upon expressing ε_1, ε_2, and ε_{12} in terms of the stress function ψ according to (52. 3) and (54. 3), this equation reduces to the form

$$\begin{gathered} \frac{\partial^4\psi}{\partial r^4} - \frac{(1-\nu)}{r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{2(1-\nu)}{r^3}\frac{\partial\psi}{\partial r} + \frac{2}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\varphi_1^2} - \frac{4}{r^3}\frac{\partial^3\psi}{\partial r\partial\varphi_1^2} + \\ + \frac{7-2\nu}{r^4}\frac{\partial^2\psi}{\partial\varphi_1^2} + \frac{1}{r^4}\frac{\partial^4\psi}{\partial\varphi_1^4} - K_{11}\frac{\partial^2 w}{\partial r^2} = 0, \quad K_{11} = E\alpha \operatorname{ctg}\gamma. \end{gathered} \tag{54.6}$$

We obtain the equilibrium equation from (7 4) by use of (52. 3), (54. 3)-(54. 5):

$$D\Delta\Delta w + \frac{1}{r}\frac{dD}{dr}\left\{2r\frac{\partial^3 w}{\partial r^3} + (2+\nu)\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r} + \frac{2}{r}\frac{\partial^3 w}{\partial r \partial \varphi_1^2} - \right.$$
$$\left. - \frac{3}{r^2}\frac{\partial^2 w}{\partial \varphi_1^2}\right\} + \frac{d^2 D}{dr^2}\left\{\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \varphi_1^2}\right\} + \frac{\operatorname{ctg}\gamma}{r}\frac{\partial^2 \psi}{\partial r^2} - \tag{54.7}$$
$$- T_1^0\frac{\partial^2 w}{\partial r^2} - T_2^0\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi_1^2}\right) = 0.$$

After substituting

$$z = \ln\frac{r}{r_0}, \quad \psi = Fe^{1/2 z}\cos n\varphi, \quad w = e^{-1/2 z} w_1 \cos n\varphi \tag{54.8}$$

and neglecting quantities of the order of unity in comparison with n_1^2, the system of equations (54.6), (54.7) is brought into the simple form

$$\frac{d^4 F}{dz^4} - 2n_1^2\frac{d^2 F}{dz^2} + n_1^4 F - K_{11}\left(\frac{d^2 w_1}{dz^2} - 2\frac{dw_1}{dz} + \frac{3}{4}w_1\right) = 0,$$
$$\frac{d^4 w_1}{dz^4} - \left(2n_1^2 - \frac{p_{11}}{2}\right)\frac{d^2 w_1}{dz^2} + (n_1^4 - p_{11}n_1^2)w_1 +$$
$$+ \frac{\operatorname{ctg}\gamma}{D_0}\left(\frac{d^2 F}{dz^2} + 2\frac{dF}{dz} + \frac{3}{4}F\right) = 0,$$
$$p_{11} = \frac{p \operatorname{tg}\gamma}{D_0}. \tag{54.9}$$

Taking wave formation to be of the form (53. 6), we arrive at the characteristic equation

$$(m_1^2 + n_1^2)^2 - p_{11}\left(n_1^2 + \frac{m_1^2}{2}\right) +$$
$$+ \frac{K_{11}}{D_0}\operatorname{ctg}\gamma\frac{m_1^4 + 2{,}5m_1^2 + 9/16}{(m_1^2 + n_1^2)^2} = 0 \tag{54.10}$$

This equation, obtained on the assumption that $n^2 > \sin^2\gamma$, is applicable provided that the shell is of medium length and small angles, as even for shells of medium thickness $n \geqslant 3$; or, if the shell length is less than the radius of its smaller base, buckling occurs with the formation of a large number of waves along the circumference.

For a short shell of not too large an angle, satisfying the conditions

$$\gamma \leqslant 30^\circ, \quad L \leqslant R_0 = r_0 \operatorname{tg}\gamma, \tag{54.11}$$

we have

$$\zeta = \ln\left(1 + \frac{L}{r_0}\right) < \frac{1}{2}; \quad m_1^2 \geqslant 4\pi^2. \tag{54.12}$$

Here, in the last term of the left-hand member of equation (54.10) one may neglect $2.5m_1^2 + 9/16$ in comparison with m_1^4, and thus obtain an approximate characteristic equation identical with the corresponding equation for some ficticious cylindrical shell

$$(m_1^2 + n_1^2)^2 - p_{11}\left(n_1^2 + \frac{m_1^2}{2}\right) +$$
$$+ K_{11}\frac{\operatorname{ctg}\gamma}{D_0}m_1^4/(m_1^2 + n_1^2)^2 = 0. \tag{54.13}$$

Let us note that here the admissable error in the value of the critical pressure is one-fourth the error tolerated in the above-mentioned term of equation (54. 10), as with the critical value of n_1 this term, according to the analysis carried out for the cylindrical shell, is one-third of the first term of the equation.

With the shape of wave formation defined by the equalities (53.6) and (54.8), the following conditions are satisfied exactly:

$$w=0,\quad T_2=0 \quad \text{at} \quad z=0,\quad z=\zeta. \tag{54.14}$$

The first of these is apparent from (53. 6). The second conditions is satisfied, as

$$T_2=\frac{\partial^2\psi}{\partial r^2}=e^{\frac{3}{2}z}\left(\frac{d^2F}{dz^2}+2\frac{dF}{dz}+\frac{3}{4}F\right)\cos n\varphi,$$

where the expression in brackets, according to the second of equations (54.9), is equal to zero at the shell edges.

When the boundary condition $T_1 = 0$ is satisfied, it follows from (54.14) that the geometric condition $\varepsilon_2 = 0$ is fulfilled. However, the boundary conditions

$$T_1=0,\ M_1=0 \quad \text{at} \quad z=0,\quad z=\zeta \tag{54.15}$$

are not satisfied exactly. The principal part of the expression for T_1 is, according to (52.3), (54.8), and (54.9),

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi_1^2}=\frac{n_1^2AK_{11}(m_1^2-3/4)}{r^2(m_1^2+n_1^2)^2}\left\{\sin m_1z+\frac{2m_1}{m_1^2-\frac{3}{4}}\cos m_1z\right\},$$

therefore, the condition $T_1 = 0$ is satisfied at every point of the edge contour only upon neglecting unity in comparison with m_1.

It can be shown that the maximum value of the unbalanced contour forces, applied to a contour element defined by the angle $d\varphi$, is m_1^{-2} times the corresponding potential energy of elongation of the middle surface of the shell. The error from the non-fulfillment of the boundary condition $M_1 = 0$ will be even smaller, namely, of the order of m_1^2/n_1^4 in comparison with unity. Without dwelling here on the proof of these assertions*, we shall only note that to obtain an approximate characteristic equation (54. 13) from equation (54. 9), one has to neglect in the latter the quantities w and F and their first derivatives with respect to z in comparison with the second derivatives. There, F will also be proportional to $\sin m_1z$, and consequently, the conditions (54. 15) will be satisfied by retaining only the higher terms. We see that tolerating a comparatively large error of the order of unity in comparison with m in the boundary conditions (apart from the principal condition $w = 0$) and neglecting the terms containing odd derivatives of the required functions sought in the differential equations leads to an error of the order of unity in comparison with m_1^2 in the determination of the critical load.

Thus, admitting the error indicated in the value of the critical load, we may consider that the boundary conditions are satisfied and the critical pressure is determined from equation (54. 13) in the same way in which it had been done in §36 for a cylindrical shell, and the formulas derived there for the cylindrical shell are transformed into formulas for a conical shell of varying thickness, by replacing the quantities t/L, t/R, Rt/L^2 respectively by

$$\alpha/\zeta,\quad \alpha\,\mathrm{ctg}\,\gamma,\quad \alpha\,\mathrm{tg}\,\gamma/\zeta^2.$$

Thus we find, for example, that

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* See Chapter II of /XII. 7/.

$$p_{\kappa}=\frac{0.85\,E\alpha\,(\alpha\,\mathrm{ctg}\,\gamma)^{3/2}}{(1-\nu^2)^{3/4}\,\zeta}\,;\left(1-\frac{0.65}{\zeta}\sqrt{\alpha\,\mathrm{tg}\,\gamma}\right) \qquad (54.16)$$

by satisfying the condition

$$\frac{\pi}{\zeta}\leqslant 1.2\sqrt{\frac{\mathrm{ctg}\,\gamma}{\alpha}}. \qquad (54.17)$$

Another limiting case is of special interest: the consideration of the stability of a conical cupola with a very small opening of radius $r_0 \sin\gamma$ for $r_0 \to 0$ /XII. 8/.

In that case

$$\zeta=\ln\left(1+\frac{L}{r_0}\right)\to\infty,\quad m_1=\frac{\pi}{\zeta}\to 0,$$

and in equations (54.9) and in the boundary conditions one may neglect all the terms containing derivatives of F and w_1 with respect to z. Then all the boundary conditions are satisfied, and from equation (54.10) we find

$$p_{11}=n_1{}^2+\frac{9}{16}\,\frac{K_{11}\,\mathrm{ctg}\,\gamma}{D_0 n_1{}^6}, \qquad (54.18)$$

$$n_{\kappa}^2=\frac{3}{\sqrt{2}}\left(\frac{r_1}{t}\right)^{1/2}(\cos\gamma)^{1/2}(\sin\gamma)^{3/2}(1-\nu^2)^{1/4},$$

$$p_{\kappa}=\frac{0.235}{(1-\nu^2)^{3/4}}\,\frac{Et}{r_1}\left(\frac{t}{r_1\,\mathrm{tg}\,\gamma}\right)^{3/2}. \qquad (54.19)$$

Our original equations had been obtained on the assumption that along the shell circumference many waves are formed. This assumption, as can be seen from (54.18), is realized only for values of γ not close to 0 or $\pi/2$.

§ 55. The Stability of a Circular Conical Shell under Torsion

We shall consider briefly the determination of the upper critical shearing stress T_{12}^{1}, uniformly distributed along end sections*. We shall solve the problem by the Ritz method, making use of our energy criterion for stability, expressed in the form of the variational equation (25. 31), where $B = r\sin\gamma$, $\varepsilon_1, \ldots, \varkappa_{12}$ are quantities given by (54. 5).

For the components of the additional displacement which appears with buckling, we take the expressions

$$
\begin{gathered}
u = C_1 e^z \left\{\sin(\mu_1 z + n\varphi) - \sin(\mu_2 z + n\varphi)\right\}, \\
v = C_2 e^z \left\{\sin(\mu_1 z + n\varphi) - \sin(\mu_2 z + n\varphi)\right\}, \\
w = -2Ce^z \sin(\mu z + n\varphi)\sin m_1 z,
\end{gathered}
\tag{55.1}
$$

where

$$
z = \ln r/r_0,\ m_1 = \frac{\pi}{\zeta} = \frac{\mu_1 - \mu_2}{2},\ \zeta = \ln\left(1 + \frac{L}{r_0}\right),\ \mu = \frac{\mu_1 + \mu_2}{2}. \tag{55.2}
$$

and C_1, C_2, C are arbitrary constants.

It is not difficult to show that the functions (55.1) are the exact integrals of the differential equations of neutral equilibrium (25.26), (25.27), in the limiting cases when $\gamma = 0$ and $\gamma = \pi/2$, and in the general case satisfy the essential boundary conditions

$$
u = v = w = 0 \quad \text{at} \quad z = 0, \qquad z = \zeta. \tag{55.3}
$$

Introducing in this section the notations

$$
\begin{gathered}
\mu^2 = \frac{2D(1 + \zeta^2/\pi^2)\zeta}{K(\rho^2 - 1)r_0^2},\ C_0' = \frac{4\tau_0(1 + \zeta^2/\pi^2)\zeta}{K(\rho^2 - 1)},\ \rho = 1 + \frac{L}{r_0}, \\
n_1 = n/\sin\gamma
\end{gathered}
\tag{55.4}
$$

and substituting from (55. 1) into (25. 31), we obtain a quantity proportional to Э

$$
\begin{gathered}
\text{Э}' = -C_0'\mu(n_1 C^2 + C_2 C\operatorname{ctg}\gamma) + C_1^2\left(\frac{1-\nu}{2}n_1^2 + \frac{\mu_1^2 + \mu_2^2 + 3}{2}\right) + \\
+ C_2^2\left[\frac{1-\nu}{2}(\mu_1^2 + \mu_2^2 + 4) + n_1^2\right] + C_1C_2\mu n_1(1 + \nu) + C^2\operatorname{ctg}^2\gamma + \\
+ 2C_2Cn_1\operatorname{ctg}\gamma + 2C_1C\nu\mu\operatorname{ctg}\gamma + C^2\eta^2\left\{\frac{\mu_1^4 + \mu_2^4}{2} + \right. \\
+ \nu\left(\mu_1^2 + \mu_2^2\right)\left(n_1^2 + \frac{C_2 n_1}{C}\operatorname{ctg}\gamma\right) + \left(n_1^2 + \frac{C_2}{C}n_1\operatorname{ctg}\gamma - 1\right)^2 + \\
\left. + (1 - \nu)\left(\mu_1^2 + \mu_2^2\right)(n_1 + C_2\operatorname{ctg}\gamma/C)^2\right\}.
\end{gathered}
$$

Here we neglect quantities of the order of unity in comparison with n_1^2 in the terms containing the small factor D/r_0^2

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* See article /XII.6/ and also § 19 of the monograph /0.13/.

Setting up the equations

$$\partial \mathcal{J}'/\partial C_1 = \partial \mathcal{J}'/\partial C_2 = \partial \mathcal{J}'/\partial C = 0,$$

eliminating C_1 and C_2 from them, after neglecting quantities of the order of unity in comparison with n_1^2, we obtain the characteristic equation

$$C_0 = \frac{C_0' \pi^2}{\zeta \varepsilon_k^{5/2} (1 - \nu^2) \operatorname{ctg}^3 \gamma} =$$

$$= \frac{\mu_0^{3/2} (1 - \varepsilon_k \mu_0/\beta)^{-\frac{1}{2}}}{\sqrt{\mu_0^2 - 1}} \left\{ \beta^{3/2} \left[1 + \frac{4\varepsilon_k^2 \mu^2}{\beta^2 (\mu^2 + m_1^2)} \right] + \right.$$

$$\left. + \frac{1}{\beta^{5/2}} \left[1 + \frac{\nu_1 \beta}{\varepsilon_k \mu_0^3} \left(1 - \frac{\mu_0 \varepsilon_k}{\beta} \right) + \frac{\nu_0}{\mu_0^3} \right] : \left[1 + \frac{(1 - \varepsilon_k \mu_0/\beta) \nu_2 \varepsilon_k}{\beta \mu_0} \right] \right\}, \tag{55.5}$$

where

$$\varepsilon_k^2 = \frac{\eta m_1^2}{\operatorname{ctg} \gamma \sqrt{1 - \nu^2}}, \quad \beta = \frac{\varepsilon_k (n_1^2 + \mu^2 + m_1^2)}{m_1 \sqrt{\mu^2 + m_1^2}}, \quad \frac{\mu^2 + m_1^2}{m_1^2} = \mu_0,$$
$$\nu_0 = \frac{\nu^2}{1 - \nu^2}, \quad \nu_1 = \frac{1}{2(1 + \nu)}, \quad \nu_2 = \frac{(1 + \nu)^2}{2(1 - \nu)}. \tag{55.6}$$

The subsequent problem consists in the determination, from (55.5), of such values of $\mu_2 = \mu_1 - 2m_1$ and n_1, for which C_0, and therefore also τ_0, has the smallest value. To simplify the solution of that problem with the aim of obtaining a computational formula, we shall limit ourselves to the consideration of the most important case, when

$$\varepsilon_k^2 << 1. \tag{55.7}$$

From the expressions (55. 6) and (55. 4) it is apparent that the condition (55. 7) is satisfied if the shell is thin and the value of γ is not close to $\pi/2$.

Further, as is shown by trial calculations, with values of ε_k, equal to 0.1, 0.05, and 0.02, the critical values of μ_0 are approximately equal to 2.0, 2.5, and 3.5, while the critical value of β is in all cases approximately equal to 1.2, and therefore (with an overestimation error of less than 2%) the quantity C_0 may be determ... from the approximate formula

$$C_0 = \frac{\mu_0^{3/2} (1 - \varepsilon_k \mu_0/\beta)^{-\frac{1}{2}}}{\sqrt{\mu_0^2 - 1}} \left\{ \beta^{3/2} + \frac{1}{\beta^{5/2}} \left[1 + \frac{\nu_1 \beta}{\varepsilon_k \mu_0^3} \left(1 - \frac{\varepsilon_k \mu_0}{\beta} \right) \right] \right\}. \tag{55.8}$$

From the conditions of minimal critical load

$$\partial C_0/\partial \mu_0 = 0, \quad \partial C_0/\partial \beta = 0$$

we find by the method of successive approximations the respective values of μ_0 and β. As a function changes slowly near its minimum, to determine the critical values of β and μ_0 we shall neglect $\varepsilon_k \mu_0/\beta$ in comparison with unity in the expression (55. 7) and set

$$\sqrt{\mu_0^2 - 1} = \mu_0 \left(1 - \frac{1}{2\mu_0^2} - \frac{1}{8\mu_0^4} - \ldots \right) = \mu_0.$$

Thus, to the first approximation

$$C_0 = C_0^1 = \mu_0^{1/2} \left\{ \beta^{3/2} \left(1 + \frac{\nu_1}{\varepsilon_k \mu_0^3} \right) + \frac{1}{\beta^{5/2}} \right\}.$$

Here from equations $\partial C_0/\partial\mu_0=0$ and $\partial C_0/\partial\beta=0$ we find

$$\beta^{I}=1.236,\ \mu_0^{I}=1.23\left(\frac{\nu_1}{\varepsilon}\right)^{1/3},\ C\approx 2.61\left(\frac{\nu_1}{\varepsilon_k}\right)^{1/6}.$$

Introducing these values of β and μ_0 in (55. 7), we obtain

$$C_0\approx C_0^{II}=2.61\left(\frac{\nu_1}{\varepsilon_k}\right)^{1/6}\left[1+\varepsilon^{2/3}\left(0.33\nu_1^{1/3}+\frac{0.33}{\nu_1^{2/3}}\right)+\right.$$
$$\left.+\varepsilon_k^{4/3}\left(0.29\nu_1^{2/3}+\frac{0.375}{\nu_1^{4/3}}+\frac{0.11}{\nu_1^{1/3}}\right)\right].$$

With $\nu = 0.3$ and $\varepsilon_k = 0.1$, we find $C_0^{11}/C_0^{1} = 1.26$. Usually $\varepsilon \leqslant 0.05$, and then $C_0^{11}/C_0^{1} \leqslant 1.15$. Regardless of the considerable difference between C_0^{11} and C_0^{1}, the error in the value of C_0, given by the formula (55.8), does not exceed 5 % even with $\varepsilon = 0.1$ and $0.25 \leqslant \nu \leqslant 0.33$, where C_0^{1} turns out to be larger than the critical value Besides, as is well known, energy criterion for stability gives an excessive value of the critical load, and therefore we shal improve the formula for C_0 by dropping the last term in the expression for C_0^{11}, which for $\varepsilon = 0.1$ constitutes 8% of the value of C_0.

Thus,

$$C_0\approx 2.61\left(\frac{\nu_1}{\varepsilon_k}\right)^{1/6}\left[1+0.33\left(\frac{\varepsilon_k}{\nu_1}\right)^{2/3}(1+\nu_1)\right],$$
$$\varepsilon\leqslant 0.1;\ 0.25\leqslant\nu\leqslant 0.33,$$
$$\tau_0=\frac{Et(\rho^2-1)\zeta}{4(\pi^2+\zeta^2)}\operatorname{ctg}^2\gamma\,\varepsilon_k^{5/2}C_0;\quad \varepsilon_k^2=\frac{\pi t}{\zeta^{3/2}r_0\operatorname{tg}\gamma\sqrt{1-\nu^2}}\left\{\frac{\pi^2+\zeta^2}{6(\rho^2-1)}\right\}^{1/2}. \qquad (55.8a)$$

For a short shell of small angle

$$R_0\approx R=r_0\sin\gamma,\quad L\operatorname{tg}\gamma<<2R_0;\quad \operatorname{tg}\gamma\approx\sin\gamma.$$

In that case

$$\rho^2-1\approx 2\frac{L\operatorname{tg}\gamma}{R},\quad \zeta=\ln\rho\approx\frac{L\operatorname{tg}\gamma}{R},\quad \varepsilon_k^2\approx\frac{2.97tR}{L^2\sqrt{1-\nu^2}}$$

and we obtain for the critical stress a formula differing from formula (38.17) for the critical shearing stress τ_1 in a cylindrical shell only by a numerical coefficient, where the range of applicability of these formulas is defined by the inequalities (38.16) or

$$0.03\leqslant\varepsilon_k\leqslant 0.12. \qquad (55.9)$$

Calculations show that the ratio of the critical shearing stress at the boundaries of the region (55. 9), found for a cylindrical shell from (55. 8a) and (38. 17) are 1.14 and 0.99, respectively.

This coincidence of solutions obtained by different methods may be considered as entirely satisfactory, if one takes into account that in this section we satisfied more rigid boundary conditions (55. 3) than in § 38.

For values of γ not close to $\pi/2$, the critical shearing stress is determined from (55. 8a), where the values of C_0 for $\nu = 0.3$ may be taken from Table XVII.

Table XVII

t_k	0.12	0.10	0.07	0.05	0.03
C_0	3.75	3.87	3.95	4.08	4.31

To our regret, we do not have experimental data for a conical shell which would allow us to judge the influence of factors not taken into account in deriving the formula (55. 8a). In the case of torsion of a cylindrical shell, to make the theoretical formulas (38. 17) agree with experimental data it is necessary to introduce a correction coefficient 0.6. It is to be assumed that such a coefficient should be also introduced in formula (55. 8a) as long as it is not established experimentally.

In the limiting case of an annular plate, when $\gamma = \pi/2$, our approximate solution is inapplicable. But in that case it is easy to integrate the equations of neutral equilibrium (52. 4). The solution of that problem which fulfills the boundary conditions has been given in /XII. 9/, in which, however, no computational formulas are given. If one limits oneself to fulfilling the more important boundary condition $w = 0$, as is done in /0. 13/ and /XII. 6/, the critical shearing stress may be determined from the approximate formula

$$T_{12}^{1} = \tau_0 = -\frac{D}{2nm_2{}^2}\left[(m_2^2+1)^2 + 2n^2(m_2^2-1) + n^4\right], \tag{55.10}$$

where n^2 and m_2 satisfy the equations

$$n^2 = \frac{1}{3}(m_2^2+2),\ m_2 = -2\zeta\left[(n^2-1)^2 + (n^2+4\lambda^2)^2\right]/(n^2+4\lambda^2)^2, \tag{55.11}$$

a and b are the inner and outer radii of the plate,

$$\rho = 1 + \frac{L}{r_0} = \frac{b}{a} < 7.40;\ \zeta = \ln b/a.$$

If n^2 is not large in comparison with unity, then instead of n one should set in (55. 10) and (55. 11) the integers closest to it and the critical stress is the smaller of the values of T_{12}^{1} thus found.

If $\zeta^2 \ll \pi^2$, then

$$n^2 \approx \frac{4\pi^2}{3\zeta^2},\ m_2 \approx -2.125\frac{\pi}{\zeta},\ T_{12,\kappa}^{1} = \frac{68.8D}{a^2} : \left(\ln\frac{b}{a}\right)^2. \tag{55.12}$$

§ 56. Lower Critical Load of a Conical Shell*. Local Stability of Shells

To determine the critical load of collapse from the state of equilibrium with large deflections, one has to use the relations of the non-linear theory of shells, which for the case of a circular conical shell may be obtained from the formulas of § 25 by setting $B = r \sin \gamma$. Let the shell have the ideal regular shape, i. e., $w^0 = 0$. Then from (25. 12) and (25. 33) we obtain the condition of compatibility and the equilibrium equation

$$\Delta\Delta\psi^{I} = Et\left[\frac{1}{r^2}\left(\frac{\partial^2 w^{I}}{\partial r\partial\varphi_1}\right)^2 - \frac{1}{r^2}\frac{\partial^2 w^{I}}{\partial r^2}\frac{\partial^2 w^{I}}{\partial \varphi_1^2} - \frac{2}{r^3}\frac{\partial w^{I}}{\partial\varphi_1}\frac{\partial^2 w^{I}}{\partial r\partial\varphi_1} + \right.$$
$$\left. + \frac{1}{r^4}\left(\frac{\partial w^{I}}{\partial\varphi_1}\right)^2 - \frac{1}{r}\frac{\partial w^{I}}{\partial r}\frac{\partial^2 w^{I}}{\partial r^2} + \frac{1}{r}\operatorname{ctg}\gamma\frac{\partial^2 w^{I}}{\partial r^2}\right], \tag{56.1}$$

$$D\Delta\Delta w^{I} + \frac{1}{r}\operatorname{ctg}\gamma\frac{\partial^2\psi^{I}}{\partial r^2} - T_1^{I}\frac{\partial^2 w^{I}}{\partial r^2} - T_2^{I}\left(\frac{1}{r}\frac{\partial w^{I}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^{I}}{\partial\varphi_1^2}\right) -$$
$$- 2T_{12}^{I}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w^{I}}{\partial\varphi_1}\right) + p = 0, \tag{56.2}$$

where T_1^I, T_2^I, T_{12}^I are expressed in terms of ψ^I according to formulas (52.3).

Introducing the substitutions

$$z = \ln(r/r_0),\ \psi^{I} = e^{z}F,\ w^{I} = r\operatorname{ctg}\gamma\, e^{z} w_1^{I}, \tag{56.3}$$

we reduce these equations to the system of two non-linear equations in F and w_1^I, whre the independent variables z and φ_1 are dimensionless quantities. As had been shown in §§ 52-53, with the loss of stability of a thin conical shell under longitudinal compression, many waves are formed in the axial and longitudinal directions, so that

$$m_1^2 \sim R/t >> 1,\ n_1^2 = n^2/\sin^2\gamma >> 1.$$

In the case of transverse pressure, at least, $n_1^2 > 1$. This means that the second derivatives of w^I and F^I with respect to $\varphi_1 = \varphi \sin\gamma$ are large in comparison with the functions themselves. Besides, if the shell is short then $\partial^2 w_1^I/\partial z^2 >> w_1^I$, and in the case of a shell of medium length, $\partial^2 w_1^I/\partial z^2 \sim w_1^I$.

Experiments show that for the state of a shell closely spaced buckles and dents are characteristic also after the buckling. Therefore, to simplify the equations (56. 1) and (56. 2) we shall neglect w_1^I and F^I in comparison with their second derivatives with respect to φ_1. Thus, we arrive at the system of equations

$$\Delta\Delta F^{I} = Etr_0\operatorname{ctg}^2\gamma\, e^{z}\left\{\left(\frac{\partial^2 w_1^{I}}{\partial z\partial\varphi_1}\right)^2 - \frac{\partial^2 w_1^{I}}{\partial\varphi_1^2}\left(\frac{\partial^2 w_1^{I}}{\partial z^2} + \frac{\partial w_1^{I}}{\partial z}\right) - \right.$$
$$\left. - \frac{\partial^2 w_1^{I}}{\partial z^2}\left(\frac{\partial w_1^{I}}{\partial z} - 1\right)\right\},$$
$$D\Delta\Delta w_1^{I} - e^{z}\left\{\left(\frac{\partial^2 F^{I}}{\partial z^2} + \frac{\partial F^{I}}{\partial z}\right)\left(\frac{\partial^2 w_1^{I}}{\partial\varphi_1^2} - 1\right) + \frac{\partial^2 F^{I}}{\partial\varphi_1^2}\left(\frac{\partial^2 w_1^{I}}{\partial z^2} + \frac{\partial w_1^{I}}{\partial z}\right) + \right.$$
$$\left. + \frac{\partial}{\partial z}\left(\frac{\partial F^{I}}{\partial z}\frac{\partial w_1^{I}}{\partial z}\right) - 2\frac{\partial^2 F^{I}}{\partial z\partial\varphi_1}\frac{\partial^2 w_1^{I}}{\partial z\partial\varphi_1}\right\} + \frac{r_0^2 e^{2z}}{\operatorname{ctg}\lambda}p = 0,\ \varphi_1 = \varphi\sin\gamma. \tag{56.4}$$

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* See article /XII.4/.

We shall consider these as the initial equations for the asymmetrical deformation of a conical shell with $n_1^2 \gg 1$. They may be considerably simplified for the longitudinal compression of short shells, as in that case w^I and φ^I increase rapidly in absolute value with the change in φ_1 as well as with the change in z. The ratio of two successive derivatives is of the order of $\sqrt{R/t}$, and therefore for thin shells, when $\sqrt{t/R} \ll 1$, we shall neglect F and w_1^I in comparison with their first derivatives. Besides, as one retains only the higher derivatives, their variable coefficients (determining the geometrical characteristics of the shell) are smoothly varying functions; the latter may be considered as constant parameters, subject to determination from the condition of minimality of the critical load. This corresponds to the consideration of the stability of an infinitesimal portion of the shell near $z = z_c$, whose metric may be considered as Euclidean. Therefore, in equations (56.4) we shall set $e^z = e^{z_c} = \text{const}$. Thus, we shall obtain the approximate equations

$$\Delta\Delta F^I = Etr_0^2 \operatorname{ctg}^2 \gamma e^{z_c} \left\{ \left(\frac{\partial^2 w_1^I}{\partial z \partial \varphi_1} \right)^2 - \frac{\partial^2 w_1^I}{\partial z^2} \frac{\partial^2 w_1^I}{\partial \varphi_1^2} - \frac{\partial^2 w_1^I}{\partial z^2} \right\},$$

$$D\Delta\Delta w_1^I = e^z \left\{ -\frac{\partial^2 F^I}{\partial z^2} + \frac{\partial^2 w_1^I}{\partial z^2} \frac{\partial^2 F^I}{\partial \varphi_1^2} + \frac{\partial^2 w_1^I}{\partial \varphi_1^2} \frac{\partial^2 F^I}{\partial z^2} - 2 \frac{\partial^2 w_1^I}{\partial z \partial \varphi_1} \frac{\partial^2 F^I}{\partial z \partial \varphi_1} \right\} - p \frac{r_0^3 e^{3z_c}}{\operatorname{ctg} \gamma}. \tag{56.5}$$

Carrying out the additional transformations

$$F^I = \psi^I e^{-z_c}, \quad w_1^I = \frac{w^I}{r_c \operatorname{ctg} \gamma}, \quad z = \frac{x}{r_c}, \quad \varphi_1 = \frac{s}{r_c}, \quad z_c = \ln \frac{r_c}{r_0}, \tag{56.6}$$

we arrive at the equations

$$\Delta\Delta\psi^I = Et \left\{ \left(\frac{\partial^2 w^I}{\partial x \partial s} \right)^2 - \frac{\partial^2 w^I}{\partial x^2} \frac{\partial^2 w^I}{\partial s^2} - \frac{1}{r_c \operatorname{tg} \gamma} \frac{\partial^2 w^I}{\partial x^2} \right\},$$

$$D\Delta\Delta w^I + \frac{1}{r_c \operatorname{tg} \gamma} \frac{\partial^2 \psi^I}{\partial x^2} - \frac{\partial^2 w^I}{\partial x^2} \frac{\partial^2 \psi^I}{\partial s^2} - \frac{\partial^2 w^I}{\partial s^2} \frac{\partial^2 \psi^I}{\partial x^2} + 2 \frac{\partial^2 w^I}{\partial x \partial s} \frac{\partial^2 \psi^I}{\partial x \partial s} + p = 0. \tag{56.7}$$

These equations coincide with the non-linear equations (40.3) and (40.4) for cylindrical shells with the fictitious radius $R_c = r_c \operatorname{tg}\gamma$. They are obtained by neglecting quantities of the order of $\sqrt{t/R}$ in comparison with unity, which is mathematically well founded, as $\sqrt{t/R} \to 0$. This is the so-called asymptotic integration of the equations of the theory of shells. In those cases, when with the loss of stability very short waves appearing have lengths of the order of $\sqrt{R/t}$, the replacement of the geometrical parameters of the zone of one half-wave by constant quantities leads to an error of the same order of magnitude. This idea, initially applied to the linear theory of the edge effect in shells, was utilized in the theory of stability of shells for the first time by I. Ya. Shtaerman in work /XII. 1/. He, by giving it an intuitive interpretation, showed that a narrow wavy belt, forming in the equatorial zone of a shell of rotation with the loss of stability, is very close to the wavy belt forming with the loss of stability in a cylindrical shell.

Later, Yu. N. Rabotnov constructed, in this setting, the general linear theory of stability called by him the theory of local stability of shells /V. 15/.

V. Z. Vlasov has generalized the theory for the case of non-linear problems /0.4/.

From the above it follows that the lower critical load for a conical shell under longitudinal compression is equal to the corresponding quantity for a cylindrical shell of radius R_c. Denoting that stress at the sections $r = r_0$ and $r = r_c$ by T_0^H and T_c^H respectively, we find from (40.25)

$$T_c^{\text{н}} \approx 0.187 \frac{Et^2}{R_c}, \quad T_0^{\text{н}} = T_c^{\text{н}} \cdot \frac{R_c}{R_0} = 0.187 \frac{Et^2}{R_0}. \tag{56.8}$$

The minimal buckling force P^{H}, acting perpendicularly to the base plane, is

$$P^{\text{н}} = T_0^{\text{н}} \cdot 2\pi R_0 \cos^2\gamma = 1.18 Et^2 \cos^2\gamma. \tag{56.9}$$

Hence it follows that the critical force which is borne by the shell does not depend on its radius, but decreases rapidly with the increase of the cone angle.

One may analogously consider the question of determining the lower critical load under longitudinal compression of other shells of zero Gaussian curvature whose surface divides under buckling into a large number of shallow parts. Then, making use of the concept of local stability, we set the principal radius of curvature equal to its maximal value. In the case of a cylindrical shell of elliptic section with small eccentricity, this radius is $R = a^2/b$, where a is the semi-major axis of the ellipse and b is the semi-minor axis. Consequently, according to the formula derived for a circular cylindrical shell, the modulus of the lower critical load is

$$|T_{\text{н}}^1| \approx 0.187 \frac{Et^2}{R} = 0.187 \frac{Et^2 b}{a^2}. \tag{56.10}$$

The corresponding value of the upper critical load is

$$|T_{\text{u}}^1| = 0.6 Et^2 b/a^2, \tag{56.11}$$

with values of eccentricity

$$e = \frac{\sqrt{a^2 - b^2}}{a} < 0.5$$

This formula is in very good agreement with an analogous formula obtained by Kh. M. Mushtari /0.13/ by a more precise method (see formula (37.22)).

The contour length of a shell of small eccentricity is approximately equal to $2\pi a(1-e^2/4)$. Consequently, the lower critical load is given by the formula

$$P^{\text{н}} = 1.18 \frac{Et^2 b}{a} \left(1 - \frac{1}{4} e^2\right). \tag{56.12}$$

Analogous quantities for a conical shell of elliptic section will be respectively,

$$|T_{\text{н}}^1| = 0.187 Et^2 \frac{b}{a^2} \cos\gamma, \quad P^{\text{н}} = 1.18 Et^2 \frac{b}{a} \left(1 - \frac{1}{4} e^2\right) \cos^2\gamma. \tag{56.13}$$

The lower critical pressure under all-round compression of a short conical shell may be also determined by starting in from equations (56. 7) and subsequently making use of the solution obtained earlier for the cylindrical shell. According to (43. 14) and the fulfillment of condition (43. 13), it is necessary to introduce the coefficient 0.68 for the determination of the value of the lower critical pressure, calculated according to the formulas of §§ 53-54. Here, one should not forget the fact that formula (43.14) had been derived when only one boundary condition, w = 0, was satisfied. This state of affairs does not have a noticeable influence on the value of the critical pressure, provided, as had been assumed in § 43, the short shell under consideration is one of the segments of a long shell, supported by intersticed transverse ribs, weakly resistant to the rotation of the skin elements and the displacements u and v in the tangent plane, since in that case the work of the unbalanced contour forces on the transverse ribs at the ends will be small in comparison with the deformation energy of the shell. For a conical shell too we assume that it is divided into equally stable short sections by transverse ribs of the indicated type.

Chapter XIII

THE STABILITY AND LARGE DEFLECTIONS OF SHELLS OF REVOLUTION

§ 57. The Stability of the Axially Symmetric Membrane Equilibrium State of a Shell of Revolution

We shall consider the equilibrium of a shell of constant thickness t, whose middle surface σ before the deformation is a surface of revolution. The position of a point on σ will be fixed by the intersection of the parallel θ = const with the meridian β = const, where θ is the angle between the axis of revolution OZ and the normal to the surface (Figure 38), R_2 = CM is the radius of curvature of the normal section θ = const, $R_1 = O_1M$ is the radius of curvature of that section of the surface β = const which is a meridian of the surface of revolution. The line element of σ is given by (25.4), where $d\alpha$ is a linear element of the meridian, and B is the radius of the parallel. In the formulas of § 25 the Gaussian coordinates α and β are taken as independent variables. To obtain the corresponding relations in the coordinates θ and β, it is necessary to take into account that

$$d\alpha = R_1 d\theta, \quad B = R_2 \sin\theta,$$
$$\frac{\partial(\ldots)}{\partial\alpha} = \frac{\partial(\ldots)}{\partial\theta}\frac{d\theta}{d\alpha} = \frac{1}{R_1}\frac{\partial(\ldots)}{\partial\theta}. \tag{57.1}$$

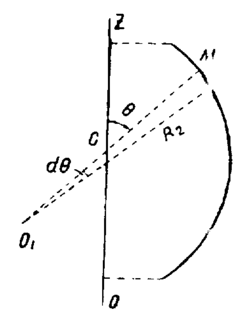

Figure 38

Let the part of the shell bounded by the parallels $\alpha = \alpha_0$ and $\alpha = \alpha_1$ be in equilibrium under the action of a normal pressure p, uniformly distributed over the surface and external meridional stresses uniformly distributed over the edge contours. It is obvious that the state of equilibrium of the shell before the loss of stability will be axially symmetrical. Neglecting the variations in curvature, we obtain from (25.11) and (25.12) the following equations for the membrane stresses in the equilibrium state:

$$\frac{d}{d\alpha}(BT_1^1) = T_1^1\frac{dB}{d\alpha}, \quad \frac{T_1^1}{R_1} + \frac{T_2^1}{R_2} + p = 0. \tag{57.2}$$

Assuming that at some critical load the stability of the axially symmetrical state is lost and a large number of waves of infinitesimal amplitude are formed on the surface, we obtain the equations of neutral equilibrium (25. 26) and (25. 27). The equations (25. 26) can be satisfied, by allowing an error inherent in the theory of shallow shells, if the additional stresses appearing with the loss of stability are expressed in terms of the stress function ψ according to formulas of the form of (25. 32).

$$T_1 = \frac{1}{B^2}\frac{\partial^2\psi}{\partial\beta^2} + \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial\psi}{d\alpha}, \quad T_2 = \frac{\partial^2\psi}{\partial\alpha^2},$$
$$T_{12} = -\frac{1}{B}\frac{\partial^2\psi}{\partial\alpha\partial\beta} + \frac{1}{B^2}\frac{dB}{d\alpha}\frac{\partial\psi}{\partial\beta}. \tag{57.3}$$

To determine ψ and the additional deflection w we have the equation of compatibility of the form (25. 23) and the equilibrium equation (25. 12)

$$\Delta\Delta\psi + Et(k_2\varkappa_1 + k_1\varkappa_2) = 0, \quad k_i = 1/R_i,$$
$$D\Delta\Delta w + T_1 k_1 + T_2 k_2 + T_1^1\varkappa_1 + T_2^1\varkappa_2 = 0. \tag{57.4}$$

If with the buckling the shell surface separates into a large number of parts, both in the direction of the meridians and along the circumference of the parallels, then, considering the local stability of the shell in the sense indicated in § 56, we may neglect the quantities ψ and w in comparison with their derivatives, and also consider the smoothly varying geometrical parameters B, k_1, and k_2 as constant quantities. Thus, taking into account (24.13) and (25.24) and passing over to the variables θ and β by means of (57.1), we shall obtain the approximate equations:

$$\Delta\Delta\psi - \frac{Et}{R_1R_2}\left(\frac{1}{R_1}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{R_2\sin^2\theta}\frac{\partial^2 w}{\partial\beta^2}\right) = 0, \tag{57.5}$$
$$\Delta\psi = \frac{1}{R_1^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{R_2^2\sin^2\theta}\frac{\partial^2\psi}{\partial\beta^2},$$
$$D\Delta\Delta w + \frac{1}{R_1R_2^2\sin^2\theta}\frac{\partial^2\psi}{\partial\beta^2} + \frac{1}{R_1^2R_2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{T_1^1}{R_1^2}\frac{\partial^2 w}{\partial\theta^2} -$$
$$-\frac{T_2^1}{R_2^2\sin^2\theta}\frac{\partial^2 w}{\partial\beta^2} = 0. \tag{57.6}$$

Here θ is considered as a constant quantity, equal to the value at which the critical stress will turn out to be minimal. Setting

$$\psi = A\sin m(\theta-\theta_0)\cos n\beta, \quad w = C\sin m(\theta-\theta_0)\cos n\beta, \tag{57.7}$$

we satisfy the equations (57.5) and (57.6), if

$$A(m_1^2+n_1^2)^2 = -CEt(k_2m_1^2+k_1n_1^2), \tag{57.8}$$
$$D(m_1^2+n_1^2)^2 + T_1^1m_1^2 + T_2^1n_1^2 +$$
$$+Et(k_2m_1^2+k_1n_1^2)^2 : (m_1^2+n_1^2)^2 = 0, \tag{57.9}$$

where we have set

$$m_1 = m/R_1, \quad n_1 = n/(R_2\sin\theta). \tag{57.10}$$

If the shell has a vertex lying on the axis of symmetry, then from equations (57. 2) and (25. 6) we obtain

$$T_1^1 = -\frac{1}{2}pR_2, \quad T_2^1 = T_1^1(2-\delta), \quad \delta = R_2/R_1. \tag{57.11}$$

We shall first consider the special case of a spherical shell, under the action of an external normal pressure p = const. Then

$$R_1' = R_2 = R, \quad T_1^1 = T_2^1 = -pR/2,$$
$$-T_1^1 = D(m_1^2 + n_1^2) + Et/[R^2(m_1^2 + n_1^2)], \tag{57.12}$$
$$T_{1k}^1 = -2\sqrt{\frac{EtD}{R^2}} \approx -0.6\frac{Et^2}{R} \text{ for } \nu = 0.3.$$

The last formula was obtained by Zoelly /XIII. 17/ and independently by L. S. Leibenzon /XIII. 1/ by an exact integration of the equations of neutral equilibrium and the subsequent neglecting of quantities of the order of t/R in comparison with unity. It should be noted that in the approximate investigation of the question, one had admitted the neglecting of quantities of the order of $\sqrt{t/R}$ in comparison with unity, and despite that the error in the value of the critical stress turned out to be a quantity of the order of t/R in comparison with unity, as in the given problem the expression for the critical stress has the form

$$-T_1^1 = \sqrt{DEt/R^2}(1 + \lambda + 1/1 + \lambda),$$

where λ is the tolerable error of the order of $\sqrt{t/R}$. But

$$1 + \lambda + 1/1 + \lambda = 1 + \lambda + 1 - \lambda + \lambda^2 - \cdots \approx 2 + \lambda^2,$$

i.e., equating the expression in brackets to its minimal value, equal to 2, we tolerate, in effect, an error of the order of t/R.

In the general case, when the equations (57.11) are valid, the equation (57.9) may be written in the form

$$p = \frac{Et^2}{R_2^2\sqrt{3(1-\nu^2)}}\left(\lambda_1 + \frac{1}{\lambda_1}\right)\lambda_2 = p'\lambda_2, \tag{57.13}$$

where

$$\lambda_1 = \sqrt{\frac{DR_2^2}{Et}}\,\frac{(m_1^2 + n_1^2)^2}{m_1^2 + \delta n_1^2}, \quad \lambda_2 = \frac{1+\delta}{\mu + 2 - \delta}, \quad \mu = \frac{m_1^2}{n_1^2}.$$

The values of λ_1 and μ, satisfying the equations $\partial p/\partial\lambda_1 = 0$, $\partial p/\partial\mu = 0$, correspond to the critical pressure. The first of these equations leads to $\lambda_1 = 1$. Further

$$\frac{\partial p}{\partial\mu} = p'\frac{\partial\lambda_2}{\partial\mu} = \frac{2p'(1-\delta)}{(\mu + 2 - \delta)^2}. \tag{a}$$

Thus, the minimum of p with respect to μ is reached only with $\delta = 1$, for $\lambda_2 = 1$. Such is the case, for example, for an ellipsoidal shell under an external normal pressure p, the shell being formed by the rotation of an ellipse with semi-axes a and b about the axis 2b. The principal radii of curvature of such an ellipsoid are

$$R_1 = \frac{a^2b^2}{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}, \quad R_2 = \frac{a^2}{(a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}}. \tag{57.14}$$

Consequently, for it

$$\delta = 1 + (a^2 - b^2)\frac{\sin^2\theta}{b^2}, \quad T_1^1 = -\frac{a^2}{2b}\frac{p}{\sqrt{3}}. \tag{57.15}$$

With $\theta = 0$ and $\theta = \pi$ we have $\delta = 1$, $R_1 = R_2 = R = a^2/b$. It is obvious that the neighborhoods of these so-called spherical points of shells of revolution are the most unstable if the shell is oblate ($a > b$). Here formula (57.13) is transformed into the formula (57.12) for a spherical shell of radius $R = a^2/b$, i.e.,

$$p_k \approx 1.2\,Et^2b^2/a^4 \text{ for } \nu = 0.3. \qquad (57.16)$$

If the shell is elongated along the axis, i.e., b>a, then $\delta<1$ everywhere with the exception of the poles. Then, according to (a) the external pressure increases monotonically with the increase in μ, and therefore for the critical value of μ one should take the smallest of its admissable values. Assuming that $\mu<<1$, $m_1^2<<n_1^2$, we find $\lambda_2\approx\delta/(2-\delta)$. Here the quantity λ_2/R_2^2 has the smallest value at the point $\theta = \pi/2$. Hence it follows that an ellipsoid of revolution, elongated along the axis, loses its stability in the equatorial zone where, according to formula (57.13), the critical external pressure is

$$p_k = \frac{2\,Et^2}{\sqrt{3(1-\nu^2)}}\;\frac{\delta}{(2-\delta)\,R_2^2}\bigg|_{\theta=\pi/2} \approx \frac{1.2\,Et^2}{2\,b^2-a^2}. \qquad (57.17)$$

If a shell in the form of an oblate ellipsoid of revolution is under the action of an inside pressure (p<0), then, as can be seen from (57.13), the meridional stresses in it are tensile ($T_1^1>0$) but the annular stresses are compressive ($T_2^1<0$) for $\delta>2$ or, according to (57.15), $(a^2-b^2)\sin^2\theta>b^2$. The quantity δ reaches it maximum at $\theta = \pi/2$, i.e., at the equator. Here the meridional tensile stress has the smallest value and the annular compressive stress has the greatest of all possible absolute values. Consequently, if in the case under consideration the phenomenon of loss of stability is possible, then the first buckles should form along the equator. Thus, let

$$\theta=\pi/2,\ \delta=a^2/b^2,\ R_2=a. \qquad (57.18)$$

As can be seen from (57.13), for p<0 we ought to have $\lambda_2<0$. At the same time, $\delta>1$ and according to formula (a) $\partial p/\partial\mu<0$, i.e., as μ increases the negative quantity p should decrease, and therefore with the increase of μ it increases in absolute value. Then to the critical pressure corresponds the smallest of the possible values of μ: $\mu^2<<1$. Thus, we have again obtained formula (57.17), which in the case of inside pressure is applicable for a>b. This formula had been obtained by us in another way in /VI.1/. It differs considerably from the erroneous formula of I.V. Gekkeler /XIII.18/, in the derivation of which the buckling zone had been considered as a compressed ring upon an elastic base, where the coefficient of flexibility of the base had been determined incorrectly.

We shall further consider a truncated shell of revolution, bounded by the edge sections

$$\theta=\theta_0 \text{ and } \theta=\pi-\theta_0, \qquad (57.19)$$

which is in equilibrium under the action of contour stresses p_1 only. In that case, using the relation (25.6), or

$$R_1\cos\theta=\frac{d}{d\theta}(R_2\sin\theta), \qquad (57.20)$$

we find expressions for the stresses before the loss of stability from equations (57.2), for p = 0

$$T_1^1=p_1\,R_2^0\sin^2\theta_0/R_2\sin^2\theta;\quad T_2^1=-p_1R_2^0\sin^2\theta_0/R_1\sin^2\theta. \qquad (57.20a)$$

Introducing them in equation (57.9), we obtain

$$p_1\,\frac{R_2^0\sin^2\theta_0}{\sin^2\theta}=\sqrt{DEt}\left(\lambda_1+\frac{1}{\lambda_1}\right)\lambda_2, \qquad (57.21)$$

where λ_1 has the same value as in formula (57.13),

$$\lambda_2 = \frac{\mu + \delta}{\delta - \mu}, \quad \frac{\partial \lambda_2}{\partial \mu} = \frac{\delta + 1\beta}{(1\delta - \mu)^2}.$$

If the shell has a positive Gaussian curvature, then $\delta > 0$, $\partial\lambda_2/\partial\mu > 0$ and at the critical compressive stress $\mu^2 >> 1, n_1 = 0, \lambda_2 = -1$. Besides, the stability loss starts at the edges, where $\sin^2\theta$ has minimum value.

Thus, in that case

$$p_{1k} = -\frac{2\sqrt{DEt}}{R_2^0}, \quad R_2^0 = R_2(\theta_0). \tag{57.22}$$

If the Gaussian curvature of the shell is negative, then $\delta < 0, \partial\lambda_2/\partial\mu < 0$. But as can be seen from the expression for the quantity λ_1, with $0 \leqslant \mu < |\delta|$ we initially have $\lambda_1 < 0$; therefore, $p_1 < 0$, if $\lambda_2 > 0$, where with the increase in μ the quantity λ_2, and therefore also $|p_1|$ at $\lambda_1 = 1$, decreases. With $\mu = -\delta$ we have $\lambda_2 = 0$ and $\lambda_1 = \infty$, i.e., one obtains an indeterminate result, which, however, is easily determined if one returns to the initial expression for p_1. Thus, for $\mu = m^2 R_2^2 \sin^2\theta / n^2 R_1^2 = -\delta$ we have

$$p_1 \frac{R_2^0 \sin^2\theta_0}{\sin^2\theta} = DR_2 n_1^2 \frac{(1+\mu)^2}{\delta - \mu} = \frac{Dn^2(1-\delta)^2}{2R_2 \sin^2\theta\delta}. \tag{57.23}$$

The smallest absolute stress is reached with $n = 2$. In particular, for a shell described along a catenoid, $\delta = R_2/R_1 = -1$ and with $\theta = \theta_0$ we have

$$p_{1k} = -2Et^2 : [R_2^0 \sin^2\theta_0 \sqrt{3(1-\nu^2)}]. \tag{57.24}$$

The absolute value of this quantity turns out to be R_2^0/t times the critical compressive stress for a shell of positive Gaussian curvature, determined according to (57.22). There according to (57.8) we have $A = 0$ for $\mu = -\delta$, i.e., the buckling of the shell occurs as a pure bending. Such a deformation, as is known from the theory of surfaces, can occur only in the case when the edge contours are entirely free. If, then, at the edges, the conditions

$$w = 0, \ T_1 = 0, \ M_1 = 0 \text{ for } \theta = \theta_0 \text{ and } \theta = \pi - \theta_0, \tag{57.25}$$

ought to be satisfied, then in the expressions (57.7) it is necessary to set

$$m = i\pi/\pi - 2\theta_0, \ i - \text{integer}. \tag{57.26}$$

Since at the minimal load without the fulfillment of the boundary conditions, $n = 2$, $\mu = m^2 R_2^2 \sin^2\theta / n^2 R_1^2 = -\delta$, i.e., in the case of the catenoid $R_2^2 = R_1^2$, $\delta = -1$, $m^2 \sin^2\theta = 4$, then also if the conditions (57.26) are satisfied, one has to take for m (in order not to depart too much from the absolute minimum of $|p_1|$) the smallest of the possible values determined from (57.24) setting $i = 1$. Thus,

$$m = \pi/\pi - 2\theta_0, \ n = 2, \ \delta = -1. \tag{57.27}$$

But when m and n are quantities of the order of unity, the theory of local stability is inapplicable, as the derivatives of ψ and w are quantities of the same order as the functions themselves, and therefore formula (57.24) can lead to quite an erroneous result. For example, as is shown by the thorough analysis carried out by N.A. Alumyae/XIII. 19/, the actual value of the critical compressive stress at the equator of a long catenoid is

$$|T_1^1| \approx 2.63\, Et\,(t/R)^{4/3}, \quad \nu = 1/3, \tag{57.28}$$

where R is the radius of the equator. However, the investigation carried out above on the stability of shells with negative Gaussian curvature shows, nevertheless, that the latter are less stable than shells with positive Gaussian curvature.

§.58. The Axially Symmetric Deformation of a Shallow Shell of Revolution under Large Deflections

In equations (25.6), (25.9), (25.11)-(25.13), and (25.33), we drop the index "I" and denote by r the distance of a point on the middle surface from the axis of revolution, and by δ—just as before—the arc distance measured from the pole along the meridian. We thus obtain

$$B = r, \frac{d}{d\alpha}(r k_2) = k_1 \frac{dr}{d\alpha}, \quad \varkappa_1 = -\frac{d^2 w}{d\alpha^2}, \quad r\varkappa_2 = -\frac{dw}{d\alpha}\frac{dr}{d\alpha}, \tag{58.1}$$

$$\frac{d}{d\alpha}(r T_1) = T_2 \frac{dr}{d\alpha}, \quad T_1 = \frac{1}{r}\frac{dr}{d\alpha}\frac{d\psi}{d\alpha}, \quad T_2 = \frac{d^2\psi}{d\alpha^2}, \tag{58.2}$$

$$\frac{1}{r}\frac{d}{d\alpha}\left\{r\frac{d}{d\alpha}(T_1 + T_2)\right\} - Et\left(k_2\frac{d^2w}{d\alpha^2} + \frac{k_1}{r}\frac{dw}{d\alpha}\frac{dr}{d\alpha} - \frac{1}{r}\frac{d^2w}{d\alpha^2}\frac{dw}{d\alpha}\frac{dr}{d\alpha}\right) = 0, \tag{58.3}$$

$$\frac{D}{r}\frac{d}{d\alpha}\left\{r\frac{d}{d\alpha}\left[\frac{1}{r}\frac{d}{d\alpha}\left(r\frac{dw}{d\alpha}\right)\right]\right\} + T_1\left(k_1 - \frac{d^2w}{d\alpha^2}\right) + T_2\left(k_2 - \frac{1}{r}\frac{dr}{d\alpha}\frac{dw}{d\alpha}\right) + p = 0. \tag{58.4}$$

Let us consider a shallow part of the shell in the neighborhood of the pole and let the solid angle $2\theta_1$ subtended by that part of the shell be small. Then, taking into account that $d\alpha = R_1 d\theta$, $r = R_2 \sin\theta$, and that according to (58.1)

$$d(r k_2) = k_2 dr, \quad \cos\theta\, d\theta = k_1\, dr,$$

we find

$$dr = R_1 \cdot \cos\theta\, d\theta = \cos\theta\, d\alpha \approx d\alpha.$$

Consequently, in the preceding equations the derivatives with respect to α may be replaced by derivatives with respect to r. Then

$$dr/d\alpha \approx 1$$

and using (58.1) and (58.2) the equations (58.3) and (58.4) may be integrated once. Thus, we obtain the equations

$$r\frac{d}{dr}(T_1 + T_2) = Et\left\{rk_2\frac{dw}{dr} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + C\right\}, \tag{58.5}$$

$$Dr\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] + T_1 r^3 k_2 - T_1\frac{dw}{dr} + \frac{pr^2}{2} + C' = 0. \tag{58.6}$$

As at the pole $r = 0$, $dw/dr = -r\varkappa_2 = 0$, $d/dr(T_1 + T_2) \neq \infty$, and hence $C = 0$. In a similar way we convince ourselves that $C' = 0$.

In what follows, we shall confine ourselves to the investigation of the equilibrium of a shallow spherical segment under the action of a uniformly distributed external pressure p. Let the radius of the base annulus of the segment be a.

We introduce the notations

$$\rho = \frac{r}{a}, \quad \frac{dw}{dr} = \theta, \quad \frac{d\psi}{dr} = -aEt\,\Phi, \quad \lambda = \frac{12(1-\nu^2)a^2}{t^2}. \tag{58.7}$$

Then from (58.5) and (58.6) we obtain, using (58.2), the equations*

$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho\Phi)\right] = \frac{1}{2}\theta^2 - \rho\gamma\theta, \quad \gamma = \frac{a}{R}, \tag{58.8}$$

$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho\theta)\right] + \lambda\Phi(\theta - \rho\gamma) + \frac{p\rho^2a^3}{2D} = 0. \tag{58.9}$$

Further, from (25.10), (58.1), and (58.2), we obtain expressions for the radial bending moment, radial stress, and peripheral elongation:

$$M_1 = -\frac{D}{a}\left(\frac{d\theta}{d\rho} + \nu\frac{\theta}{\rho}\right), \quad T_1 = -\frac{Et\Phi}{\rho}, \quad \varepsilon_2 = \nu\frac{\Phi}{\rho} - \frac{d\Phi}{d\rho}. \tag{58.10}$$

We shall consider the generalized boundary conditions, assuming that the shell has elastically flexible fastenings at the contour.

We shall take the bending moment M_1 along the contour as proportional to the angle of rotation λ_1 of the contour of the shell

$$M_1 = \lambda_1\theta_1 = \lambda_1\frac{dw}{dr} \text{ for } \rho = r/a = 1, \tag{58.11}$$

where λ_1 is the constant of proportionality.

Hence, using the expression for M_1 from (58.10), we obtain the first boundary condition θ_1:

$$\left(\frac{d\theta}{d\rho}\right)_{\rho=1} + m\theta_1 = 0, \quad m = \nu + \frac{1}{C_1}, \quad C_1 = \frac{D}{a\lambda_1}. \tag{58.12}$$

Here C_1 is the characteristic of flexibility to rotation of the support fastening. With a hinged fastening $C_1 = \infty$, with the absence of rotation on the contour $C_1 = 0$.

We shall consider the stress T_1 at the contour to be proportional to the displacement u at the contour, i.e.,

$$T_1 = \lambda_2 u \text{ for } \rho = 1, \tag{58.13}$$

where λ_2 is the constant of proportionality.

In the case of axially symmetrical deformation the annular elongation ε_2 is determined by the formula

$$\varepsilon_2 = \frac{u}{r} + \frac{w}{R}.$$

At the contour, let the condition

$$w = 0 \text{ for } \rho = 1. \tag{58.14}$$

be satisfied. Then

$$u = a\varepsilon_2 \text{ for } \rho = 1.$$

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* See /XIII.23/, /XIII.6, and /0.19/.

Consequently, using the expression for ε_2 from (58.10), we bring the boundary condition (58.13) into the form*

$$\left(\frac{d\Phi}{d\rho}\right)_{\rho=1}+n(\Phi)_{\rho=1}=0,\ n=-\nu-C_2;\ C_2=\frac{Et}{\lambda_2 a}. \qquad (58.15)$$

The quantity C_2 above is called the characteristic of flexibility to displacement of the support fastening.

If the support contour does not hinder the displacements u, then $C_2 = \infty$; if at the contour $u = 0$, then $C_2 = 0$.

Apart from the boundary conditions (58.12), (58.14), and (58.15), owing to the axial symmetry we have

$$\vartheta=0 \text{ for } \rho=0 \qquad (58.16)$$

One can satisfy all the boundary conditions relating to deflection by setting in the first approximation

$$\vartheta=\frac{\omega a}{R}(\rho^3-\nu_1\rho),\quad \nu_1=\frac{3+m}{1+m}, \qquad (58.17)$$

where ω is a constant yet to be determined.

Introducing this expression into the equation (58.8) and integrating twice, we obtain Φ, where the constants of integration are determined from condition (58.15) and from the condition of the boundedness of the solution at the point $\rho = 0$. After substituting the Φ so obtained and the expression (58.17) in equation (58.9), we integrate it by the Galerkin method. For this we multiply the left-hand member of the equation by $\vartheta\, d\rho$ and equate to zero the integral between the limits 0 and 1 of the expression obtained. Thus we obtain a cubic equation in ω:

$$s_1\omega^3+s_2\omega^2+s_3\omega=s_4 p, \qquad (58.18)$$

where

$$\begin{gathered}
s_1=-\frac{1}{96}\left[\frac{1}{14}-\frac{\nu_1}{2}+\frac{3}{2}\nu_1^2-2\nu_1^3+\nu_1^4-\frac{(n\nu_2+\nu_3)\nu_4}{4(1+n)}\right],\\
s_2=\frac{1}{96}\left[\frac{5}{12}-\frac{5}{2}\nu_1+\frac{19}{4}\nu_1^2-3\nu_1^3-\frac{(n\nu_5+\nu_6)\nu_4+(n\nu_2+\nu_3)\nu_7}{1+n}\right],\\
s_3=-\frac{1}{24}\left[\frac{4}{1-\nu^2}\,\frac{\nu_7}{g^2}+\frac{1}{10}-\frac{\nu_1}{2}+\frac{\nu_1^2}{2}-\frac{(n\nu_5+\nu_6)\nu_7}{1+n}\right],\\
s_4=\frac{1}{4}\,\frac{R^2}{Et^3}\nu_7,\ g=\frac{a^2}{2Rt},\ \nu_7=\frac{1}{6}-\frac{\nu_1}{4},\\
\nu_6=5-9\nu_1,\ \nu_5=1-3\nu_1,\ \nu_3=7-20\nu_1+18\nu_1^2,\\
\nu_4=\frac{1}{2}-\frac{4}{3}\nu_1+\nu_1^2,\ \nu_2=1-4\nu_1+6\nu_1^2.
\end{gathered} \qquad (58.19)$$

From (58.7) we have

$$\vartheta=\frac{dw}{a\,d\rho},\qquad w=a\int_0^\rho \vartheta\, d\rho+\text{const}.$$

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* In /VI.7/ this condition is written in an erroneous form, and therefore in the expression for n the sign obtained for C_2 is incorrect.

At the shell contour, with $\rho = 1$, the deflection is zero; consequently

$$\text{const} = -a \int_0^1 \vartheta d\rho, \quad w = a \int_1^\rho \vartheta d\rho.$$

The relative deflection δ of the center of the segment is given by the formula

$$\delta = \frac{|w_0|}{t} = \frac{a}{t} \left| \int_1^0 \vartheta d\rho \right| = g \left| \left(\frac{1}{2} - \nu_1 \right) \omega \right|. \tag{58.20}$$

The extremum condition for the pressure has the form

$$\frac{dp}{d\vartheta} = \frac{dp}{d\omega} \cdot \frac{d\omega}{d\vartheta} = 0, \quad \frac{dp}{d\omega} = 0. \tag{58.21}$$

Substituting in this equation the expression for p in terms of ω from the equation (58.18), we obtain the values of ω corresponding to the extremal values of the load

$$\omega_1 = \frac{-s_2 - \sqrt{s_2^2 - 3s_1 s_3}}{3s_1}, \quad \omega_2 = \frac{-s_2 + \sqrt{s_2^2 - 3s_1 s_3}}{3s_1}. \tag{58.22}$$

As

$$\frac{d^2p}{d\omega^2} = 6\omega \frac{s_1}{s_4} + 2\frac{s_2}{s_4} = -2\left[\left(\frac{s_2}{s_4} \right)^2 - 3\frac{s_1 s_3}{s_4^2} \right]^{1/2} < 0 \text{ for } \omega = \omega_1,$$

Consequently the value $\omega = \omega_1$ corresponds to the maximal value of pressure p_K^{max}.

Therefore,

$$p_K^{max} = \frac{s_1}{s_4} \omega_1^3 + \frac{s_2}{s_4} \omega_1^2 + \frac{s_3}{s_4} \omega_1, \quad p_K^{max} = \frac{s_1}{s_4} \omega_2^3 + \frac{s_2}{s_4} \omega_2^2 + \frac{s_3}{s_4} \omega_2. \tag{58.23}$$

The dependence of p on ω is shown in Figure 39.

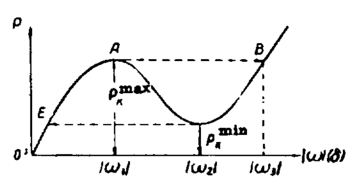

Figure 39

As can be seen from the graph, after reaching the pressure p_K^{max}, the further increase in the quantity $|\omega|$ from $|\omega_1|$ to $|\omega_2|$ occurs discontinuously (by a bang), i.e., there is an instantaneous loss of stability. If, after this, one reduces the pressure, then the deflection is gradually reduced and the value of $|\omega|$ decreases from $|\omega_3|$ to $|\omega_2|$. When the pressure becomes p_K^{min}, the bent form of equilibrium becomes unstable for $\omega = \omega_2$ and collapse occurs (from the position D to the position E).

Following the terminology proposed in the article of G. A. Geniev and N. S. Chausov /VI. 7/, we shall call the buckling (twisting) of the shell at p_K^{max} the loss of stability of the second kind.

But it can occur that already for $p<p_K^{max}$ the shell loses stability of its axially symmetrical state and passes into a moment state, which is usually not axially symmetrical. This phenomenon is called the stability loss of the first kind. The corresponding critical pressure is given by the formula

$$p_K = \frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{R^2},$$

in the derivation of which it is assumed that the radius of the shell does not change under loading. In practice, in the case of a shallow segment even before the loss of stability of the axially symmetrical state of equilibrium there is a considerable increase in the radius of curvature of the shell, in particular near the pole, due to which at the instant of buckling the radius of curvature of the shell at the pole is equal to some quantity $R_K>R$, and the preceding formula should be expressed in the form

$$p_K = \frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{R_K^2}, \tag{58.24}$$

where R_K is the radius of curvature of the deformed shell at $p = p_K$.

Attention has been drawn to this fact in /VI. 7/, where it is proposed to determine R_K from the formula

$$R_K = \frac{a^2+H_K^2}{2H_K}, \quad H_K = H - |w_{0K}|.$$

Here H is the initial altitude of the segment and (W_{OK}) is the deflection of the pole under critical load. It is obvious that in this way we find an average value of the radius of curvature of the deformed surface, whereas the radius of curvature in the neighborhood of the pole is greater than this quantity.

According to the theory of local stability, the stability loss of the first kind in a complete shell should start precisely in the region of the pole, and therefore, by substituting in (58. 24) the indicated mean value of the radius of curvature instead of R_K, we shall obtain a larger value of p_K. Taking into account that with the stability loss of the first kind small waves are formed—the length of each of which constitutes only a part of the quantity 2a—and that near the pole the curvature changes slowly*, we propose to take as R_K the radius of curvature of the deformed shell with $\rho = 0$. Here, for the type of deformation considered, we shall obtain a somewhat smaller value of p_K.

From (58. 1) and (58. 17) we obtain

$$\varkappa_1 = -\frac{d^2w}{dr^2} = -\frac{1}{a}\frac{d\vartheta}{d\rho} = -\frac{\omega}{R}(3\rho^2 - \nu_1), \quad \varkappa_{1(\rho=0)} = \nu_1\frac{\omega}{R}.$$

Consequently,

$$\frac{1}{R_K} = \frac{1}{R}(1+\nu_1\omega), \quad p_K = \frac{2Et^2}{\sqrt{3(1-\nu^2)}R^2}(1+\nu_1\omega)^2. \tag{58.25}$$

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* For example, with hinged fastening $\nu_1 \approx 2.54$, if $\nu = 0.3$, and in the expression for $\varkappa_1$ the term $3\rho^2$ is considerably smaller than the constant part, even for $\rho = 1/2$.

Equating the value of p_K to the value of p given by (58. 18), we obtain an equation analogous to that used in /VI. 7/:

$$\frac{s_1}{s_4}\omega^3+\frac{s_2}{s_4}\omega^2+\frac{s_3}{s_4}\omega=\frac{2Et^2}{\sqrt{3(1-\nu^2)}\,R^2}(1+\nu_1\omega)^2. \qquad (58.26)$$

Calculations show that one can neglect the first term of the left-hand member of this equation. Thus, to determine the value of $\omega = \omega_K$ (and therefore also that of the deflection w_{OK} for which loss of stability of the first kind occurs) we have a quadratic equation. Substituting for ω_K in (58. 25), we obtain p_K. If it turns out that the values of ω_K are complex, then stability loss will be of the second kind.

The computations carried out show that the critical maximum pressure determined from (58. 25) or (58. 24), is half or one-third the critical pressure found from the formula of Zoelly and Leibenzon. Initial irregularities in the shell shape also exert a very strong influence on the critical pressure in the case of a stability loss of the first kind.

We shall consider here the approximate determination of the critical pressure for a shallow spherical segment on the assumption that the segment has a symmetrical initial dent with respect to the pole, whose depth is f_0 at the pole, and the radius of whose circumference is a_0. When such a dent exists, the radius of curvature of the segment near the pole will be greater than away from it; therefore, a local stability loss of that region is possible, provided the diameter of the dent is not smaller than the wavelength of the anticipated buckling which is nearly symmetrical with respect to the pole, and is of an infinitesimal amplitude.

In equation (57. 9) we set $n^2 < m^2$, $k_2 = k_1 = 1/R_K$, where R_K is the radius of curvature of the dent region at the instant of buckling starts, and obtain

$$-T_1^1 = \frac{Dm^2}{R_\kappa^2}+\frac{Et}{m^2}.$$

From the condition $\partial T_1^I/\partial m = 0$ we find

$$m_\kappa^2=\sqrt{R_\kappa^2\,12(1-\nu^2)/t^2}.$$

But the length of the expected wave, as is apparent from (57. 7), is

$$2\pi R_\kappa/m_\kappa = 2\pi\sqrt{R_\kappa t}:\sqrt[4]{12(1-\nu^2)}.$$

Consequently,

$$a_0=\pi\sqrt{R_\kappa t}/\sqrt[4]{12(1-\nu^2)}.$$

Let f be the altitude of the segment of radius a_0 without considering the dent, i. e.,

$$f=a_0^2/2R.$$

The actual altitude of that segment before applying the load is approximately

$$(a_0^2/2R)-f_0.$$

Consequently, its curvature may be obtained from the formula

$$\frac{1}{R_0} = 2\left(\frac{a_0^2}{2R} - f_0\right) : a_0^2 = \frac{1}{R} - \frac{2f_0}{a_0^2}.$$

At the instant of buckling, the curvature of the segment changes owing to the applied pressure (p_K) to the value

$$\varkappa_1(\rho = 0) = \omega\nu_1/R.$$

Thus, the curvature of the pole region at the instant of buckling will be

$$\frac{1}{R_\kappa} = \frac{1}{R} - \frac{2f_0}{a_0^2} + \frac{\omega\nu_1}{R} = \frac{1}{R} - \frac{2f_0}{\pi^2 R_\kappa t}\sqrt{12(1-\nu^2)} + \frac{\nu_1\omega}{R}.$$

Consequently,

$$\frac{1}{R_\kappa} \simeq \frac{1}{R}(1 + \nu_1\omega) : \left(1 + \frac{2f_0}{\pi^2 t}\sqrt{12(1-\nu^2)}\right);$$

$$p_\kappa = \frac{2}{\sqrt{3(1-\nu^2)}}\,\frac{Et^2}{R_\kappa^2}. \qquad (58.27)$$

Here as before we find the quantity $\omega = \omega_K$ from equation (58.26). The utilization of the last equation for the determination of the deflection due to the load for a segment with an initial dent should not lead to a considerable error, since for a shallow shell the deflection of the pole will be due not so much to the bending in the dent region as to the general bending of the entire shell with the given boundary conditions.

The solution given here was obtained in the first approximation. Making it more exact by approximating the quantity θ by polynomials of higher degrees than (58.17), we shall obtain smaller values of the upper critical pressure. The limiting value p_K differs, as some calculations show, from the first approximation by 10-20% if one limits oneself to axially symmetrical deformations. The solution of the non-linear, non-axially symmetrical problem is so far unknown to us, and therefore, at the present state of the theory it is risky to carry out stability calculations for shallow spherical shells without experimental verification. To our regret, in the literature there are no data on serious experimental investigations of shallow spherical domes. We did not manage, so far, to acquaint ourselves with the recently-published work of this kind /XIII. 13/. In work /XIII. 14/, known to us, the experiments were carried out on spherical domes of considerable depth, with base diameters of 2a = 40 cm, radii of curvature R = 25 cm and R = 52 cm, and thicknesses varying between the limits 400<R/t<2,000. For the critical pressure at which the dome loses stability the authors of that investigation obtained the empirical formula

$$p_\kappa^{э} = 0.3\alpha\frac{Et^2}{R^2}\quad\left(400 \leqslant \frac{R}{t} \leqslant 2000,\;\; 20^\circ \leqslant \theta^\circ \leqslant 60^\circ\right),$$

$$\alpha = \left(1 - 0.175\,\frac{\theta^\circ - 20^\circ}{20^\circ}\right)\left(1 - 0.07\,\frac{R}{400t}\right), \qquad (58.28)$$

where θ° is the rise angle in degrees at the edges, where the maximum discrepancy between the authors' experimental data and the results calculated according to that formula reached 20% (in those cases when deviations from the ideal spherical shape were noticeable to the naked eye).

At the boundary of the region of applicability of the empirical formula, $\theta = 20°$

We shall apply the above theory to the determination of the critical pressure in such a shell. Let us take the data:

$$a = 20\ \text{cm},\ R = 52\ \text{cm},\ \frac{R}{t} = 800,\ \ \nu = 0.3;\ E = 2.1 \cdot 10^6\ \text{kg/cm}^2.$$

It turns out that in this case symmetrical loss of stability of the second kind can occur only under very large pressures.

The critical curvature at the stability loss of the first kind is

$$\frac{1}{R_K} = 0.983 / R\left(1 + 0.67\frac{f_0}{t}\right).$$

Hence with $f_0 = 0$, from (58.24) we find $p_K = 1.16Et^2/R^2$ and with $f_0 = 1.5t$ we find $p_K = 0.28Et^2/R^2$.

At the same time, according to (58.28) $p_K = 0.258Et^2/R^2$. Thus, even with an initial bending equal to 1.5t, our theoretical solution gives a somewhat higher critical pressure than its experimental value. This discrepancy may be partially accounted for by deficiencies in the experiment, as in work /XIII. 14/ it turns out that in some cases the initial irregularities were clearly noticeable to the naked eye. But the main reason lies, apparently, in the fact that the shells tested were rather deep, whose stability loss cannot occur in the axially symmetrical form if there are no large axially symmetrical irregularities.

The testing of two series of shallower segments has been carried out by R. G. Surkin in the mechanics department of the Kazan' section of the USSR Academy of Sciences. The segments were portions of a sphere with the diameter of the base ring 2a = 200 mm; the mean values of the radius of curvature of the surface and of the thickness were: for the first series with seven shells tested, R = 358 mm and t = 0.454 mm; for the second series with four shells R = 495 mm and t = 0.460 mm. The samples of both series were prepared by hydraulic stretching of sheet brass ($E = 10^6\,\text{kg/cm}^2$).

In the preparation of the shells, circular symmetry was well preserved.

All the specimens tested lost stability abruptly under loading, i. e., by snapping. Then there occurred a complete inversion of the spherical segment with some additional stretching as a result of the high speed of the inversion.

As a result of the experiment, the mean value of the critical pressure for the first series turned out to be $p^э = 0.787$ atm, which is 2/5 times the value of the critical pressure according to the linear theory, and for the second series $p = 0.374$ atm, i. e., 5/14 times the value of the critical pressure according to the linear theory.

The calculated results and experimental data are given in Table XVIII for $\nu = 0.3$.

Table XVIII

No of the series	t in mm	R in mm	ω	f_0 in mm	R_K according to (58.27)	p_K according to (58.25) in atm	p_K according to (58.27) in atm	p_K in atm
I	0.454	358	−0.144	0.049	605	0.767	0.683	0.787
II	0.460	495	−0.139	0.057	828	0.438	0.374	0.374

A comparison between the theoretical values of the critical pressure taking into account the initial irregularities according to (28. 27), and the experimental values, shows that in the given case theory and experiment are in satisfactory agreement.

In conclusion, we draw the reader's attention to the article of Hu Hai-Chang /XIII. 25/ in which he considers the stability of a hinged shallow spherical segment whose polar region is loaded by a symmetrically distributed normal pressure, and also gives a detailed investigation of the stability of such a segment under the action of bending moments uniformly distributed over the contour, where in both cases the deformation is assumed to be axially symmetrical.

§ 59. Shallow Spherical Membrane under the Action of Internal Pressure

The exact solution of the problem of determining large deflections of an absolutely flexible spherical membrane, whose rigidity to bending may be neglected, was given in works /XIII. 8/ and /XIII. 15/.

$$\xi = r^2/a^2, \quad \omega = \frac{1}{2r}\frac{dw}{dr} = \frac{1}{a^2}\frac{dw}{d\xi}, \; k = \frac{1}{R}. \tag{59.1}$$

Equations (58. 5) and (58. 6), in which one should set C = C' = 0, take the form

$$\frac{1}{a^2}\frac{d}{d\xi}(T_1 + T_2) = Et(\omega k - \omega^2), \quad \omega = \frac{k}{2} - \frac{q}{4T_1}. \tag{59.2}$$

According to (58.2) and (59.1)

$$T_2 = \frac{d}{dr}(rT_1), \quad T_1 + T_2 = 2\frac{d}{d\xi}(\xi T_1).$$

Introducing this expression and eliminating ω from (59.2), we obtain

$$\frac{2}{a^2}\frac{d^2}{d\xi^2}(\xi T_1) = Et\left(\frac{k^2}{4} - \frac{q^2}{16T_1}\right).$$

Setting

$$k^* = E\frac{k^2a^2}{4}(Eq^2a^2/t^2)^{-1/3}, \quad T_1 = \frac{1}{4}(Eq^2a^2t)^{1/3}\sigma_1, \tag{59.3}$$

we bring the last equation into the form of the non-linear differential equation in σ_1:

$$\frac{d^2}{d\xi^2}(\xi\sigma_1) - 2k^* + \frac{2}{\sigma_1^2} = 0. \tag{59.4}$$

We shall seek the solution of this equation in the form of the power series

$$\sigma_1 = b_0 + b_1\xi + \ldots = \sum_{n=0}^{\infty} b_n\xi^n. \tag{59.5}$$

We shall assume that this series converges in the region $0 \leqslant \xi \leqslant 1$, and that its sum satisfies the necessary boundary conditions. Let

$$\chi = \xi\sigma_1 = b_0\xi + b_1\xi^2 + \ldots = \sum_{n=0}^{\infty} b_n\xi^{n+1}. \tag{59.6}$$

Here (59.4) may be written in the form of the equation

$$\frac{d^2\chi}{d\xi^2} = 2k^* - 2\xi^2/\chi^2.$$

Integrating that equation once, we find

$$\frac{d\chi}{d\xi} = 2k^*\xi - 2\int_0^{\xi} \frac{\xi^2}{\chi^2}\, d\xi + c. \tag{59.7}$$

Here $c = b_0$, so that according to (59. 6)

$$\frac{d\chi}{d\xi} = b_0 \quad \text{for} \quad \xi = 0.$$

Besides

$$\int_0^{\xi} \frac{\xi^2}{\chi^2}\, d\xi = \int_0^{\xi} \frac{1}{\sigma_1^2}\, d\xi = \int_0^{\xi} d\xi \Big/ \left[\sum_{n=0}^{\infty} b_n \xi^n\right]^2,$$

$$\frac{1}{\left[\sum\limits_{n=0}^{\infty} b_n \xi^n\right]^2} = \frac{1}{b_0^2}\left[\sum_{n=0}^{\infty} c_n \xi^n\right]^2 = \frac{1}{b_0^2}\sum_{n=0}^{\infty} d_n \xi^n, \tag{59. 8}$$

where

$$c_0 = 1, \quad d_0 = c_0^2, \quad c_i + \frac{1}{b_0}\sum_{n=1}^{i} c_{i-n} b_n = 0 \ \text{for} \ i \geqslant 1,$$

$$d_m = \frac{1}{m}\sum_{n=1}^{m} (3n - m)\, c_n d_{m-n} \ \text{for} \ m \geqslant 1. \tag{59. 9}$$

Introducing these expressions in (59. 7) and integrating, we obtain

$$\sigma_1 = \frac{\chi}{\xi} = b_0 + k^*\xi - \frac{2}{b_0^2}\sum_{n=0}^{\infty} \frac{d_n}{(n+1)(n+2)}\xi^{n+1}. \tag{59.10}$$

Equating the series (59.10) and (59.5) we obtain the relations

$$b_1 = k^* - \frac{d_0}{b_0^2}, \quad b_2 = -\frac{d_1}{3b_0^2}, \ldots, \ b_i = -\frac{2d_{i-1}}{b_0^2 i(i+1)}. \tag{59.11}$$

Further, from formulas (59.9) we calculate the coefficients

$$d_1 = -\frac{2b_0}{b_1}, \quad d_2 = \frac{3b_1^2 - 2b_0 b_2}{b_0^2},$$

$$d_3 = -\frac{2}{b_0^3}(2b_1^3 - 3b_0 b_1 b_2 + b_0^2 b_3). \tag{59.12}$$

Introducing these in (59. 11) we obtain

$$b_1 = -\frac{1-\lambda}{b_0^2}, \quad b_2 = -\frac{2}{3}\frac{(1-\lambda)}{b_0^3}, \quad b_3 = -\frac{(1-\lambda)(13-9\lambda)}{18b_0^4}\ldots, \tag{59.13}$$

where

$$\lambda = k^* b_0^3 \tag{59.14}$$

is proportional to the curvature of the spherical membrane. Thus, according to (59.5) and (59.13) T_1 is expressed in terms of b_0 and λ.

Here

$$T_2 = \frac{d}{dr}(rT_1) = T_1 + 2\xi\frac{dT_1}{d\xi} =$$
$$= \frac{1}{4}\sqrt[3]{Eq^2a^2t}\,[b_0 + 3b_1\xi + \ldots + (2i+1)b_i\xi^i + \ldots]. \qquad (59.15)$$

To determine the deflection function w, we make use of equation (59.2), which by use means of (59.1) and (59.3) is brought into the form

$$\frac{dw}{d\xi} = \frac{a^2k}{2} - a\sqrt[3]{\frac{qa}{Et}}\,\frac{1}{\sigma_1}.$$

Introducing here the expression (59.5), integrating with respect to ξ, and setting

$$a_0 = \frac{1}{b_0},\quad a_1 = -\frac{b_1}{2b_0^2} = \frac{1-\lambda}{2b_0^4},\quad a_2 = \frac{1}{3}\left(\frac{b_1^2}{b_0^3} - \frac{b_2}{b_0^2}\right),$$
$$a_3 = -\frac{1}{4}\left(\frac{b_1^3}{b_0^4} - 2\frac{b_1b_2}{b_0^3} + \frac{b_3}{b_0^2}\right), \qquad (59.16)$$

we obtain

$$w = \frac{a^2k}{2}\xi - a\xi\sqrt[3]{\frac{qa}{Et}}(a_0 + a_1\xi + a_2\xi^2 + \ldots) + w_0.$$

The constant of integration w_0 is equal to the deflection of the center of the membrane, i.e., $w = w_0$ for $\xi = 0$. On the other hand, let the membrane contour be clamped, i.e., $w = 0$ for $\xi = 1$.

Consequently,

$$w_0 = -\frac{a^2k}{2} + a\sqrt[3]{\frac{qa}{Et}}\sum_{n=0}^{\infty}a_n,$$
$$w = -\frac{a^2k}{2}(1-\xi) + a\sqrt[3]{\frac{qa}{Et}}\left(\sum_{n=0}^{\infty}a_n - \xi\sum_{n=0}^{\infty}a_n\xi^n\right). \qquad (59.17)$$

At $\xi = 0$ the series (59.10) converges to the value of b_0, where $b_0 > 0$ should hold, as T_1 is the tensile stress.

In the general case

$$\sigma_1 = b_0 + k^*\xi - \frac{2\xi}{b_0^2}\left[\sum_{i=0}^{n}\frac{d_i}{(i+1)(i+2)}\xi^i + R_n\right],$$

where

$$|R_n| = \left|\sum_{i=n+1}^{\infty}\frac{d_i\xi^i}{(i+1)(i+2)}\right| < \frac{1}{(n+2)(n+3)}\left|\sum_{i=n+1}^{\infty}d_i\xi^i\right|.$$

On the other hand

$$\frac{b_0^2}{\sigma_1^2} - \sum_{i=0}^{\infty}d_i\xi^i = \sum_{i=0}^{n}d_i\xi^i + \sum_{i=n+1}^{\infty}d_i\xi^i.$$

Consequently,

$$|R_n| < \left| \left(\frac{b_0^3}{a_1^2} - \sum_{i=0}^{n} d_i \xi^i \right) \right| : [(n+2)(n+3)]. \tag{59.18}$$

For $\xi = 1$, this gives an estimate of the residual term

$$|R_n(1)| < \left| \frac{b_0^3}{a_{11}^2} - \sum_{i=0}^{n} d_i \right| : [(n+2)(n+3)], \quad a_{11} = \sum_{i=0}^{n} b_i.$$

In what follows we shall carry out the calculations by limiting ourselves to four terms of the series (59.5) or (59.10). The estimate of the residual term of the series according to (59. 18) shows that the error thus admitted is not large if $\lambda < 1$. The remaining undetermined coefficient b_0 may be found from the condition that the annular elongation ε_2 be zero at the edge $\xi = 1$.

This gives the dependence

$$T_2 - \nu T_1 = 0 \text{ for } \xi = 1,$$

which after the substitution of the expressions for T_1 and T_2 from (59. 13) is reduced to the equation

$$\frac{(1-\nu)}{1-\lambda} b_0^9 - (3-\nu) b_0^6 - \frac{2}{3}(5-\nu) b_0^3 - \\ - \frac{(7-\nu)}{18}(13 - 9\lambda) = 0 \tag{59.19}$$

Some results calculated according to these formulas are given for $\nu = 0.3$ in Table XIX.

Table XIX

λ	b_0	$-b_1$	$-b_2$	$-b_3$	$\lvert R_3(1)\rvert <$	k^*
0	1.713	0.341	0.0452	0.0[illegible]97	0.0009	0
0.4	1.500	0.260	0.0493	0.0[illegible]10	0.0016	0.173
0.9	1.012	0.098	0.0628	0.0[illegible]47	0.0021	0.879

We determine the stresses T_{10}, T_{20} at the center and the stresses T_{1a}, T_{2a} at the edge from the formulas

$$T_{10} = T_{20} = \frac{b_0}{4} Q, \quad Q = (Eq^2 a^2 t)^{1/3},$$

$$T_{1a} = \frac{Q}{4} \sum_{i=0}^{3} b_i, \quad T_{2a} = \frac{Q}{4} \sum_{i=0}^{3} (2i+1) b_i,$$

$$w_0 = -\frac{a^2 k}{2} + a \left(\frac{qa}{Et} \right)^{1/3} \sum_{i=0}^{3} a_i \tag{59. 20}$$

The numerical results found from these formulas are summarized in Table XX.

Table XX

λ	T_{10}/Q	T_{1a}/Q	$\frac{w_0}{a}(Et/qa)^{1/3}$
0	0.428	0.328	0.662
0.2	0.409	0.321	0.952
0.4	0.375	0.292	1.157
0.6	0.340	0.267	1.386
0.8	0.291	0.233	1.706
0.9	0.253	0.206	2.006

From this table it follows that with increasing k* the stresses at the center and at the edge of the spherical membrane decrease.

In the special case when k* = 0, i.e., k = 0, we obtain the well-known solution of G. Genki for a flat membrane with a uniformly distributed load (see, for example, /XIII. 24/. Calculations carried out by S. A. Alekseev /XIII. 26/, have shown that the influence of the coefficient of transverse compression ν on the value of the deflection and stresses is quite considerable.

§ 60. Making the General Theory of Shell-Snap More Accurate

The theory of shallow shells which we have used from § 17 is based on the assumption that the condition $\theta_0^2 < 1$ is satisfied, where θ_0 is the rise angle of the part of the shell considered. The loss of stability of non-shallow shells is frequently accompanied by a dent, corresponding to a body angle $2\theta_0$. This can be considered as small only in the first approximation. For example, E. Zechler and V. Bolley of the California Institute of Technology, while testing hemispherical shells under the action of an external uniformly distributed pressure p have found that

$$\sigma_s = 0.154\frac{Et}{R}, \quad 2\theta_0 \approx 0.3; \quad \frac{|w_0|}{t} \approx 12.5, \tag{60. 1}$$

where $|w_0|$ is the dent depth and σ^s is the critical stress. Already in this case the application of the theory of shallow shells for the determination of elongations of the middle surface may lead, each time the stress functions, etc. are introduced to an error exceeding 2%, and therefore the total possible error of the solution will considerably exceed the error admissable in the thepry of small deformation of this shells which, as was shown above, is a quantity of the order of the relative elongation ε_p (within the limits of elasticity), in comparison with unity.

At the same time, there is a tendency to broaden the range of applicability of the above theory of shallow shells, allowing a shell to be called shallow if

$$\frac{H}{2a} \approx \frac{a}{4R} \leqslant \frac{1}{5} \text{ or } \theta_0 \leqslant \frac{4}{5}. \tag{60. 2}$$

In § 26 it had been noted that at the boundary of the region (60. 2) the admissible error in the theory of shallow shells can be very considerable. Here, we shall consider this question in great detail, fixing our attention on the possible loss of precision in the determination of elongations for a medium deflection, which was already pointed out in § 15.

We shall refer the middle surface of the shell to the lines of curvature. Let a line element of that surface be defined by the formula

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2, \tag{60. 3}$$

where A and B are quantities of the same order as the radii of curvature of the shell.

For small deformations with arbitrary displacements, the elongation and shear of the middle surface are given by (3. 5):

$$\varepsilon_1 = e_{11} + \frac{1}{2}(e_{11}^2 + e_{12}^2 + \omega_1^2), \quad \varepsilon_2 = e_{22} + \frac{1}{2}(e_{22}^2 + e_{21}^2 + \omega_2^2), \tag{60. 4}$$

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* See article /XIII. 7/.

$$\varepsilon_{12}=(1+e_{11})e_{21}+(1+e_{22})e_{12}+\omega_1\omega_2,$$
$$e_{11}=\frac{1}{A}\frac{\partial u}{\partial \theta}+\frac{v}{AB}\frac{\partial A}{\partial \beta}+\frac{w}{R_1},\quad e_{21}=\frac{1}{B}\frac{\partial u}{\partial \beta}-\frac{v}{AB}\frac{\partial B}{\partial \theta},$$
$$\omega_1=\frac{1}{A}\frac{\partial w}{\partial \theta}+\frac{u}{R_1}\qquad \overleftrightarrow{1,2} \tag{60.5}$$

Unfortunately, the expressions (60.4) contain squares of the normal as well as the tangential components of displacement. They may be somewhat simplified if the displacement components are comparable with the shell thickness, but are small in comparison with the other linear dimensions. Let the shell be thin, i.e.,

$$\frac{t}{R}\sim\varepsilon_p. \tag{60.6}$$

For ordinary metals

$$\varepsilon_p\sim 0.001-0.003.$$

As can be seen from (60.1)

$$\frac{|w_0|}{t}\sim\varepsilon_p^{-1/3},\quad \theta_0\sim\varepsilon_p^{1/3}. \tag{60.7}$$

Since in the snapping phenomenon weare dealing with displacements and stresses which attenuate rapidly from the center of the region toward its edges (the radius of the region being small), then according to (60. 7)

$$\frac{\partial|w|}{\partial\theta}\sim\frac{|w|}{\theta_0}\sim|w|\varepsilon_p^{-1/3},\quad \frac{|w|}{R}\lesssim\frac{|w_0|}{R}=\frac{|w_0|}{t}\cdot\frac{t}{R}\sim\varepsilon_p^{2/3},$$
$$\frac{1}{A^2}\left(\frac{\partial w}{\partial\theta}\right)^2\sim\frac{1}{R^2}\left(\frac{\partial w}{\partial\theta}\right)^2\sim\varepsilon_p^{2/3}.$$

As has been shown in § 15, the tangential displacement components are small in comparison with w. If, for example, $u\varepsilon w_p^{1/3}$ then

$$\frac{1}{A}\frac{\partial u}{\partial\theta}\sim\frac{1}{R}\frac{\partial u}{\partial\theta}\lesssim\frac{1}{R}\frac{\partial w}{\partial\theta}\varepsilon_p^{1/3}\sim\varepsilon_p^{2/3},\quad \frac{u}{A}\leq\varepsilon_p\quad \overleftrightarrow{1,2}. \tag{60.8}$$

Thus, ε_1, ε_2, ε_{12} may be differences of quantities of the order of $\varepsilon_p^{2/3}$ Therefore in determining the elongations there may be a loss pf precision if one is not careful in retaining quantities of higher orders of magnitude.

From (60.4) it follows that

$$e_{11}=-\frac{1}{2}\omega_1^2-\frac{1}{2}e_{12}^2+\varepsilon_1-\frac{1}{2}e_{11}^2;\ \ldots$$

Consequently, as

$$\varepsilon_1\sim\varepsilon_p,\quad e_{12}\sim e_{11}\sim\varepsilon_p^{2/3},\quad \omega_1\sim\varepsilon_p^{1/3}\qquad \overleftrightarrow{1,2},$$

then, neglecting ε_p in comparison with unity, we find

$$\varepsilon_1\approx e_{11}+\frac{1}{2}e_{12}^2+\frac{1}{2}\omega_1^2+\frac{1}{8}\omega_1^4,$$
$$\varepsilon_{12}\approx e_{12}\left(1-\frac{1}{2}\omega_1^2\right)+e_{21}\left(1-\frac{1}{2}\omega_2^2\right)+\omega_1\omega_2.\qquad \overleftrightarrow{1,2} \tag{60.9}$$

Further, according to (60.8) and (60.5) we have

$$e_{12}=\frac{1}{B}\frac{\partial v}{\partial\beta}+\frac{w}{R_2},\quad \frac{1}{B}\frac{\partial v}{\partial\beta}\approx-\frac{w}{R_2}+\varepsilon_2-\frac{1}{2}e_{21}^2-\frac{1}{2}\omega_1^2-\frac{1}{8}\omega_1^4.$$

Taking into account that

$$\frac{w}{R}\sim\varepsilon_p^{2/3},\quad \varepsilon_2\sim\varepsilon_p,\ \ldots,$$

in the first approximation we may take

$$\frac{\partial v}{\partial\beta}\approx-\frac{Bw}{R_2}\quad\text{or}\quad v=-\int_0^\beta\frac{Bw}{R_2}\,d\beta, \tag{60.10}$$

if v = 0 at β = 0. Consequently, with the assumed accuracy

$$\omega_2=-\frac{1}{B}\frac{\partial w}{\partial\beta}-\frac{1}{R_2}\int_0^\beta\frac{Bw\,d\beta}{R_2}\qquad \overleftrightarrow{1,2}. \tag{60.11}$$

Now we have

$$\omega_2^2\approx\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right)^2+\frac{2}{BR_2}\frac{\partial w}{\partial\beta}\int_0^\beta\frac{Bw}{R_2}\,d\beta,\quad \omega_2^4\approx\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right)^4,\ldots \tag{60.12}$$

Let $e_{11}^{I}, \ldots, \omega_2^{I}$ be the respective quantities for the state of the shell before snapping, ε_1^*, ε_2^*, ε_{12}^* the elongations in the buckled state, and u, v, w the additional displacements. Then

$$\varepsilon_1^*=e_{11}^{I}+e_{11}+\frac{1}{2}(e_{12}^{I}+e_{12})^2+\frac{1}{2}(\omega_1^{I}+\omega_1)^2+\frac{1}{8}(\omega_1^{I}+\omega_1)^4.$$

Assuming that the first equilibrium state may be considered as close to the membrane state we have

$$\varepsilon_1^*\approx\varepsilon_1^{I}+\varepsilon_1,\quad \varepsilon_{12}^*\approx\varepsilon_{12}^{I}+\varepsilon_{12}.$$

If, besides the principal equilibrium state, a stable equilibrium state under the same external load is possible after the snapping, then the total energy of snapping must also be minimal in the latter position. Thus, the problem is reduced to the minimization of the functional

$$\begin{aligned}\Im=\iint_{(\sigma)}\Big\{\frac{K}{2}\Big[(\varepsilon_1^{I}+\varepsilon_1)^2+(\varepsilon_2^{I}+\varepsilon_2)^2+2\nu(\varepsilon_1^{I}+\varepsilon_1)(\varepsilon_2^{I}+\varepsilon_2)+\\+\frac{(1-\nu)}{2}(\varepsilon_{12}^{I}+\varepsilon_{12})^2\Big]+\frac{D}{2}\left[\varkappa_1^2+\varkappa_2^2+2\nu\varkappa_1\varkappa_2+2(1-\nu)\varkappa_{12}^2\right]-\\-W\Big\}AB\,d\alpha\,d\beta\end{aligned}$$

$$K=Et/(1-\nu^2),\quad D=Et^3/12(1-\nu^2). \tag{60.13}$$

Here the integral is taken over the entire middle surface of the shell; W is the specific work of the external surface forces; the work of the contour external forces is considered to be zero.

We determine the changes in curvature from the formulas

$$\varkappa_1 = \frac{1}{AB}\frac{\partial A}{\partial \beta}\omega_2 + \frac{1}{A}\frac{\partial \omega_1}{\partial \alpha}, \quad \varkappa_{12} = \frac{1}{A}\frac{\partial \omega_2}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\omega_1 \qquad \overleftarrow{\overrightarrow{1, 2}} \tag{60.14}$$

The elongations are expressed linearly in terms of u and v with the aid of (60. 10), and therefore the latter enter in the expression for the total energy in a power not higher than the second. Due to this one obtains, in minimizing the energy, equations which are linear in the amplitude of the tangential displacement.

§ 61. The Lower Critical Pressure for a Complete Spherical Shell

The upper critical stress for an ideal spherical shell under uniform external pressure is given by the formula

$$\sigma_U = \frac{p_U R}{2t} = \frac{Et}{R\sqrt{3(1-\nu^2)}}. \qquad (61.1)$$

It is almost four times greater than the experimental value (60.1).

In putting forward their explanation of the discrepancy between experiment and the classical theory, Th. von Karman and Hsue Shen-Tsien made the assumption /XIII. 2/ that when pressures are considerably smaller than p_U, then besides the state of stable equilibrium of the shell, there is possible a stable equilibrium state which retains the spherical shape with the formation of a dent; also, a "jump" or "snap" into that buckled state is possible if the shape of the shell is imperfect or if the applied pressure is pulsating. However, these authors did not succeed in giving a satisfactory solution to the problem of determining the minimum pressure in a non-linear state. K. O. Friedrichs also had to admit, at the end of his work /XIII. 4/ devoted to that topic, that the question of the minimum pressure, called the lower critical point p_H, remains open. In the exposition of this question we shall follow our article /XIII. 7/.

We shall take the center of the snap region as the pole of the spherical shell and meridians and parallels for the coordinate lines. Then in the formulas of the preceding section

$$A = R,\ B = R\sin\theta,\ \varepsilon_1^{\mathrm{I}} = \varepsilon_2^{\mathrm{I}} = \varepsilon^{\mathrm{I}} = w^{\mathrm{I}}/R;\ \varepsilon_{12}^{\mathrm{I}} = 0.$$

Let the snapping be symmetrical with respect to the pole. Consequently,

$$v = 0,\ \ \varepsilon_{12} = \varkappa_{12} = e_{12} = e_{21} = 0,\ \ \varepsilon_{11} = \left(\frac{du}{d\theta} + w\right) : R;$$

$$\varepsilon_2 = e_{22} = \frac{1}{R}(u \operatorname{ctg}\theta + w),\ \ u = (R\varepsilon_2 - w)\operatorname{tg}\theta,\ \ \varkappa_1 \approx \frac{1}{R_2}\frac{d^2 w}{d\theta^2}, \qquad (61.2)$$

$$\omega_1 = \frac{1}{R}\left(-\frac{dw}{d\theta} + u\right) \approx -\frac{1}{R}\left(\frac{dw}{d\theta} + w\theta\right),$$

$$\varkappa_2 = \frac{\omega_1 \operatorname{ctg}\theta}{R} \approx -\frac{1}{R^2\theta}\frac{dw}{d\theta}.$$

No loss of precision can occur in the determination of the work of the external forces because, limiting ourselves to the first approximation and setting $\sin\theta \approx \theta$, $\cos\theta \approx 1$, we have

$$W = -p(w + w^{\mathrm{I}}) = \frac{2T^{\mathrm{I}}}{R}(w + w^{\mathrm{I}}) = \frac{2K(1+\nu)\varepsilon^{\mathrm{I}}}{R}(w + \varepsilon^{\mathrm{I}} R).$$

For quantities depending only on uniform compression, θ varies between 0 and π.

Quantities characterizing snapping can be different from zero only in the region $0 \leqslant \theta \leqslant \theta_0$, and therefore according to (60.13)

$$\frac{\mathcal{E}}{2\pi}=-\int_0^{\pi} K(1+\nu)\,\varepsilon^{12} R^2 \sin\theta\, d\theta+\int_0^{\theta_0}\Big\{K\Big[\frac{1}{2}(\varepsilon_1^2+\varepsilon_2^2+2\nu\varepsilon_1\varepsilon_2)+$$
$$+(1+\nu)\,\varepsilon^1(\varepsilon_1+\varepsilon_2)-2(1+\nu)\,\varepsilon^1\frac{w}{R}\Big]+\frac{D}{2R^4}\Big[\Big(\frac{d^2w}{d\theta^2}\Big)^2+\frac{1}{\theta^2}\Big(\frac{dw}{d\theta}\Big)^2+$$
$$+\frac{2\nu}{\theta}\,\frac{d^2w}{d\theta^2}\,\frac{dw}{d\theta}\Big]\Big\}R^2\theta\, d\theta.$$

We assume that

$$u=0,\quad w=0,\quad \frac{dw}{d\theta}=0 \text{ for } \theta=\theta_0. \tag{61.3}$$

This is equivalent to the assumption of a rigid annulus at the boundary of the expected snapping region. According to the symmetry conditions we have

$$u=0,\quad \frac{dw}{d\theta}=0 \text{ for } \theta=0. \tag{61.4}$$

Besides, one may set

$$\varepsilon^1\left(\frac{1}{2}\omega_1^2+\frac{1}{8}\omega_1^4\right)=\frac{1}{2}\varepsilon^1\omega_1^2.$$

Consequently,

$$\frac{\mathcal{E}}{2\pi}=-2K(1+\nu)R^2\varepsilon^{12}+\int_0^{\theta_0}\Big\{\frac{K}{2}(\varepsilon_1^2+\varepsilon_2^2+2\nu\varepsilon_1\varepsilon_2)+$$
$$+K(1+\nu)\frac{\varepsilon^1}{2R^2}\Big(\frac{dw}{d\theta}\Big)^2+\frac{D}{2R^4}\Big[\Big(\frac{d^2w}{d\theta^2}\Big)^2+\frac{1}{\theta^2}\Big(\frac{dw}{d\theta}\Big)^2\Big]\Big\}R^2\theta\, d\theta. \tag{61.5}$$

The total energy in the first form of equilibrium is

$$\mathcal{E}^1=-4\pi K(1+\nu)R^2\varepsilon^{12}.$$

The equilibrium state is stable if the energy functional $\mathcal{E}$ in that state has a minimal value. The equilibrium state before the snap (zero state) is stable if $p< p_U$. As was shown in /XIII. 2/, other stable states of equilibrium are also possible if the pressure exceeds some minimal value p_H, where $p_H< p_U$. It is obvious that with $p>p_H$ there are at least three possible equilibrium states: a stable "zero state", a stable non-linear state, and an unstable state in which the graph of the quantity $\mathcal{E}$ as a function of the depth of the dent has a saddle-point (Figure 40).

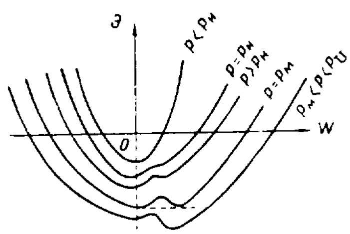

Figure 40

At $p = p_H$ the stable and unstable states of equilibrium merge into one state, in which $\mathcal{E}$ has a parabolic point, i.e., the first and the second variations of $\mathcal{E}$ are zero. The total energy in that state is greater than in the zero state. Consequently, if p is only a little larger than p_H the energy of the non-linear state is greater than in the state before the snap.

At some value of p greater than p_H, the total energy of the non-linear state becomes equal to the energy of the zero state. We shall denote this value of the pressure by p_M. In the case under consideration, p_M differs but little from p_H. The computation of p_M is simpler, however, than that of p_H, and therefore in what follows we shall frequently limit ourselves to the determination of p_M. Let us also note that, as shown in Figure 40, in order to pass from the zero state to the stable non-linear state, the shell must overcome an energy barrier which tends to zero as the pressure p approaches the value p_U. But this barrier may also be surmounted for $p_H \leqslant p \leqslant p_U$ if there exist initial irregularities unfavorable to stability, pressure pulsations, etc. In the case of a complete shell or a steeply sloping segment, the surmounting of the energy barrier may also be facilitated by a redistribution of the energy between the dent region and the remainder of the shell, and, in the case of a shallow segment, by a redistribution of the energy between the shell and the support, which is always elastic in practice. It is also possible that so far our theory has not taken into account the most accessible non-linear forms of buckling (for example, non-symmetrical forms), in which the barrier is small and can be surmounted for very small irregularities of the shell which always exist in practice. In any case, we cannot consider the mechanism of the "snap" phenomenon as entirely established, although many attempts have been made in that direction*. The determination of the upper critical load, taking into consideration all the real conditions of the problem, is, as before, the main problem of the theory of shell stability. However, the investigation of the non-linear equilibrium states—in particular, the determination of p_H--is also necessary, as it extends our knowledge of the loading capacity of shells.

According to the method outlined for solving the problem, we set up the expression $\mathcal{E}' = \mathcal{E} - \mathcal{E}^1$, retaining, together with the principal terms, also the terms of the order of $(dw/d\theta)^6$. Introducing the notations

$$\zeta = \theta^2/\theta_0^2, \quad \theta_0^2 = b \tag{61.6}$$

and taking expressions for u and v satisfying the conditions (61.3) and (61.4), namely, setting

$$u = c\theta_0 \sqrt{\zeta} R h(\zeta), \quad w = aRg(\zeta), \quad h(1) = g(1) = \frac{dg}{d\zeta}(\zeta = 1) = 0, \tag{61.7}$$

we set up, according to the Ritz method, equations expressing the condition that the first variation of the total energy is zero:

$$\frac{\partial \mathcal{E}'}{\partial a} = 0, \quad \frac{\partial \mathcal{E}'}{\partial b} = 0, \quad \frac{\partial \mathcal{E}'}{\partial c} = 0. \tag{61.8}$$

Besides, to determine p_M one has to fulfill the condition:

$$\mathcal{E} = \mathcal{E}^1, \text{ i.e.}, \mathcal{E}' = 0, \tag{61.9}$$

and at $p = p_H$ the second variation of the total energy should be zero, i.e., the additional condition

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* See, for example, articles /XIII. 3/, /XIII. 5/, /XIII. 9/, /XIII. 14/.

$$\begin{vmatrix} \partial^2\mathcal{E}/\partial a^2, & \partial^2\mathcal{E}/\partial a\partial b, & \partial^2\mathcal{E}/\partial a\partial c \\ \partial^2\mathcal{E}/\partial a\partial b, & \partial^2\mathcal{E}/\partial b^2, & \partial^2\mathcal{E}/\partial b\partial c \\ \partial^2\mathcal{E}/\partial a\partial c, & \partial^2\mathcal{E}/\partial b\partial c, & \partial^2\mathcal{E}/\partial c^2 \end{vmatrix} = 0. \tag{61.10}$$

should be satisfied. For a series of functions h and g the corresponding values of p_M had been determined to the first approximation, i.e., by the usual theory of shallow shells, and to the second approximation by the more accurate theory of § 60. It turned out that the second approximation gives a value of p_M smaller by 5-7.5% than the first. Here, it should not be forgotten that in the case under consideration, according to (60.1) $\theta_0 \approx 0.15$, i.e., is less than one-fifth the boundary of the region (60.2). In our opinion, the application of the ordinary theory of shallow shells should be limited at most to the region

$$\frac{H}{2a} \approx \frac{a}{4R} \leqslant 0.1. \tag{61.11}$$

From the considered functions h and g, the smallest value of p_M is obtained from the functions

$$h = (1-\zeta)(1-1.2\zeta),\ g = (1-\zeta)^2(1+0.5\zeta). \tag{61.12}$$

for which in the first approximation

$$|\sigma_M| = \frac{p_M R}{2t} = 0.22\frac{Et}{R\sqrt{1-\nu^2}}. \tag{61.13}$$

We also found the value

$$|\sigma_H| = \frac{p_H R}{2t} \approx 0.193\frac{Et}{R\sqrt{1-\nu^2}}. \tag{61.14}$$

This result had been obtained on the assumption of rigid fixing of the edge of the snap region. In reality, however, the snap region interacts elastically with the remaining portion of the shell, and therefore we shall give another determination of the quantity p_H, where we shall confine ourselves to the solution of the problem in the first approximation*.

From (60.9) and (61.2), setting $\cos\theta \approx 1$, $\sin\theta \approx \theta$, we find:

$$\varepsilon_1 = e_{11} + \frac{1}{2}\omega_1^2 = \frac{1}{R}\left(\frac{du}{d\theta} + w\right) + \frac{1}{2R^2}\left(\frac{dw}{d\theta}\right)^2,\quad \varepsilon_2 = \frac{1}{R}\left(\frac{u}{\theta} + w\right).$$

Eliminating u, we obtain:

$$\varepsilon_1 = \frac{d}{d\theta}(\varepsilon_2\theta) + f,\quad f = -\frac{1}{R}\frac{dw}{d\theta} + \frac{1}{2R^2}\left(\frac{dw}{d\theta}\right)^2. \tag{61.15}$$

Whence, and from the equilibrium equation

$$\frac{d}{d\theta}(T_1\theta) = T_2,$$

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* For a more comprehensive exposition of the contents of this section see article /XIII.7/.

where T_1 and T_2 are the additional membrane stresses, we find the relation

$$\theta \frac{d}{d\theta}(T_1 + T_2) = -Etf, \quad T_1 + T_2 = -Et\int^{\theta} \frac{f d\theta}{\theta} + C.$$

Let

$$T_1 + T_2 = 0 \text{ for } \theta = \theta_0. \tag{61.16}$$

Then

$$T_1 + T_2 = Et\int_{\theta}^{\theta_0} \frac{f d\theta}{\theta}.$$

Further

$$\int_0^{\theta_0} T_1 T_2 \theta d\theta = \int_0^{\theta_0} T_1 \theta \frac{d}{d\theta}(T_1\theta)\, d\theta = \frac{1}{2}(T_1\theta)^2_{\theta=\theta_0} = 0,$$

if at the boundary of the snap region the condition

$$T_1 = 0 \text{ for } \theta = \theta_0. \tag{61.17}$$

is satisfied. Therefore

$$\frac{K}{2}\int_0^{\theta_0} (\varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2)\,\theta d\theta =$$

$$= \frac{1}{2Et}\int_0^{\theta_0} [(T_1 + T_2)^2 - 2T_1T_2(1+\nu)]\,\theta d\theta = \frac{Et}{2}\int_0^{\theta_0}\left\{\int_{\theta}^{\theta_0}\frac{f d\theta}{\theta}\right\}^2 \theta d\theta.$$

Let

$$w = aRg(\zeta), \quad \theta^2 = b. \tag{61.18}$$

Introducing these quantities in (61.5), we obtain after rather lengthy calculations

$$\mathcal{E}_1 = \frac{\mathcal{E}'}{2\pi R^2 K} = C_{11}a^2 b + 2C_{12}a^3 + C_{22}\frac{a^4}{b} + \frac{C_1 a^2 \tau_1^2}{b} - eC_2 a^2, \tag{61.19}$$

where

$$\tau_1 = \frac{t}{R\sqrt{12(1-\nu^2)}}, \quad e = \frac{|T_1^0|}{Et}, \quad C_1 = 16\int_0^1 \zeta^2 \frac{d^2 g}{d\zeta^2}\, d\zeta,$$

$$C_2 = 4\int_0^1 \zeta\left(\frac{dg}{d\zeta}\right)^2 d\zeta, \quad C_{11} = \int_0^1 g^2 d\zeta, \quad C_{12} = \int_0^1 g\left\{\int_{\zeta}^1\left(\frac{dg}{d\zeta}\right)^2 d\zeta\right\} d\zeta,$$

$$C_{22} = \int_0^1\left\{\int_{\zeta}^1\left(\frac{dg}{d\zeta}\right)^2 d\zeta\right\}^2 d\zeta. \tag{61.20}$$

Let us introduce the new notations

$$\lambda = \frac{a}{b}\sqrt{\frac{2C_{12}}{C_{11}}}, \quad \mu = \frac{\tau_1}{b}\sqrt{\frac{C_1}{C_{11}}}, \quad \theta_1 = \sqrt{\frac{2C_{12}}{C_{11}}},$$

$$\theta_2 = \frac{C_{22}}{2C_{12}}, \quad e^* = \frac{e}{\tau_1}\,\frac{C_2}{\sqrt{C_1 C_{11}}}. \tag{61.21}$$

Then the energy functional is brought into the form

$$Э^* = \frac{2Э_1 C_{11}}{C_1^{3/2} C_{11}^{1/2} \tau_1^3} = \frac{1}{\mu^3}(\lambda^2 + \theta_1\lambda^3 + \theta_2\lambda^4) + \frac{\lambda^2}{\mu} - \frac{\lambda^2}{\mu^2} e^*. \qquad (61.22)$$

We determine the value of p_H from the equations

$$\frac{\partial Э^*}{\partial \lambda} = 0, \quad \frac{\partial Э^*}{\partial \mu} = 0, \qquad (61.23)$$

$$\frac{\partial^2 Э^*}{\partial \lambda^2} \cdot \frac{\partial^2 Э^*}{\partial \mu^2} - \left(\frac{\partial^2 Э^*}{\partial \lambda \partial \mu}\right)^2 = 0, \qquad (61.24)$$

from which, after some minor transformations, we obtain

$$\mu^2 = 1 - \theta_2\lambda^2, \quad -\lambda^3 + \frac{4}{\theta_2}\lambda + \frac{3}{2}\frac{\theta_1}{\theta_2^2} = 0,$$

$$e^* = \frac{1}{\mu}\left(2 + \frac{3}{2}\theta_1\lambda + \theta_2\lambda^2\right). \qquad (61.25)$$

For the numerical determination of p_H we shall assign to the snap region the damping function

$$g = e^{-n\zeta}(1 - k\zeta). \qquad (61.26)$$

This choice is explained by the fact that in practice the influence of the snap extends over the entire shell and the parts of the snap region are conditional in the sense that with $\theta \geqslant \theta_0$ or, what amounts to the same, with $\zeta \geqslant 1$, the buckling becomes small and experimenters do not distinguish it. We shall assume that in the expression (61.26) the number taken for n was such that for $\zeta \geqslant 1$ the deflection becomes negligibly small. Then, in the given case the conditions (61.16) and (61.17) are approximately satisfied and in formulas (61.20) one may take infinity instead of unity as the upper limits of the integrals.

On calculation we find $k \approx 0.7n$,

$$\sigma_H = \frac{p_H R}{2t} \approx \frac{0.16}{\sqrt{1-\nu^2}}\frac{Et}{R}, \quad \theta_0 \approx 8° \text{ for } n = 4,$$
$$\theta_0 \approx 9° \text{ for } n = 5. \qquad (61.27)$$

In /XIII. 8/ and /XIII. 16/, R. G. Surkin has considered the generalization of the theory given here to the case of an ellipsoidal shell, elongated along the axis. Obviously, the stability loss of such a shell should begin in the equatorial region, where the radii of curvature R_1 and R_2 have the greatest values. In consideration of this fact, the author assumes that a local stability loss occurs in the equatorial zone, and an elongated dent with elliptic base is formed whose greater diameter is oriented parallel to the axis of revolution of the shell. Let $\alpha = \frac{\pi}{2} - \theta$ and β be the Gaussian coordinates of the middle surface of the shell. Then the dent region may be defined by the inequalities:

$$-\alpha_0 \leqslant \alpha \leqslant \alpha_0, \quad -\beta_0 \leqslant \beta \leqslant \beta_0, \qquad (61.28)$$

where $\alpha_0 = x_0/R_1$, $\beta_0 = y_0/R_2$, if $2x_0$ and $2y_0$ are the linear dimensions of the snap region in the direction of the meridian and the equator of the shell. With the introduction of the independent variables $\xi = \alpha^2/\alpha_0^2$, $\eta = \beta^2/\beta_0^2$, the contour of the snap region is determined by the ellipse $\xi + \eta = 1$.

As from geometrical considerations

$$\delta = R_2/R_1 = \alpha_0^2/\beta_0^2, \tag{61.29}$$

the dent region may be determined if one knows one of the quantities α_0 and β_0.

We shall approximate the displacement components by functions of the form

$$u = \rho_1 \lambda \alpha_0^2 R_1 h(\xi, \eta), \quad v = \rho_2 \lambda \alpha_0^2 \beta_0 R_1 j(\xi, \eta), \quad w = \lambda \alpha_0^2 R_1 g(\xi, \eta), \tag{61.30}$$

where ρ_1, ρ_2, λ are the parameters required, and the functions h, j, and g should satisfy the boundary conditions:

$$\begin{gathered} h(\xi, \eta) = j(\xi, \eta) = 0 \text{ for } \xi = \eta = 0, \quad \xi + \eta = 1, \\ g(\xi, \eta) = 1 \text{ for } \xi = \eta = 0, \quad g(\xi, \eta) = 0 \text{ for } \xi + \eta = 1. \end{gathered} \tag{61.31}$$

The smallest value of the lower critical pressure p_H was obtained with the following functions, characterizing the snap shape:

$$\begin{aligned} h(\xi, \eta) &= e^{-n(\xi+\eta)}[1 - k(\xi + \eta) - k_3(\xi^2 + \eta^2)], \\ j(\xi, \eta) &= e^{-n(\xi+\eta)}[1 - k_1(\xi + \eta) - k_4(\xi^2 + \eta^2)], \\ g(\xi, \eta) &= e^{-n(\xi+\eta)}[1 - k_2(\xi + \eta)]. \end{aligned} \tag{61.32}$$

Here the boundary conditions (61.31) are approximately satisfied, if for n one takes such a number for which the deflection at the boundary of the region of buckling becomes negligibly small.

The parameters $\rho_1, \rho_2, \lambda, \alpha_0$, and the value of p_H corresponding to them, were found from the condition of minimum total energy of the system, where k, $k_1, \ldots, k_4$ were taken as equal to the numbers found by trial for a spherical shell.

Results of calculations carried out for various values of δ are summarized in Table XXI.

Table XXI

δ	1	0.5	0.333
$p_H R_2^2/Et^2$	0.414	0.284	0.222

Comparison of the data in this table with the values of the upper critical pressure calculated by formula (57.17) shows that with a decrease in δ, the ratio indicated increases, and with $\delta = 0.333$ it becomes almost equal to unity. This shows that with the increase of the shell elongation the influence of the non-linear factor decreases.

In conclusion, let us note that the solution given here for an elongated ellipsoidal shell should be considered only as a first approximation, whose error increases with the increase in eccentricity of the generating ellipse.

§ 62. Some Remarks on the Method of Solving the Problem of § 61

In solving the problem, von Karman and Tsien /XIII. 2/ tolerated, as was also noted by Friedrichs /XIII. 4/, two substantial errors. First, they arbitrarily assumed that $\varepsilon_2 = 0$. Second, the problem of determining the snap form and the pressures under which the functional of the total energy has the minimal value was replaced by them by the problem of determining the smallest values of the pressure without concerning themselves with the minimization of the energy (the Lagrange-Dirichlet principle notwithstanding). According to formulas (61. 22) and (61. 20) this function has the form

$$\mathcal{J}^* = f_1(\lambda, \mu) - p f_2(\lambda, \mu).$$

According to the method used by von Karman and Tsien, we find consecutively

$$\frac{\partial \mathcal{J}^*}{\partial \lambda} = \frac{\partial f_1}{\partial \lambda} - p\frac{\partial f_2}{\partial \lambda} = 0, \quad p = \frac{\partial f_1}{\partial \lambda} : \frac{\partial f_2}{\partial \lambda}, \tag{62. 1}$$

$$\frac{\partial p}{\partial \lambda} = \left(\frac{\partial^2 f_1}{\partial \lambda^2} \cdot \frac{\partial f_2}{\partial \lambda} - \frac{\partial f_1}{\partial \lambda}\, \frac{\partial^2 f_2}{\partial \lambda^2}\right) : \left(\frac{\partial f_2}{\partial \lambda}\right)^2 = 0, \tag{62. 2}$$

$$\frac{\partial p}{\partial \mu} = \left(\frac{\partial^2 f_1}{\partial \lambda \partial \mu} \cdot \frac{\partial f_2}{\partial \lambda} - \frac{\partial f_1}{\partial \lambda}\, \frac{\partial^2 f_2}{\partial \lambda \partial \mu}\right) : \left(\frac{\partial f_2}{\partial \lambda}\right)^2 = 0. \tag{62. 3}$$

The equations obtained for the determination of λ, μ and p_H differ considerably from the equations (61. 25), with the exception of equation (62. 1).

Generally speaking, both procedures lead to identical results only in the case of small displacements, as then the terms of order higher than the second in the deflection amplitude drop out of the energy expression.

In that case we have

$$\mathcal{J}^* = \lambda^2 [F_1(\mu) - pF_2(\mu)]$$

and, following von Karman, we find

$$\frac{\partial \mathcal{J}^*}{\partial \lambda} = 2\lambda (F_1 - pF_2) = 0, \qquad p = \frac{F_1}{F_2}, \quad \frac{\partial p}{\partial \lambda} = 0,$$

$$\frac{\partial p}{\partial \mu} = 0 \quad \text{or} \quad \frac{\partial F_1}{\partial \mu} - p\frac{\partial F_2}{\partial \mu} = 0.$$

This latter coincides with the equation $\partial \mathcal{J}^*/\partial \mu = 0$. Besides, (61. 24) is satisfied, and also the equation $\mathcal{J}^* = 0$. Thus it turns out that in linearizing the problem, one may with equal justification make use of the equations (61. 23) or (62. 1)-(62. 3), where the difference between p_U, p_H, and p_M vanishes.

Turning to work /XIII. 2/, it should be noted that as a result of the superposition of the two errors indicated above, a "solution" was incidentally obtained, close to ours (61. 27). However, retaining the assumption $\varepsilon_2 = 0$ and the snap shape $g(\zeta) = (1 - \zeta)^2$, assumed in that work, and applying the correct procedure for the minimization of the energy, one obtains:

$$\sigma_H = \frac{p_H R}{2t} = \frac{0.47}{\sqrt{1 - \nu^2}}\, \frac{Et}{R}.$$

An error of another kind was tolerated by Friedrichs /XIII.4/: in deriving formula (7) of his work, he relied, in fact, on the condition (61.16), without concerning himself with the actual fulfillment of that condition, and therefore the value obtained by him for the lower critical pressure $\sigma_H = 0.13\, Et/R\sqrt{1-\nu^2}$ is unfounded. Note that in determining p_H one may limit oneself to equations (61.23), and replace (61.24) by an equation obtained by minimization with respect to λ or μ of the expression for p, obtained from any one of the equations (61.23). We propose that the reader convince himself of the correctness of that assertion.

In article /XIII.10/ it had been proposed to consider the whole spherical shell after the snap as consisting of a dent with a circular contour and of the remaining part of the shell, where it was assumed that within the dent region the deformations are attenuated according to the law of the edge effect. The joining of both portions of the shell had been provided for in such a way that the displacement, angle of rotation and bending moment vary continuously on passing across their boundary, and there is a discontinuity in the shearing stress which is taken into account in smoothing out the imbalances of the approximate solution by the Galerkin method. Unfortunately, in his solution the author did not consider it necessary to vary the dent radius in addition to the deflection amplitude, and therefore he obtained a negative value for the quantity p_H. We arrived, by correctly solving the problem in the same setting in /XIII. 11/, at the value

$$\sigma_H = \frac{p_H R}{2t} = 0.10\, \frac{Et}{R\sqrt{1-\nu^2}}. \tag{62.4}$$

Apparently, the difference between that formula and formula (61.27) is explained by the fact that in deriving the former, one had not ensured, as indicated above, the smooth variation of the shearing stress.

In work /XIII. 12/ has been considered the determination of p_H by integrating the equilibrium equation by the Galerkin method, the Galerkin equation being set up in the form (25.23), as it is obtained from the principle of virtual displacements. Here the equation of the components normal to the surface of the shell was multiplied during the integration, not by the deflection function as is usual, but by a variation of that function taking into account the variation of the deflection amplitude and the snap region.

The deflection function was approximated by the expression (61.26). The value of p_H thus obtained differs from (61.27) by less than 5 %.

We also call the reader's attention to article /XIII.27/, in which it is demonstrated—proceeding from general considerations—that in determining the critical load by integrating the equilibrium equation by the Galerkin method, it is necessary, in order to obtain a solution with an error of the second order of magnitude in comparison with the error tolerated in the choice of the approximating function, to multiply the left-hand member of that equation precisely by the variation of the approximating function.

Chapter XIV

A METHOD FOR SOLVING BOUNDARY VALUE PROBLEMS FOR NON-LINEAR EQUATIONS IN THE THEORY OF SHALLOW SHELLS

In the preceding chapters we considered approximate variational methods of solving non-linear problems in the theory of shells. The method set forth below for solving non linear equations of a cylindrical shell, based on the application of the method of integration in series form, is really a general method for solving non-linear boundary value problems in the theory of shallow shells. The method is illustrated by an example of a cylindrical strip, rectangular in the plane.

§ 63. Large Deflections of a Rectangular Cylindrical Strip, Rigidly Fastened at All Edges

1°. We shall consider a cylindrical strip, rectangular in the plane, subjected to the action of a uniform external pressure and rigidly fixed at all the edges. Let R be the radius of the shell, 2a the length of the strip along the generators, and 2b its width. We shall take the origin of coordinates at the center of the strip, the ox-axis along the generators, and the oy axis along the directrix. Since the generating and directing cylindrical surfaces are geodesic lines, $\kappa = 0$, and consequently, the fundamental relations of §24 are considerably simplified. Satisfying all the geometrical boundary conditions, $u = v = w = \partial w/\partial x = 0$ for $x = \pm a$; $u = v = w = \partial w/\partial y = 0$ for $y = \pm b$ we find from (24.36) and (24.53b) the following boundary conditions for ψ and M_{ik}:

at the edges x = ±a:

$$h_1(\psi) = \frac{\partial^2\psi}{\partial x^2} - \nu\frac{\partial^2\psi}{\partial y^2} = 0, \quad g_1(\psi) = \frac{\partial^3\psi}{\partial x^3} + (2+\nu)\frac{\partial^3\psi}{\partial x \partial y^2} = 0, \tag{63.1}$$

$$M_{22} - \nu M_{11} = 0, \qquad M_{12} = 0; \tag{63.1a}$$

at the edges y = ±b:

$$h_2(\psi) = \frac{\partial^2\psi}{\partial y^2} - \nu\frac{\partial^2\psi}{\partial x^2} = 0, \quad g_2(\psi) = \frac{\partial^3\psi}{\partial y^3} + (2+\nu)\frac{\partial^3\psi}{\partial y \partial x^2} = 0, \tag{63.2}$$

$$M_{11} - \nu M_{22} = 0, \qquad M_{12} = 0. \tag{63.2a}$$

The boundary conditions $h_1(\psi) = 0$, $h_2(\psi) = 0$ mean that the respective edges are inextensible.

The functions

$$M_{22} - \nu M_{11} = \sum_{m,n} f_{mn}(-1)^n \frac{\varphi_m(x)\cos\beta_n y}{\alpha_m^2},$$

$$M_{11} - \nu M_{22} = \sum_{m,n} f_{mn}(-1)^m \frac{\psi_n(y)\cos\alpha_m x}{\beta_n^2},$$

$$M_{12}(1+\nu) = \sum_{m,n} f_{mn}(-1)^{m+n} \frac{\sin\alpha_m x \sin\beta_n y}{\alpha_m\beta_n},$$

where

$$\varphi_m(x) = 1-(-1)^m\cos\alpha_m x,\quad \varphi_n(y) = 1-(-1)^n\cos\beta_n y,$$
$$m,\ n = 1,\ 2,\ 3,\ \dots;\ \alpha_m = \frac{n\pi}{a},\quad \beta_n = \frac{n\pi}{b} \tag{63.4}$$

satisfy the boundary conditions for the moments and the Codazzi compatibility conditions expressed by the moments:

$$\frac{\partial(M_{22}-\nu M_{11})}{\partial x} - (1+\nu)\frac{\partial M_{12}}{\partial y} = 0,$$
$$\frac{\partial(M_{11}-\nu M_{22})}{\partial y} - (1+\nu)\frac{\partial M_{12}}{\partial x} = 0. \tag{63.5}$$

If one denotes the deflection parameters by w_{mn}, then

$$w_{mn} = D' f_{mn}(\alpha_m\beta_n)^2,\quad D' = 12/Et^3.$$

The functions

$$M_{11}^0 = \frac{p}{4}(x^2-a^2),\quad M_{22}^0 = \frac{p}{4}(y^2-b^2),\quad M_{12}^0 = 0 \tag{63.6}$$

satisfy the third equilibrium equation (15.9) without taking into account the tangential stresses. Therefore the general solutions of that equation will be

$$M_{11} = M_{11}^0 + \frac{\partial\psi_2}{\partial y} + \left(\frac{1}{R} + \hat{\varkappa}_{22}\right)\psi;\quad M_{22} = M_{22}^0 + \frac{\partial\psi_1}{\partial x} + \hat{\varkappa}_{11}\psi,$$
$$-2M_{12} = \frac{\partial\psi_1}{\partial y} + \frac{\partial\psi_2}{\partial x} + 2\hat{\varkappa}_{12}\psi, \tag{63.7}$$

where ψ_1 and ψ_2 are arbitrary functions, and $\hat{\varkappa}_{ik}$ is expressed in terms of M_{ik} by the elasticity relations (24.19a). Introducing (63.3) into (24.19a), we find for $\hat{\varkappa}_{ik}$ the expressions

$$\hat{\varkappa}_{11} = D'\sum_{m,n} f_{mn}(-1)^m\frac{\varphi_n(y)\cos\alpha_m x}{\beta_n^2},$$
$$\hat{\varkappa}_{22} = D'\sum_{m,n} f_{mn}(-1)^n\frac{\varphi_m(x)\cos\beta_n y}{\alpha_m^2},$$
$$\hat{\varkappa}_{12} = D'\sum_{m,n} f_{mn}(-1)^{m+n}\frac{\sin\alpha_m x\sin\beta_n y}{\alpha_m\beta_n}. \tag{63.8}$$

Let us find the stress function ψ.

The function

$$\psi = \frac{1}{2}(p_2x^2 + p_1y^2) + \sum_{m=1}^{\infty}[A_m L_m(x)\cos\beta_m y +$$
$$+ B_m M_m(y)\cos\alpha_m x] + \sum_{m,n=0,1,2} C_{mn}\cos\alpha_m x\cos\beta_n y,\quad C_{00} = 0, \tag{63.9}$$

where

$$L_m(x) = \left(\frac{1-\nu}{1+\nu}\cdot\frac{1}{\beta_m} - a\operatorname{cth} a\beta_m\right)\operatorname{ch}\beta_m x + x\operatorname{sh}\beta_m x,$$
$$M_m(y) = \left(\frac{1-\nu}{1+\nu}\cdot\frac{1}{\alpha_m} - b\operatorname{cth} b\alpha_m\right)\operatorname{ch}\alpha_m y + y\operatorname{sh}\alpha_m y \tag{63.10}$$

exactly satisfies the boundary conditions $g_1(\psi)=g_2(\psi)=0$ related to the angle of rotation $\partial w/\partial n$.

Substituting $h_1(\psi)=h_2(\psi)=0$ in the boundary conditions, and expanding in a Fourier cosine series, we obtain a system of equations for determining the unknown constants p_1, p_2, A_k, B_k:

$$p_2-\nu p_1=\sum_{m=1}^{\infty} C_{m0}(-1)^m\alpha_m^2;\quad p_1-\nu p_2=\sum_{n=1}^{\infty} C_{0n}(-1)^n\beta_n^2; \tag{63.11}$$

$$f_kB_k-\frac{4(1+\nu)}{a}\sum_{m=1}^{\infty}\frac{A_m(-1)^{m+k}\operatorname{sh} a\beta_m}{\left(\frac{\beta_m}{\alpha_k}+\frac{\alpha_k}{\beta_m}\right)^2}=\sum_{n=0}^{\infty} C_{kn}(-1)^n(\beta_n^2-\nu\alpha_k^2);$$

$$f_k'A_k-\frac{4(1+\nu)}{b}\sum_{m=1}^{\infty} B_m\frac{(-1)^{m+k}\operatorname{sh} b\alpha_m}{\left(\frac{\alpha_m}{\beta_k}+\frac{\beta_k}{\alpha_m}\right)^2}=\sum_{m=0}^{\infty} C_{mk}(-1)^m(\alpha_m^2-\nu\beta_k^2), \tag{63.12}$$

where f_k and f_k' are the known quantities

$$f_k=(3-\nu)\alpha_k\operatorname{ch}\alpha_k b-\frac{b(1+\nu)\alpha_k^2}{\operatorname{sh}\alpha_k b};$$

$$f_k'=(3-\nu)\beta_k\operatorname{ch}\beta_k a-\frac{a(1+\nu)\beta_k^2}{\operatorname{sh}\beta_k a}. \tag{63.13}$$

Substituting for ψ and $\hat{\varkappa}_{ik}$ in (24. 26a), we obtain

$$K'\sum_{m,\,n=0,\,1,\,2,\,3,\,\ldots} C_{mn}(\alpha_m^2+\beta_n^2)^2\cos\alpha_m x\cos\beta_n y+$$
$$+\frac{D'}{R}\sum_{m,\,n=1,\,2,\,3,\,\ldots} f_{mn}(-1)^m\frac{\cos\alpha_m x\,\varphi_n(y)}{\beta_n^2}=$$
$$=D'^2\sum_{m,\,n,\,r,\,s=1,\,2\,\ldots} f_{mn}f_{rs}\left[\frac{(-1)^{m+s}\cos\alpha_m x\cos\beta_s y\,\varphi_n(y)\,\varphi_r(x)}{\beta_n^2\alpha_r^2}-\right.$$
$$\left.-\frac{(-1)^{m+n+r+s}\sin\alpha_m x\sin\beta_n y\sin\alpha_r x\sin\beta_s y}{\alpha_m\beta_n\alpha_r\beta_s}\right]. \tag{63.14}$$

Here, taking into account the formulas

$$2\cos^2\alpha_m x=1+\cos 2\alpha_m x;\ 2\sin^2\alpha_m x=1-\cos 2\alpha_m x$$

and after the cancellation of terms, the expression in the square bracket reduces to the double series

$$\sum_{m,\,n=0,\,1,\,2,\,\ldots}\bar{D}_{mn}\cos\alpha_m x\cos\beta_n y,\ \bar{D}_{00}=0.$$

Consequently, equating the coefficients of the same cosines, we shall find an expression for the constants C_{mn} in terms of f_{mn}. Thus, the compatibility condition will be exactly satisfied.

For the coefficients C_{mn}, we obtain from (63. 14) the expressions

$$K'\alpha_m^4 C_{m0} + \frac{D'}{R}\sum_{n=1,2,3,\dots} f_{mn}\frac{(-1)^m}{\beta_n^2} = D'^2\overline{D}_{m0},$$
$$K'\beta_n^4 C_{0n} = D'^2\overline{D}_{0n}, \quad m, n = 1, 2, 3, \dots,$$
$$K'(\alpha_m^2+\beta_n^2)^2 C_{mn} - \frac{D'}{R}\frac{f_{mn}(-1)^{m+n}}{\beta_n^2} = D'^2\overline{D}_{mn}, \tag{63.15}$$

where

$$2ab\overline{D}_{k0} = \int_{-a}^{a}\int_{-b}^{b}\sum_{m,n,r,s=1,2,\dots} f_{mn}f_{rs}[*]\cos\alpha_k x\,dx\,dy \quad k = 1, 2, 3, \dots,$$

$$2ab\overline{D}_{0k} = \int_{-a}^{a}\int_{-b}^{b}\sum_{m,n,r,s=1,2,\dots} f_{mn}f_{rs}[*]\cos\beta_k y\,dx\,dy,$$

$$ab\overline{D}_{ik} = \int_{-a}^{a}\int_{-b}^{b}\sum_{m,n,r,s=1,2,\dots} f_{mn}f_{rs}[*]\cos\alpha_i x\cos\beta_k y\,dx\,dy,$$
$$i, k = 1, 2, 3, \dots.$$

In these equations [*] stands for the square bracket appearing in the right-hand member of (63.14).

Hence, after some transformations, we obtain:

$$2\overline{D}_{k0} = \sum_{m,n=1,2,3,\dots} f_{kn}f_{mn}\frac{(-1)^{k+1}}{\alpha_m^2\beta_n^2} +$$
$$+\frac{1}{a}\sum_{m,n,r}\frac{f_{mn}f_{rn}(-1)^{m+r}}{\alpha_r\beta_n^2}\left[\frac{I_{mrk}(a)}{\alpha_r} - \frac{I'_{mrk}(a)}{\alpha_m}\right],$$
$$2\overline{D}_{0k} = \sum_{m,n=1,2,3,\dots} f_{mk}f_{mn}\frac{(-1)^{k+1}}{\alpha_m^2\beta_n^2} +$$
$$+\frac{1}{b}\sum_{m,n,r}\frac{f_{mn}f_{mr}(-1)^{n+r}}{\alpha_m^2\beta_n}\left[\frac{I_{nrk}(b)}{\beta_n} - \frac{I'_{nrk}(b)}{\beta_r}\right];$$
$$ab\overline{D}_{ik} = ab\sum_{m,n}\frac{f_{mk}f_{in}(-1)^{i+k}}{\alpha_m^2\beta_n^2} -$$
$$-\sum_{m,n,r}\frac{(-1)^r}{\alpha_m^2\beta_n^2}\left[a(-1)^{i+n}f_{in}f_{mr}I_{nrk}(b) + b(-1)^{m+k}f_{rn}f_{mk}I_{mri}(a)\right] +$$
$$+\sum_{m,n,r,s=1,2,3,\dots}\frac{(-1)^{m+n+r+s}f_{mn}f_{rs}}{\alpha_r\beta_n}\left[\frac{I_{mri}(a)I_{nsk}(b)}{\alpha_r\beta_s} - \frac{I'_{mri}(a)I'_{nsk}(b)}{\alpha_m\beta_s}\right]. \tag{63.16}$$

where for the integrals we wrote

$$I_{mrk}(a) = \int_{-a}^{a}\cos\alpha_m x\cos\alpha_r x\cos\alpha_k x\,dx =$$
$$= \frac{a}{2}\left(\delta^0_{r+k-m} + \delta^0_{m+k-r} - \delta^0_{m+r-k}\right),$$

$$I''_{mrk}(a) = \int_{-a}^{a} \sin \alpha_m x \sin \alpha_r x \cos \alpha_k x \, dx =$$
$$= \frac{a}{2} (\delta^0_{r+k-m} + \delta^0_{m+k-r} - \delta^0_{m+r-k})$$
$$(\delta_i^0 = 1, \text{ for } i = 0, \ \delta_i^0 = 0, \text{ for } i \neq 0).$$

Thus the compatibility condition (24.26a) and the boundary conditions of the problem are satisfied exactly, by series.

Now we have to determine the functions of moments ψ_1 and ψ_2 from the general solutions (63.7). We shall seek the functions ψ_i in series form

$$\psi_1 = x \sum_{n=0}^{\infty} A_{0n} \cos \beta_n y - \frac{p}{4} x (y^2 - b^2) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin \alpha_m x \cos \beta_n y,$$
$$\psi_2 = y \sum_{m=0}^{\infty} B_{m0} \cos \alpha_m x - \frac{p}{4} y (x^2 - a^2) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos \alpha_m x \sin \beta_n y,$$

where A_{mn} and B_{mn} (m, n = 0, 1, 2, 3, ...) are unknown coefficients. Substituting in the relations (63.7), we obtain

$$\left.\begin{aligned} &\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \beta_n \cos \alpha_m x \cos \beta_n y + \sum_{m=0}^{\infty} B_{m0} \cos \alpha_m x + \\ &\qquad + \left(\frac{1}{R} + \hat{\varkappa}_{22}\right) \psi - M_{11} = 0, \\ &\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A_{mn} \alpha_m \cos \alpha_m x \cos \beta_n y + \sum_{n=0}^{\infty} A_{0n} \cos \beta_n y + \\ &\qquad + \hat{\varkappa}_{11} \psi - M_{22} = 0, \end{aligned}\right\} \quad (63.17)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} \beta_n + B_{mn} \alpha_m) \sin \alpha_m x \sin \beta_n y + x \sum_{n=1}^{\infty} A_{0n} \beta_n \sin \beta_n y +$$
$$+ y \sum_{m=1}^{\infty} B_{m0} \alpha_m \sin \alpha_m x + pxy - 2 \hat{\varkappa}_{12} \psi - 2 M_{12} = 0. \quad (63.18)$$

In the given problem, the function ψ is an even function of the coordinates. Therefore the left-hand terms of (63.17) will be even functions of the coordinates, and the left-hand member of (63.18) will be an odd function, i.e., the choice of functions ψ_1 and ψ_2 accords with the properties of the functions φ and M_{ik}. From (63.17) we find for the Fourier coefficients the expressions:

$$\left.\begin{aligned} &A_{mn} \alpha_m ab = \\ &= \int_{-a}^{a} \int_{-b}^{b} (M_{22} - \hat{\varkappa}_{11} \psi) \cos \alpha_m x \cos \beta_n y \, dx dy \\ &B_{mn} \beta_n ab = \\ &= \int_{-a}^{a} \int_{-b}^{b} \left(M_{11} - \frac{\psi}{R} - \hat{\varkappa}_{22} \psi\right) \cos \alpha_m x \cos \beta_n y \, dx dy \end{aligned}\right\} \quad m, n = 1, 2, 3, \ldots;$$

$$\left.\begin{aligned} &2ab \, A_{0n} = \int_{-a}^{a} \int_{-b}^{b} (M_{22} - \hat{\varkappa}_{11} \psi) \cos \beta_n y \, dx dy \\ &2ab \, B_{m0} = \\ &= \int_{-a}^{a} \int_{-b}^{b} \left(M_{11} - \frac{\psi}{R} - \hat{\varkappa}_{12} \psi\right) \cos \alpha_m x \, dx dy \end{aligned}\right\} \quad m, n = 1, 2, 3, \ldots \quad (63.19)$$

Multiplying (63.18) by $\sin\alpha_m x \sin\beta_n y$ and integrating, we find the relations between the coefficients

$$ab\,(A_{mn}\beta_n + B_{mn}\alpha_m) - 2ab\left[A_{0n}\frac{\beta_n}{\alpha_m}(-1)^m + B_{m0}\frac{\alpha_m}{\beta_n}(-1)^n\right] =$$
$$= \int_{-a}^{a}\int_{-b}^{b}(2\hat{\varkappa}_{12}\psi + 2M_{12} - pxy)\sin\alpha_m x \sin\beta_n y\,dxdy$$
$$(m, n = 1, 2, 3, \ldots); \qquad (63.20)$$

the Fourier coefficients of $\sin\alpha_m x$ and $\sin\beta_n y$ of the left-hand member of (63.18) vanish identically.

Eliminating the Fourier coefficients from (63.19) and (63.20), we obtain

$$\int_{-a}^{a}\int_{-b}^{b}\left\{\cos\alpha_m x\cos\beta_n y\left[\frac{\beta_n}{\alpha_m}(M_{22} - \hat{\varkappa}_{11}\psi) + \frac{\alpha_m}{\beta_n}\left(M_{11} - \frac{\psi}{R} - \hat{\varkappa}_{22}\psi\right)\right] + \right.$$
$$+ (pxy - 2\hat{\varkappa}_{12}\psi - 2M_{12})\sin\alpha_m x\sin\beta_n y -$$
$$- \frac{\beta_n}{\alpha_m}(-1)^m(M_{22} - \hat{\varkappa}_{1}\psi)\cos\beta_n y -$$
$$\left. - \frac{\alpha_m}{\beta_n}(-1)^n\left(M_{11} - \frac{\psi}{R} - \hat{\varkappa}_{22}\psi\right)\cos\alpha_m x\right\}dxdy = 0. \qquad (63.21)$$

This equation is, in effect, another form of the third equilibrium equation of a cylindrical strip rigidly fastened at the edges, since by eliminating the functions ψ_1 and ψ_2 from (63.7) we shall obtain the equilibrium equation. At the same time, (63.21) is the integrability condition for the equations (63.7) with respect to the functions ψ_1 and ψ_2. It is important to note that with our choice of the functions M_{ik} and ψ in the form (63.3) and (63.9), it is impossible to satisfy the third equation of equilibrium directly as for that one would have to expand p in a cosine series. In (63.21) the pressure p is not expanded in series. Another advantage of (63.21) consists in the fact that M_{ik} and ψ are not differentiated twice, as is the case in the third equilibrium equation.

Substituting for M_{ik} and $\hat{\varkappa}_{ik}$ from (63.3) and (63.8) in (63.21) and integrating, we obtain

$$(-1)^{m+n}\left[\left(\frac{\alpha_m}{\beta_n} + \frac{\beta_n}{\alpha_m}\right)^2 f_{mn} + 2\sum_{k=}^{\infty}{}'\left(f_{kn}\frac{\beta_n^2}{\alpha_k^2} + f_{mk}\frac{\alpha_m^2}{\beta_k^2}\right) - \right.$$
$$\left. -4(1-\nu^2)p\right] + \frac{1-\nu^2}{abR}\alpha_m^2\left[I_{mn} - (-1)^n I_m\right] +$$
$$+ \frac{1}{Dab}\int_{-a}^{a}\int_{-b}^{b}\sum_{r,s=1,2,3\ldots}\Phi_{mnrs}\psi\,dxdy = 0, \qquad (63.22)$$

where D is the flexural rigidity, and

$$\Phi_{mnrs} = -f_{rs}\left[(-1)^{m+r}\left(\frac{\beta_n}{\beta_s}\right)^2\cos\alpha_r x\cos\beta_n y\,\varphi_s(y)\,\varphi_m(x) + \right.$$
$$+ (-1)^{n+s}\left(\frac{\alpha_m}{\alpha_r}\right)^2\cos\alpha_m x\cos\beta_s y\,\varphi_n(y)\,\varphi_r(x) -$$
$$\left. - 2(-1)^{r+s}\frac{\alpha_m\beta_n}{\alpha_r\beta_s}\sin\alpha_m x\sin\beta_n y\sin\alpha_r x\sin\beta_s y\right]; \qquad (63.23)$$

$$I_{mn} = \int_{-a}^{a}\int_{-b}^{b} \psi \cos \alpha_m x \cos \beta_n y \, dx dy; \quad I_m = \int_{-a}^{a}\int_{-b}^{b} \psi \cos \alpha_m x \, dx dy;$$

$$I_n' = \int_{-a}^{a}\int_{-b}^{b} \psi \cos \beta_n y \, dx dy. \qquad (63.24)$$

If one expands Φ_{mnrs} in a double Fourier series, then the integral in (63.22) may be written in the form

$$\int_{-a}^{a}\int_{-b}^{b} \sum_{r, s=1, 2, 3 \dots} \Phi_{mnrs} \psi \, dx dy = \frac{1}{2} \sum_{r=1, 2, \dots}^{\infty} \left[\left(\frac{\alpha_m}{\alpha_r}\right)^2 f_{rn} (-1)^n I_m + \right.$$
$$+ \left(\frac{\beta_n}{\beta_r}\right)^2 f_{mr} (-1)^m I_n' + (-1)^{r+m+n} (I_r f_{rn} + I_r' f_{mr}) \Big] +$$
$$+ \sum_{r, s, k=1, 2, 3 \dots} f_{rs} \left\{ \frac{I_{mrk}(a)(-1)^r}{a} \left[\left(\frac{\alpha_m}{\alpha_r}\right)^2 (-1)^{n+s} I_{ks} + \right. \right.$$
$$+ \left(\frac{\beta_n}{\beta_s}\right)^2 I_{kn} \Big] + \frac{I_{nsk}(b)(-1)^s}{b} \left[\left(\frac{\beta_n}{\beta_s}\right)^2 (-1)^{m+r} I_{rk} + \left(\frac{\alpha_m}{\alpha_r}\right)^2 I_{mk} \right] \Big\} +$$
$$+ \frac{1}{ab} \sum_{r, s, i, k=1, 2, 3, \dots} (-1)^{r+s} f_{rs} A_{mnrsik} I_{ik} + \sum_{r, s=1, 2, \dots} \Big[f_{rn} A_{mnrs} I_s +$$
$$+ f_{mr} B_{mnrs} I_s' - (-1)^{m+r} \left(\frac{\beta_n}{\beta_s}\right)^2 f_{rs} I_{rn} - (-1)^{n+s} \left(\frac{\alpha_m}{\alpha_r}\right)^2 f_{rs} I_{ms} \Big],$$

(63.25)

where we set

$$A_{mnrs} = \frac{(-1)^{r+n}}{2a} \left[\frac{2\alpha_m}{\alpha_r} I_{mrs}'(a) - \left(1 + \frac{\alpha_m^2}{\alpha_r^2}\right) I_{mrs}(a) \right],$$
$$B_{mnrs} = \frac{(-1)^{r+m}}{2b} \left[\frac{2\beta_n}{\beta_r} I_{nsr}'(b) - \left(1 + \frac{\beta_n^2}{\beta_r^2}\right) I_{nrs}(b) \right],$$
$$A_{mnrsik} = \frac{2\alpha_m \beta_n}{\alpha_r \beta_s} I_{mri}'(a) I_{nsk}'(b) - I_{mri}(a) I_{nsk}(b) \left(\frac{\beta_n^2}{\beta_s^2} + \frac{\alpha_m^2}{\alpha_r^2}\right). \qquad (63.26)$$

Now we shall calculate the integrals (63.24). First we shall expand ψ in a Fourier double series.

Utilizing the formulas

$$x^2 = \frac{a^2}{3} + 4 \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha_m^2} \cos \alpha_m x; \quad y^2 = \frac{b^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{\beta_n^2} \cos \beta_n y;$$
$$\int_{-a}^{a} L_k(x) \cos \alpha_m x \, dx = \frac{4(-1)^m \operatorname{sh} a\beta_k (\alpha_m^2 - \nu\beta_k^2)}{(1+\nu)(\alpha_m^2 + \beta_k^2)^2} \quad m = 0, 1, 2, \dots;$$
$$\int_{-b}^{b} M_k(y) \cos \beta_n y \, dy = \frac{4(-1)^n \operatorname{sh} b\alpha_k (\beta_n^2 - \nu\alpha_k^2)}{(1+\nu)(\beta_n^2 + \alpha_k^2)^2} \quad n = 0, 1, 2, \dots,$$

we obtain

$$\psi = \sum_{m, n=0, 1, 2, \dots} \overline{C}_{mn} \cos \alpha_m x \cos \beta_n y, \qquad (63.27)$$

where we set

$$6\,\overline{C}_{00} = p_2 a^2 + p_1 b^2;\quad \overline{C}_{k0} = C_{k0} + \frac{2p_2(-1)^k}{\alpha_k^2} - \frac{2\nu}{1+\nu}\frac{B_k}{b\alpha_k^2}\,\mathrm{sh}\, b\alpha_k,$$

$$\overline{C}_{0k} = C_{0k} + \frac{2p_1(-1)^k}{\beta_k^2} - \frac{2\nu}{1+\nu}\frac{A_k}{a\beta_k^2}\,\mathrm{sh}\, a\beta_k,$$

$$\overline{C}_{ik} = C_{ik} + \frac{4}{(1+\nu)(\alpha_i^2+\beta_k^2)^2}\left[\frac{(-1)^i A_k(\alpha_i^2 - \nu\beta_k^2)}{a}\,\mathrm{sh}\, a\beta_k + \right.$$

$$\left. + \frac{(-1)^k B_i(\beta_k^2 - \nu\alpha_i^2)}{b}\,\mathrm{sh}\, b\alpha_i\right], \tag{63.28}$$

C_{0k}, C_{k0}, C_{ik} are given by (63.15). Introducing (63.27) in (63.24), we obtain

$$I_{mn} = ab\,\overline{C}_{mn};\quad I_m = 2ab\,\overline{C}_{m0};\quad I_n' = 2ab\,\overline{C}_{0n}. \tag{63.29}$$

Substituting these integrals in (63.25), and the latter in (63.22), we find the pressure as a function of the deflection parameters.

2°. Let us give the approximate solution of the problem.

The function

$$\psi = \sum_{m,\,n=1,2,3,\dots} C_{mn}\cos\alpha_m x\cos\beta_n y \tag{63.30}$$

exactly satisfies the boundary conditions $g_i(\psi) = 0$ and satisfies in the mean tne conditions $h_i(\psi) = 0$. This function, however, cannot satisfy the compatibility conditions exactly. Therefore, we shall satisfy it in the variational form

$$\int_{-a}^{a}\int_{-b}^{b}\left(K'\Delta\Delta\psi - \hat{x}_{12}^2 + \hat{x}_{11}\hat{x}_{22} + \frac{\hat{x}_{11}}{R}\right)\delta\psi\,dx\,dy = 0, \tag{63.31}$$

where the contour integral (24.27) vanishes with (63.30), as it equals

$$I_1 = 2K'\int_{-a}^{a}\left[g_2(\psi)\delta\psi - h_2(\psi)\frac{\partial\delta\psi}{\partial y}\right]_{y=-b}^{y=b} dx +$$

$$+ 2K'\int_{-b}^{b}\left[g_1(\psi)\delta\psi - h_1(\psi)\frac{\partial\delta\psi}{\partial x}\right]_{x=-a}^{x=a} dy. \tag{63.32}$$

Substituting for $\hat{x}_{ik}$ from (63.8) and for $\delta\psi$ from (63.30) in (63.31), we obtain

$$K'(\alpha_m^2 + \beta_n^2)^2 C_{mn} - \frac{D'}{R}\frac{f_{mn}(-1)^{n+n}}{\beta_n^2} = D'^2\,\overline{D}_{mn}, \tag{63.33}$$

where m, n = 1, 2, 3, ..., $\overline{D}$ are given in terms of f_{mn} by (63.16).

We shall obtain another relation between C_{mn} and f_{mn} from (63.23) in the form

$$
(-1)^{m+n}\Big[\Big(\frac{\alpha_m}{\beta_n}+\frac{\beta_n}{\alpha_m}\Big)^2 f_{mn}+2\sum_{k=1}^{\infty}\Big(f_{kn}\frac{\beta_n^2}{\alpha_k^2}+f_{mk}\frac{\alpha_m^2}{\beta_k^2}\Big)-
$$

$$
-4(1-\nu^2)p\Big]+\frac{(1-\nu^2)\alpha_m^2}{R}C_{mn}+
$$

$$
+\frac{1}{D}\sum_{r,s,i,k=1,2,3\dots}(-1)^{r+s}f_{rs}C_{ik}A_{mnrsik}+
$$

$$
+\frac{1}{D}\sum_{r,s,k=1,2,3\dots}f_{rs}\Big[\frac{(-1)^r}{a}\Big(\frac{\beta_n}{\beta_s}\Big)^2 C_{kn}I_{mrk}(a)+\frac{(-1)^s}{b}\Big(\frac{\alpha_m}{\alpha_r}\Big)^2 C_{mk}I_{nsk}(b)+
$$

$$
+\frac{I_{mrk}(a)(-1)^{n+r+s}}{a}\Big(\frac{\alpha_m}{\alpha_r}\Big)^2 C_{ks}+\frac{I_{nsk}(b)(-1)^{m+r+s}}{b}\Big(\frac{\beta_n}{\beta_s}\Big)^2 C_{rk}\Big]-
$$

$$
-\frac{1}{D}\sum_{r,s}f_{rs}\Big[(-1)^{m+r}\Big(\frac{\beta_n}{\beta_s}\Big)^2 C_{rn}+(-1)^{n+s}\Big(\frac{\alpha_m}{\alpha_r}\Big)^2 C_{ms}\Big]=0, \qquad (63.34)
$$

where I_{mrk} and I'_{mrk} are given by (63.17). Owing to these, considerable simplifications are possible here.

§ 64. Exact Series-Solution for a Strip Hinged at All Edges

1°. The boundary conditions of hinged support

$$
u=v=w=\frac{\partial^2 w}{\partial x^2}=0\ (x=\pm a);\quad u=v=w=\frac{\partial^2 w}{\partial y^2}=0\ (y=\pm b),
$$

are, according to (24.52a) and (24.52c), equivalent to the boundary conditions in the stresses and moments:

$$
\begin{gathered}
h_1(\psi)=0,\ RK'g_1(\psi)=-D'(1+\nu)\int^{y} M_{12}\,dy\quad (x=\pm a),\\
h_2(\psi)=0,\quad g_2(\psi)=0\quad (y=\pm\ )\\
M_{11}=M_{22}=0 \text{ (on the edges)}
\end{gathered}
\qquad (64.1)
$$

The constant of integration in the second of the conditions (64.1) turns out to be zero owing to the symmetry of the strip. The boundary conditions for the moments and the Codazzi conditions are satisfied by the functions

$$
M_{11}=\frac{1}{1-\nu^2}\sum_{m,n}f_{mn}\cos\alpha_m x\cos\beta_n y\Big(\frac{1}{\beta_n^2}+\frac{\nu}{\alpha_m^2}\Big),
$$

$$
M_{22}=\frac{1}{1-\nu^2}\sum_{m,n}f_{mn}\cos\alpha_m x\cos\beta_n y\Big(\frac{1}{\alpha_m^2}+\frac{\nu}{\beta_n^2}\Big),
$$

$$
M_{12}=-\frac{1}{1+\nu}\sum_{m,n}f_{mn}\frac{\sin\alpha_m x\sin\beta_n y}{\alpha_m\beta_n},
$$

where $\qquad m,n=1,3,5,\dots;\ \alpha_m=m\pi/2a,\ \beta_n=n\pi/2b. \qquad$ (64.2)

Hence for $\hat{\varkappa}_{ik}$ we obtain the expressions:

$$
\hat{\varkappa}_{11}=D'\sum_{m,n}\frac{f_{mn}}{\beta_n^2}\cos\alpha_m x\cos\beta_n y;\quad \hat{\varkappa}_{22}=D'\sum_{m,n}\frac{f_{mn}}{\alpha_m^2}\cos\alpha_m x\cos\beta_n y;
$$

$$
\hat{\varkappa}_{12}=-D'\sum_{m,n}\frac{f_{mn}}{\alpha_m\beta_n}\sin\alpha_m x\sin\beta_n y;\quad w_{mn}=D'f_{mn}/(\alpha_m\beta_n)^2, \qquad (64.3)
$$

where w_{mn} are the deflection parameters.

We shall seek the stress function in the form

$$\psi=\psi^{*}+\psi^{**}+\sum_{m,\,n=1,\,3,\,5}D_{mn}\cos\alpha_m x\cos\beta_n y;$$

$$D_{mn}=-\frac{D'}{K'R}\cdot\frac{f_{mn}}{\beta_n^2(\alpha_m^2-\beta_n^2)^2}, \tag{64.4}$$

where ψ^* satisfies the homgeneous boundary conditions $h_i(\psi^*)=0,\ g_i(\psi^*)=0,\ (i=1,2)$ and the compatibility conditions for the plate

$$K'\Delta\Delta\psi^*=\overset{\Lambda}{\varkappa}{}^2_{12}-\overset{\Lambda}{\varkappa}_{11}\overset{\Lambda}{\varkappa}_{22}. \tag{64.5}$$

ψ^{**} is a biharmonic function of the form

$$\psi^{**}=\sum_{m=1}^{\infty}[f_m(x)\cos\beta_m y+F_m(y)\cos\alpha_m x],\ \alpha_m=\frac{m\pi}{2a},\ \beta_m=\frac{m\pi}{2b},$$

$$m=1,3,5\ldots,$$

$$f_m(x)=A'_m\operatorname{ch}\beta_m x+B'_m x\operatorname{sh}\beta_m x;$$

$$F_m(y)=A''_m\operatorname{ch}\alpha_m y+B''_m y\operatorname{sh}\alpha_m y,$$

satisfying the boundary conditions

$$h_1(\psi^{**})=0;\ g_1(\psi^{**})=\sum_{m,\,n=1,\,3,\,5\ldots}D_{mn}\frac{\beta_n^2(\beta_n^2-\nu\alpha_m^2)}{\alpha_m}\sin\alpha_m a\cos\beta_n y$$

$$(x=\pm a),$$

$$h_2(\psi^{**})=0;\ g_2(\psi^{**})=\sum_{m,\,n=1,\,3,\,5\ldots}D_{mn}[\beta_n^3+$$

$$+(2+\nu)\beta_n\alpha_m^2]\sin\beta_n b\cos\alpha_m x\quad(y=\pm b). \tag{64.6}$$

From the boundary condition $h_1(\psi^{**})=0$ it follows that

$$f''_m(a)+\nu\beta_m^2 f_m(a)=0,$$

whence

$$A'_m=-B'_m\left[a\operatorname{th}a\beta_m+\frac{2}{(1-\nu)\beta_m}\right];$$

consequently,

$$\psi^{**}=\sum_{m=1,\,3,\,5\ldots}[B'_m\overset{\Lambda}{L}_m(x)\cos\beta_m y+B''_m\overset{\Lambda}{M}_m(y)\cos\alpha_m x], \tag{64.7}$$

where we have set

$$\overset{\Lambda}{L}_m(x)=x\operatorname{sh}\beta_m x-\left[a\operatorname{th}\beta_m a+\frac{2}{(1+\nu)\beta_m}\right]\operatorname{ch}\beta_m x,$$

$$\overset{\Lambda}{M}_m(y)=y\operatorname{sh}\alpha_m y-\left[b\operatorname{th}\alpha_m a+\frac{2}{(1+\nu)\alpha_m}\right]\operatorname{ch}\alpha_m y. \tag{64.8}$$

Introducing (64.7) in the boundary conditions (64.6), we obtain

$$\sum_{m=1}^{\infty}\Big\{B''_m\alpha_m^2(1+\nu)\sin\alpha_m a\,[\operatorname{ch}\alpha_m y(b\alpha_m\operatorname{th}\alpha_m b-2)-\alpha_m y\operatorname{sh}\alpha_m y]+$$

$$+B'_m\left[\beta_m^2(1-\nu)\operatorname{sh}\beta_m a-\frac{a(1+\nu)}{\operatorname{ch}\beta_m a}\beta_m^3\right]\cos\beta_m y\Big\}=$$

$$=\sum_{m,\,n=1,\,3,\,5\ldots}D_{mn}\frac{\beta_n^2(\beta_n^2-\nu\alpha_m^2)}{\alpha_m}\sin\alpha_m a\cos\beta_n y,$$

from which, considering that

$$b \operatorname{ch} \alpha_m y (\alpha_m b \operatorname{th} \alpha_m b - 2) - b\alpha_m y \operatorname{sh} \alpha_m y =$$
$$= -4 \sum_{n=1,3,5\ldots} \frac{\beta_n^3 \sin \beta_n b \operatorname{ch} \alpha_m b \cos \beta_n y}{(\alpha_m^2 + \beta_n^2)^2},$$

we obtain for B_n' and B_n'' the infinite system of linear equations

$$B_n'\lambda_n' - \frac{4(1+\nu)\beta_n \sin \beta_n b}{b} \sum_{m=1,3,5\ldots} \frac{B_m'' \alpha_m^2 \sin a\alpha_m \operatorname{ch} \alpha_m b}{(\alpha_m^2 + \beta_n^2)^2} =$$
$$= \sum_{m=1,3,5\ldots} D_{mn} \frac{\sin a\alpha_m (\beta_n^2 - \nu\alpha_m^2)}{\alpha_m}; \quad \lambda_n' = (3-\nu) \operatorname{sh} a\beta_n - \frac{a(1+\nu)\beta_n}{\operatorname{ch} a\beta_n};$$
$$B_n''\lambda_n'' - \frac{4(1+\nu)\alpha_n \sin \alpha_n a}{a} \sum_{m=1,3,5\ldots} \frac{B_m' \beta_m^2 \sin \beta_m b \operatorname{ch} \beta_m a}{(\alpha_n^2 + \beta_m^2)^2} =$$
$$= \sum_{m=1,3,5\ldots} D_{mn} \left[\frac{\beta_m^3}{\alpha_n^2} + (2+\nu)\beta_m \right] \sin \beta_m b;$$

$$\lambda_n'' = (3-\nu) \operatorname{sh} \alpha_n b - \frac{b(1+\nu)\alpha_n}{\operatorname{ch} b\alpha_n}, \tag{64.9}$$

where the second one is obtained by analogy to the first, and λ_n' and λ_n'' are known coefficients.

For ψ^* we shall take the function (63.9) of the problem of the preceding section. Introducing it in the compatibility conditions (24.26a), we obtain

$$K' \sum_{m,n=0,1,2\ldots} C_{mn} (\alpha_m'^2 + \beta_n'^2)^2 \cos \alpha_m' x \cos \beta_n' y =$$
$$= D'^2 \sum_{r,s,l,k=1,3,5\ldots} f_{lk} f_{rs} \left[\frac{\cos \alpha_l x \cos \alpha_r x \cos \beta_k y \cos \beta_s y}{\alpha_r^2 \beta_k^2} - \right.$$
$$\left. - \frac{\sin \alpha_l x \sin \alpha_r x \sin \beta_k y \sin \beta_s y}{\alpha_l \beta_k \alpha_r \beta_s} \right],$$

where

$$\alpha_m' = m\pi/a, \quad \beta_n' = n\pi/b, \quad m, n = 0, 1, 2, 3, \ldots .$$

The right-hand member of the equation obtain has the form

$$D'^2 \sum_{m,n=0,1,2\ldots} \overline{D}_{mn} \cos \alpha_m' x \cos \beta_n' y, \quad \overline{D}_{00} = 0.$$

Consequently,

$$K' C_{mn} (\alpha_m'^2 + \beta_n'^2)^2 = D'^2 \overline{D}_{mn}, \quad m, n = 0, 1, 2 \ldots, \tag{64.10}$$

where the coefficients $\overline{D}_{mn}$ are determined from the formulas

$$ab\overline{D}_{mn} = \sum_{r,s,l,k=1,3,5\ldots} f_{lk} f_{rs} \left[\frac{I_{lr,m}(a) I_{sk,n}(b)}{\beta_k^2 \alpha_r^2} - \frac{I'_{lr,m}(a) I'_{sk,n}(b)}{\alpha_l \beta_k \alpha_r \beta_s} \right] \tag{64.11}$$
$$(m, n = 1, 2, 3, \ldots);$$
$$2b\overline{D}_{0n} = \sum_{r,s,k=1,3,5\ldots} f_{rk} f_{rs} \left[\frac{I_{sk,n}(b)}{\alpha_r^2 \beta_k^2} - \frac{I'_{sk,n}(b)}{\alpha_r^2 \beta_k \beta_s} \right]$$
$$n = 1, 2, 3 \ldots;$$
$$2a\overline{D}_{m0} = \sum_{r,s,k=1,3,5\ldots} f_{sk} f_{rk} \left[\frac{I_{rs,m}(a)}{\alpha_r^2 \beta_k^2} - \frac{I'_{rs,m}(b)}{\alpha_r \alpha_s \beta_k^2} \right], \quad m = 1, 2, 3 \ldots \tag{64.11a}$$

Here we have set

$$I_{rs,m}(a)=\int_{-a}^{a}\cos\frac{r\pi x}{2a}\cos\frac{s\pi x}{2a}\cos\frac{m\pi x}{a}\,dx,\quad r,s=1,3,5\ldots,$$
$$m=1,2,3,\ldots;$$

$$I'_{rs,m}(a)=\int_{-a}^{a}\sin\frac{r\pi x}{2a}\sin\frac{s\pi x}{2a}\cos\frac{m\pi x}{a}\,dx. \tag{64.12}$$

Substituting for C_{mn} from (64. 10) in (63. 12), we shall find the coefficients A_k and B_k of the function ψ^*. Then the stress function ψ will be expressed in terms of the deflection parameters. Thus, for the exact fulfillment of the boundary conditions with respect to ψ one has to solve two infinite systems of linear equations (63.12) and (64.9).

We shall investigate the equilibrium equation. By making the following substitutions in the general solutions (63. 7)

$$\psi_1=-\frac{p}{4}x(y^2-b^2)+$$
$$+\frac{1}{1-\nu^2}\sum_{m,n=1,3,5\ldots}f_{mn}\frac{\sin\alpha_m x\cos\beta_n y}{\alpha_m}\left(\frac{1}{\alpha_m^2}+\frac{\nu}{\beta_n^2}\right)+\hat{\psi}_1,$$
$$\psi_2=-\frac{p}{4}y(x^2-a^2)+$$
$$+\frac{1}{1-\nu^2}\sum_{m,n=1,3,5\ldots}f_{mn}\frac{\cos\alpha_m x\sin\beta_n y}{\beta_n}\left(\frac{1}{\beta_n^2}+\frac{\nu}{\alpha_m^2}\right)+\hat{\psi}_2, \tag{64.13}$$

and using the relations (64. 2), we obtain instead of (63. 7) the equation

$$\frac{\partial\hat{\psi}_2}{\partial y}+\left(\frac{1}{R}+\hat{\varkappa}_{22}\right)\psi=0;\quad \frac{\partial\hat{\psi}_1}{\partial x}+\hat{\varkappa}_{11}\psi=0; \tag{64.14}$$

$$\sum_{m,n=1,3,5\ldots}\frac{f_{mn}\left(\alpha_m^2+\beta_n^2\right)}{\alpha_m^3\beta_n^3}\cdot\frac{\sin\alpha_m x\,\sin\beta_n y}{1-\nu^2}=$$
$$=\frac{\partial\hat{\psi}_1}{\partial y}+\frac{\partial\hat{\psi}_2}{\partial x}+2\hat{\varkappa}_{12}\psi-pxy, \tag{64.15}$$

where $\hat{\psi}_i$ are the new moment functions. Taking nto consideration the form of the functions $\hat{\varkappa}_{ik}$ and ψ, we may set

$$\left(\frac{1}{R}+\hat{\varkappa}_{22}\right)\psi=\sum_{k=1}^{N}\frac{d\omega_2^k(y)}{dy}\Omega_2^k(x);\quad \hat{\varkappa}_{11}\psi=\sum_{k=1}^{N}\frac{d\omega_1^k(x)}{dx}\Omega_1^k(y),\ (*)$$

where N is some finite number, and the functions

$$\Omega_1^k(x),\ \Omega_2^k(y),\ \frac{d\omega_1^k(x)}{dx},\ \frac{d\omega_2^k(y)}{dy}$$

are even with respect to their arguments and incl de also the infinite series.

With these notations we obtain from (64. 14)

$$\hat{\psi}_1 = -\sum_{k=1}^{N} \omega_1^k(x)\Omega_1^k(y) + f_1(y);$$
$$\hat{\psi}_2 = -\sum_{k=1}^{N} \omega_2^k(y)\Omega_2^k(x) + f_2(x), \tag{64.16}$$

where f_1 and f_2 are arbitrary functions.

Introducing them in (64.15), we obtain

$$f(x, y) = -\sum_{k=1}^{N}\left[\omega_1^k(x)\frac{d\Omega_1^k(y)}{dy} + \omega_2^k(y)\frac{d\Omega_2^k(x)}{dx}\right] + \\ + 2\hat{x}_{12}\psi - pxy + f_1'(y) + f_2'(x), \tag{64.17}$$

where f(x, y) is the left-hand member of equation (64.15).

In order to clarify the nature of the functions $f_1(y)$ and $f_2(x)$ we shall multiply the above equality by $\sin \alpha_r x$ and then by $\sin \beta_s y$, where r, s = 1, 3, 5..., and integrate the results obtained over the surface of the strip. Taking into account the properties of the functions ω_i^k and Ω_i^k, we find:

$$\int_{-a}^{a} f_2'(x)\sin\alpha_r x dx = 0, \quad \int_{-b}^{b} f_1'(y)\sin\beta_s y dy = 0,$$

i.e., the functions $f_2'(x)$ and $f_1'(y)$ are even with respect to their arguments. Further, we set x = ±a and y = ±b in (64.17). This yields

$$f_1'(y) + f_2'(a) = 0, \quad f_2'(x) + f_1'(b) = 0,$$

whence

$$f_1'(y) + f_2'(x) = -f_2'(a) - f_1'(b).$$

Integrating (64.17) over the surface of the strip we have

$$f_2'(a) + f_1'(b) = 0.$$

Taking that into consideration, we obtain

$$f_1'(y) = -f_2'(x) = C,$$

where C is a constant.

Consequently, in (64.17) the sum of the arbitrary functions is zero.

Now we shall multiply (64.17) by $\sin \alpha_r x \sin \beta_s y$ and integrate over the surface of the strip. Then, by virtue of orthogonality of the system of functions $\sin \alpha_r x$, we obtain

$$\frac{f_{mn}\left(\alpha_m^2 + \beta_n^2\right)^2}{(1-\nu^2)\alpha_m^3\beta_n^3}ab = \int_{-a}^{a}\int_{-b}^{b}\left\{-\sum_{k=1}^{N}\left[\omega_1^k(x)\frac{d\Omega_1^k(y)}{dy} + \omega_2^k(y)\frac{d\Omega_2^k(x)}{dx}\right] + \right. \\ \left. + 2\hat{x}_{12}\psi - pxy\right\}\sin\alpha_m x \sin\beta_n y dx dy. \tag{64.18}$$

Let us integrate by parts the terms containing ω and Ω. We have:

$$\int_{-b}^{b} \sin\beta_n y \frac{d\Omega_1^k(y)}{dy}\, dy = \sin\beta_n y\, \Omega^k(y)\Big|_{y=-b}^{y=b} -$$

$$-\beta_n \int_{-b}^{b} \cos\beta_n y\, \Omega_1^k(y)\, dy,$$

$$\int_{-a}^{a} \omega_1^k(x) \sin\alpha_m x\, dx = \frac{1}{\alpha_m}\int_{-a}^{a} \cos\alpha_m x \frac{d\omega_1^k(x)}{dx}\, dx.$$

Consequently,

$$\int_{-a}^{a}\int_{-b}^{b}\sum_{k=1}^{N} \omega_1^k(x) \frac{d\Omega_1^k(y)}{dy} \sin\alpha_m x \sin\beta_n y\, dx dy =$$

$$= \frac{1}{\alpha_m}\int_{-a}^{a} \cos\alpha_m x \sin\beta_n y\, \hat{\varkappa}_{11}\psi\Big|_{y=-b}^{y=b} dx -$$

$$-\frac{\beta_n}{\alpha_m}\int_{-a}^{a}\int_{-b}^{b} \hat{\varkappa}_{11}\psi \cos\alpha_m x \cos\beta_n y\, dx dy.$$

Here we have made us of the notations (*). Here, $\hat{\varkappa}_{11}$ vanishes at the edges $y = \pm b$. Analogously one can obtain:

$$\int_{-a}^{a}\int_{-b}^{b}\sum_{k=1}^{N} \omega_2^k(y) \frac{d\Omega_2^k(x)}{dx} \sin\alpha_m x \sin\beta_n y\, dx dy =$$

$$= 2\frac{(-1)^{\frac{m-1}{2}}}{R\beta_n}\int_{-b}^{b} \cos\beta_n y \psi(a, y)\, dy -$$

$$-\frac{\alpha_m}{\beta_n}\int_{-a}^{a}\int_{-b}^{b} \cos\alpha_m x \cos\beta_n y \left(\frac{1}{R} + \hat{\varkappa}_{22}\right)\psi\, dx dy;$$

after taking into account that $\hat{\varkappa}_{22} = 0$ at the edges $x = \pm a$.

Utilizing the equations obtained, we shall represent (64.18) in the form

$$\frac{f_{mn}(\alpha_m^2 + \beta_n^2)^2 ab}{(1-\nu^2)\alpha_m^3\beta_n^3} - \frac{4(-1)^{\frac{m+n}{2}} p}{\alpha_m^2\beta_n^2} = -\frac{2(-1)^{\frac{m-1}{2}}}{\beta_n R}\int_{-b}^{b} \cos\beta_n y \psi(a, y)\, dy +$$

$$+\int_{-a}^{a}\int_{-b}^{b}\left\{\cos\alpha_m x \cos\beta_n y\left[\frac{\beta_n}{\alpha_m}\hat{\varkappa}_{11} + \frac{\alpha_m}{\beta_n}\left(\frac{1}{R} + \hat{\varkappa}_{22}\right)\right] +\right.$$

$$\left. + 2\hat{\varkappa}_{12} \sin\alpha_m x \sin\beta_n y\right\}\psi\, dx dy. \qquad (64.19)$$

Substituting for $\hat{\varkappa}_{ik}$ from (64.3) in (64.19), we obtain by integration

$$abf_{mn}\left(\frac{\alpha_m}{\beta_n} + \frac{\beta_n}{\alpha_m}\right)^2 - \frac{4p(1-\nu^2)(-1)^{\frac{m+n}{2}}}{\alpha_m\beta_n} + \frac{2\alpha_m}{R}(-1)^{\frac{m-1}{2}} I_1 =$$

$$= \frac{(1-\nu^2)\alpha_m^2 I_{mn}}{R} + \frac{1}{D}\sum_{r,s=1,3,5\ldots} f_{rs}\left[I_{mnrs}\left(\frac{\beta_n^2}{\beta_s^2} + \frac{\alpha_m^2}{\alpha_s^2}\right) - \right.$$

$$\left. - 2I'_{mnrs}\frac{\alpha_m\beta_n}{\alpha_r\beta_s}\right], \qquad (64.20)$$

where for the integrals we introduced the notations

$$I_{mnrs} = \int_{-a}^{a}\int_{-b}^{b} \psi \cos \alpha_m x \cos \beta_n y \cos \alpha_r x \cos \beta_s y \, dx dy;$$

$$I'_{mnrs} = \int_{-a}^{a}\int_{-b}^{b} \psi \sin \alpha_m x \sin \beta_n y \sin \alpha_r x \sin \beta_s y \, dx dy;$$

$$I'_{mn} = \int_{-a}^{a}\int_{-b}^{b} \psi \cos \alpha_m x \cos \beta_n y \, dx dy; \quad I_1 = \int_{-b}^{b} \psi \cos \beta_n y(a, y) \, dy, \tag{64.21}$$

and ψ is given by (64.4).

Making use of the integral

$$\int_{-a}^{a} \hat{L}_m(x) \cos \alpha_k x dx = \frac{4\alpha_k \operatorname{ch} \beta_m a \cdot \sin a\alpha_k \cdot (\alpha_k^2 - \nu\beta_m^2)}{\beta_m (1+\nu) \cdot (\alpha_k^2 + \beta_m^2)^2},$$

one may expand the function $L_m(x)$ in a series in cosines of half the argument

$$\hat{L}_m(x) = \frac{4 \operatorname{ch} a\beta_m}{(1+\nu) a\beta_m} \sum_{k=1,3,5\ldots} \frac{\alpha_k \sin a\alpha_k (\alpha_k^2 - \nu\beta_m^2) \cos \alpha_k x}{(\alpha_k^2 + \beta_m^2)^2}.$$

The function $\hat{M}_m(y)$ can be similarly expanded in terms of $\cos \beta_k y$. Then the stress function ψ will be reduced to the form

$$\psi = \psi^* + \sum_{i,k=1,3,5\ldots} a_{ik} \cos \alpha_i x \cos \beta_k y, \tag{64.22}$$

where

$$a_{ik} = D_{ik} + \frac{4}{(1+\nu)(\alpha_i^2 + \beta_k^2)^2} \left[\frac{\beta_k (-1)^{\frac{k-1}{2}} \operatorname{ch} b\alpha_i \cdot (\beta_k^2 - \nu\alpha_i^2) B_i''}{b\alpha_i} + \right.$$
$$\left. + \frac{\alpha_i (-1)^{\frac{i-1}{2}} \operatorname{ch} a\beta_k \cdot (\alpha_i^2 - \nu\beta_k^2) \dot{B}_k}{a\beta_k} \right], \tag{64.23}$$

and the function ψ^* is given by the series (63.9). Introducing (64.22) in the integrals (64.21), we obtain

$$I_{mnrs} = \sum_{k=1,2,3\ldots} [b\delta_{ns} \bar{C}_{k0} I_{mr,k}(a) + a\delta_{mr} \bar{C}_{0k} I_{ns,k}(b)] +$$
$$+ \sum_{i,k=1,2,3\ldots} \bar{C}_{ik} I_{mr,i}(a) I_{ns,k}(b) + \sum_{i,k=1,3,5\ldots} a_{ik} \hat{I}_{mri}(a) \hat{I}_{nsk}(b);$$
$$I'_{mnrs} = \sum_{k=1,2,3\ldots} [b\delta_{ns} \bar{C}_{k0} I'_{mr,k}(a) + a\delta_{mr} \bar{C}_{0k} I'_{ns,k}(b)] +$$
$$+ \sum_{i,k=1,2,3\ldots} \bar{C}_{ik} I'_{mr,i}(a) I_{ns,k}(b) + \sum_{i,k=1,3,5\ldots} a_{ik} \hat{\hat{I}}_{mri}(a) \hat{\hat{I}}_{nsk}(b), \tag{64.24}$$

δ_{mr} are the Kronecker deltas, $\bar{C}_{ik}$ are given by (63.28),

$$I'_{mn} = ab \, a_{mn} + \int_{-a}^{a}\int_{-b}^{b} \psi^* \cos \alpha_m x \cos \beta_n y \, dx dy,$$

and we have introduced the new integrals

$$\hat{I}_{mri}(a)=\int_{-a}^{a}\cos\frac{m\pi x}{2a}\cos\frac{r\pi x}{2a}\cos\frac{i\pi x}{2a}\,dx;$$

$$\hat{\hat{I}}_{mri}(a)=\int_{-a}^{a}\sin\frac{m\pi x}{2a}\sin\frac{r\pi x}{2a}\cos\frac{i\pi x}{2a}\,dx. \tag{64.25}$$

2°. We shall give the approximate solution of the problem.

The function

$$\psi=\sum_{m,\,n=1,\,3,\,5\,\ldots}C_{mn}\cos\alpha_m x\cos\beta_n y,\quad \alpha_m=\frac{m\pi}{2a},\quad \beta_n=\frac{n\pi}{2b} \tag{64.26}$$

exactly satisfies the boundary conditions $h_i(\psi) = 0$ and satisfies in the mean the conditions

$$RK'g_1(\psi)=-D'(1+\nu)\int^{y}M_{12}d\,y,\quad g_2(\psi)=0.$$

The contour integral (24. 27) is expressed in the form

$$I_1=2K'\int_{-a}^{a}\left[g_2(\psi)\,\delta\psi-h_2(\psi)\frac{\partial\delta\psi}{\partial y}\right]\Bigg|_{y=-b}^{y=b}dx+$$

$$+2K'\int_{-b}^{b}\left[g_1(\psi)\,\delta\psi-\frac{1}{RK'}\cdot\frac{\partial w}{\partial x}\cdot\delta\psi-h_1(\psi)\frac{\partial\delta\psi}{\partial x}\right]\Bigg|_{x=-a}^{x=a}dy.$$

With (64. 26), it vanishes.

In that case, in (64.20) one should substitute

$$I_{mnrs}=\sum_{i,\,k=1,\,3,\,5\,\ldots}C_{ik}\hat{I}_{mri}(a)\,\hat{I}_{nsk}(b),\quad I_1=0,$$

$$I'_{mnrs}=\sum_{i,\,k=1,\,3,\,5\,\ldots}C_{ik}\hat{\hat{I}}_{mri}(a)\,\hat{\hat{I}}_{nsk}(b),\quad I'_{mn}=abC_m.$$

Thereby the equilibrium equation will be exactly satisfied.

Integrating the compatibility conditions by the Bubnov-Galerkin method, we obtain:

$$K'(\alpha_m^2+\beta_n^2)^2C_{mn}-\frac{D'}{R}\frac{f_{mn}}{\beta_n}=$$

$$=\frac{D'}{ab}\sum_{r,\,s,\,i,\,k=1,\,3,\,5\,\ldots}f_{rs}f_{ik}\left(\frac{\hat{\hat{I}}_{rim}(a)\,\hat{\hat{I}}_{skn}(b)}{\alpha_r\beta_s\alpha_i\beta_k}-\frac{\hat{I}_{rim}(a)\,\hat{I}_{skn}(b)}{\beta_s^2\alpha_i^2}\right)$$

$$(m,\ n=1,\ 3,\ 5,\ \ldots),$$

where the integrals (64.25) have been introduced.

§ 65. Freely-Supported Cylindrical Strip under Uniform Compression or Elongation and External Normal Pressure

Let the strip be freely supported at all edges and be under the action of an uniform external pressure and contour forces $T_{11} = p_1$ and $T_{22} = p_2$, in the absence of the tangential stress T_{12}. The functions ψ and M_{ik} satisfy the boundary conditions

$$\frac{\partial^2\psi}{\partial y^2} = p_1, \quad \frac{\partial^2\psi}{\partial x\partial y} = 0, \quad M_{11} = M_{22} = 0 \quad (\text{for } x = \pm a),$$
$$\frac{\partial^2\psi}{\partial x^2} = p_2, \quad \frac{\partial^2\psi}{\partial x\partial y} = 0, \quad M_{11} = M_{22} = 0 \quad (\text{for } y = \pm b). \tag{65.1}$$

We shall consider the function:

$$\psi = \frac{1}{2}(p_2x^2 + p_1y^2) + \sum_{m,\,n=0,\,1,\,2,\ldots} C_{mn}\cos\alpha'_m x\cos\beta_n' y +$$
$$+ \sum_{m,\,n=1,\,3,\,5\ldots} D_{mn}\cos\alpha_m x\cos\beta_n y +$$
$$+ \sum_{m=1,\,2,\,3\ldots} [A_mF_m(x)\cos\beta_m y + B_mF_m^0(y)\cos\alpha_m x] +$$
$$+ \sum_{m=1,\,2,\,3\ldots} [C_mE_m(x)\cos\beta'_m y + D_mE_m^0(y)\cos\alpha'_m x]$$
$$(\alpha'_m = m\pi/a,\ \beta'_m = m\pi/b;\quad \alpha_m = m\pi/2a,\ \beta_m = m\pi/2b). \tag{65.2}$$

Here D_{mn} and C_{mn} are determined from (64.4) and (64.10) respectively and the single sums are biharmonic functions. Consequently, (65.2) satisfies the compatibility conditions. The functions entering into that expression have the form

$$E_m(x)\operatorname{sh} a\beta'_m = \beta'_m x\operatorname{sh}\beta'_m x - \operatorname{ch}\beta'_m x\,(1 + a\beta'_m\operatorname{cth} a\beta'_m),$$
$$m = 1, 2, 3, \ldots \tag{65.3}$$
$$F_m(x)\operatorname{sh} a\beta'_m = x\operatorname{sh}\beta_m x\operatorname{ch}\beta_m a - a\operatorname{ch}\beta_m x\operatorname{sh}\beta_m a,$$

and the functions $E_m^0(y)$ and $F_m^0(y)$ are obtained from $E_m(x)$ and $F_m(x)$ by replacing s by y, a by b, and b by a. Introducing (65.2) in the boundary conditions (65.1) we obtain for the determination of the coefficients A_m, B_m, C_m, D_m the infinite system of linear equations

$$A_mF'_m(a) - \frac{4\beta_m}{b}\sum_{k=1,\,3,\,5\ldots}^{\infty}\frac{B_k\alpha_k^2(-1)^{\frac{m+k}{2}}\operatorname{cth}\alpha_k b}{(\alpha_k^2 + \beta_m^2)^2} = \sum_{k=1,\,3,\,5\ldots} D_{km}\alpha_k(-1)^{\frac{k-1}{2}};$$
$$B_mF_m^{0\prime}(b) - \frac{4\alpha_m}{a}\sum_{k=1,\,3,\,5\ldots}^{\infty}\frac{A_k\beta_k^2(-1)^{\frac{m+k}{2}}\operatorname{cth} a\beta_k}{(\alpha_m^2 + \beta_k^2)^2} = \sum_{k=1,\,3,\,6\ldots} D_{mk}\beta_k(-1)^{\frac{k-1}{2}}, \tag{65.4}$$

where m = 1, 3, 5, ...; D_{mn} are expressed in terms of the deflection parameters (64.4)

$$F'_m(a) = 0.5 + a\beta_m(\operatorname{sh} a\beta_m)^{-1}; \quad F(a) = 0;$$

$$-C_k E_k(a) + \frac{4}{b}\sum_{m=1,2,3\ldots} D_m \frac{(-1)^{m+k}\alpha'^3_m}{(\alpha'^2_m + \beta'^2_k)^2} = \sum_{m=0,1,2,3\ldots} C_{mk}(-1)^m;$$

$$-D_k E_k^0(b) + \frac{4}{a}\sum_{m=1,2,3\ldots} C_m \frac{(-1)^{m+k}\beta'^3_r}{(\alpha'^2_k + \beta'^2_m)^2} = \sum_{m=0,1,2,3\ldots} C_{km}(-1)^m, \tag{65.5}$$

where k = 1, 2, 3, ...; C_{mn} are expressed in terms of the deflection parameters (64.10);

$$E_k(a) = a\beta'_m - \operatorname{cth} a\beta'_m(1 + a\beta'_m \operatorname{cth} a\beta'_m).$$

In deriving (65.4) and (65.5) we made use of the formulas:

$$\int_{-a}^{a} F_m(x)\cos\alpha_k x\,dx = -\frac{4\alpha_k\beta_m(-1)^{\frac{k-1}{2}}\operatorname{cth} a\beta_m}{(\alpha_k^2 + \beta_m^2)^2} \qquad m,\ k = 1, 3, 5, \ldots$$

$$\int_{-a}^{a} E_m(x)\cos\alpha_k' dx = -\frac{4(-1)^k\beta'^3_m}{(\alpha'^2_k + \beta'^2_m)^2} \qquad m,\ k = 1, 2, 3, \ldots.$$

Introducing the stress function (65. 2) in (64. 21) and inserting the obtained expressions in (64. 20), we shall obtain a system of cubic equations in the parameters f_{mn}.

In conclusion, let us note that the above method of solving non-linear boundary problems is general and may be extended to the investigation of the stability of shallow shells, rectangular in the plane. By using this method, one can extend the domain of solved problems by expanding the required functions in complete systems of other special functions. The numerical evaluation of the exact solutions obtained in series presents no difficulties of principle, but involves cumbersome computations, due to the necessity of solving a system of non-linear algebraic equations for the required parameters. The successful overcoming of these difficulties requires the use of computers. The solution of the same problems by the method of P. F. Papkovich was given in article /XIV. 1/.

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